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THE NUMBER OF GENERATORS OF THE ALGEBRA OF KÄHLER DIFFERENTIALS

In this paper R denotes an integrity domain (i.e. R is a commutative ring with unit without zero divisors), $\tilde{R}$ is the field of fractions of R and A is an associative and commutative R-algebra with a unit and containing R as a subalgebra.

Associated with A we have the A-module of Kähler differentials $\Omega_R(A)$ defined by

$$\Omega_R(A) = I/I^2$$

where I is the kernel of the multiplication $A \otimes_R A \to A$.

The purpose of this paper is to find lower bounds for the number of generators of $\Omega_R(A)$. In particular we show that for most algebras of real continuous functions the cardinal of any set of generators of $\Omega_R(A)$ is at least that of the real numbers. This is in contrast to the A-module of derivations of A in A, $\mathrm{Der}_R(A,A)$, which is the dual of $\Omega_R(A)$ and is zero for $A = C_{\mathbb{R}}(X)$ (algebra of real continuous functions on any topological space X).

(1) Definition of $n(\mathfrak{p})$. Let $\mathfrak{p}$ be a prime ideal of A such that $\mathfrak{p} \cap R = 0$ and let $k_{\mathfrak{p}}$ be the field of fractions of $A/\mathfrak{p}$ (which is canonically isomorphic to the residue field of the local ring $A_{\mathfrak{p}}$). We define then $n(\mathfrak{p})$ as the degree of transcendence of the canonical inclusion $\tilde{R} \subset k_{\mathfrak{p}}$.

(2) Theorem. Let $\mathfrak{p}$ be as above, then $n(\mathfrak{p})$ is a lower bound for the cardinal of any set of generators of $\Omega_R(A)$.

Proof. Let $R_{\mathfrak{p}}$ be the algebraic closure of $\tilde{R}$ in $k_{\mathfrak{p}}$ and consider the homomorphisms of $k_{\mathfrak{p}}$-vector spaces

$$\Omega_R(k_{\mathfrak{p}}) \xrightarrow{\cong} \Omega_{\tilde{R}}(k\ ) \longrightarrow \Omega_{R_{\mathfrak{p}}}(k_{\mathfrak{p}}).$$

As a consequence of the first fundamental exact sequence, see page 186 of [3], we obtain that the left map is an isomorphism (because $\Omega_R(\tilde{R}) = 0$) and the right map is an epimorphism, which is also an isomorphism whenever the characteristic of R is zero.

Therefore

(3) $$\dim_{k_{\mathfrak{p}}} \Omega_R(k_{\mathfrak{p}}) \geq \dim \Omega_{R_{\mathfrak{p}}}(k_{\mathfrak{p}}) = n(\mathfrak{p}).$$

We consider next the composite of the following sequence of homomorphisms of $k_{\mathfrak{p}}$-vector spaces, which are defined in the obvious way,

(4) $$k_{\mathfrak{p}} \otimes_A \Omega_R(A) \xrightarrow[(a)]{\cong} k_{\mathfrak{p}} \otimes_A (A/\mathfrak{p} \otimes_A \Omega_R(A)) \xrightarrow[(b)]{} k_{\mathfrak{p}} \otimes_A \Omega_R(A/\mathfrak{p}) \xrightarrow[(c)]{\cong} \Omega_R(k_{\mathfrak{p}})$$

where (a) is clearly an isomorphism, (b) is an epimorphism because of the second fundamental exact sequence, see page 187 of [3], and (c) is an isomorphism since Ω_R commutes with formation of fractions.

Therefore (3) and (4) yield

(5) $$\dim_{k_{\mathfrak{p}}} (k_{\mathfrak{p}} \otimes_A \Omega_R(A)) \geq n(\mathfrak{p}).$$

Finally, if $(\alpha_i)_{i \in I}$ is any set of generators of $\Omega_R(A)$ as an A-module we have an epimorphism $\bigoplus_{i \in I} A \longrightarrow \Omega_R(A)$ which sends $(a_i)_{i \in I}$ to $\sum_{i \in I} a_i \alpha_i$, and so we obtain an epimorphism of $k_{\mathfrak{p}}$ vector spaces $\bigoplus_{i \in I} k_{\mathfrak{p}} \longrightarrow k_{\mathfrak{p}} \otimes_A \Omega_R(A)$ and this, together with (5), shows that $|I| \geq n(\mathfrak{p})$, where $|I|$ denotes the cardinal of I. Q.E.D.

(6) **Proposition.** Let $\{a_i\}_{i\in I}$ be a set of elements of A such that the canonical map $R[(X_i)_{i\in I}] \longrightarrow R[(a_i)_{i\in I}]$ is an isomorphism (i.e. the elements a_i are algebraically independent over R). There exists then a prime ideal $\mathfrak{p}$ of A with $\mathfrak{p} \cap R = 0$ and such that the canonical image of the elements a_i in $k_{\mathfrak{p}}$ are algebraically independent over R and so $|I| \leqslant n(\mathfrak{p})$. In particular we have, because of theorem (2), that $|I|$ is a lower bound for the cardinal of any set of generators of $\Omega_R(A)$.

Proof. Let $\mathfrak{p}$ be an ideal of A which is maximal among those ideals $\mathfrak{p}$ of A such that $R[(a_i)_{i\in I}] \cap \mathfrak{p} = 0$. The ideal $\mathfrak{p}$ is clearly prime because of the hypothesis on $\{a_i\}_{i\in I}$ and the canonical images of the a_i in $k_{\mathfrak{p}}$ are algebraically independent over R because $\mathfrak{p} \cap R[(a_i)_{i\in I}] = 0$. Q.E.D.

Remark. The idea for considering $A/\mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal satisfying $\mathfrak{p} \cap R[(a_i)]_{i\in I} = 0$, was given to me by R. Swan.

(7) **Proposition.** If $da = 0$ for $a \in A$, then a is a zero of a nonzero polynomial in $R[X]$. Here d denotes the universal derivation on A, $d: A \longrightarrow \Omega_R(A)$, which sends x to dx = class of $(x \otimes 1 - 1 \otimes x)$ modulo I^2 (with I = kernel of the multiplication $A \otimes_R A \longrightarrow A$).

Proof. If a was not a zero of any nonzero polynomial of $R[X]$, proposition (6) would imply the existence of a prime ideal $\mathfrak{p}$ of A such that the image $\bar{a}$ of a in $k_{\mathfrak{p}}$ would be transcendental over $\tilde{R}$. Therefore $d\bar{a} \neq 0$ and so $da \neq 0$, which would contradict our hypothesis.

(8) **Theorem.** Let A be a subalgebra of the algebra $C_{\mathbf{R}}(X)$ of real continuous functions on a topological space X. Then $\Omega_{\mathbf{R}}(A) = 0$ if and only if each element of A has finite image.

Proof. Assume $\Omega_{\mathbf{R}}(A) = 0$. Thus $df = 0$ for all $f \in A$ and now (7) implies that we have $\lambda_0 + \lambda_1 f + \dots + \lambda_n f^n = 0$ with $\lambda_i \in \mathbf{R}$ $(i=0,\dots,n)$, $\lambda_n \neq 0$. This shows that f takes only a finite number of values.

Conversely assume that each element of A takes only a finite number of values. Let $\lambda_1,\dots,\lambda_n$ be the nonzero values taken by $f \in A$ with $\lambda_i \neq \lambda_j$ if $i \neq j$. The sets $X_i = \{x \in X | f(x) = \lambda_i\}$ $(i=1,\dots,n)$ are both open and closed and so they are union of connected components of X. Then $1,f,\dots,f^{n-1}$ is a basis of the real vector space $\mathbf{R}[f]$ because of the following determinant (Vandermonde) being nonzero:

$$\begin{vmatrix} 1 & \dots & 1 \\ \lambda_1 & \dots & \lambda_n \\ \vdots & & \\ \lambda_1^{n-1} & \dots & \lambda_n^{n-1} \end{vmatrix} = \prod_{i>j} (\lambda_i - \lambda_j).$$

In particular the functions e_i $(i=1,\dots,n)$ equal 1 on X_i and 0 on $X-X_i$ belong to $\mathbf{R}[f]$ and they consitute a basis. Now, since $f = \sum_{i=1}^{n} \lambda_i e_i$ and $e_i^2 = e_i$ we have $df = 0$ in view of the following lemma. Q.E.D.

(9) **Lemma.** Let x be an idempotent element of A (i.e. $x^2 = x$), then $dx = 0$. In particular $\Omega_R(A) = 0$ if A is generated by idempotent elements.

Proof. $x \otimes 1 - 1 \otimes x = (x \otimes 1 - x \otimes x)^2 - (1 \otimes x - x \otimes x)^2 \in I^2$ where I is the kernel of the multiplication $A \otimes_R A \longrightarrow A$. Therefore $dx = 0$. Q.E.D.

(10) **Theorem.** Let A be a subalgebra of the algebra of real continuous functions $C_{\mathbf{R}}(X)$, on an arbitrary topological space X and assume that there exists a function $f \in A$ satisfying the following three properties:

a) f is nonnegative,

b) $f^r \in A$ for each positive real number r,

c) f takes arbitrarily small positive values.

In these hypotheses any set of generators of $\Omega_{\mathbf{R}}(A)$ has cardinal at least that of the real numbers.

Proof. The idea for this proof is taken from theorem on page 173 of [2].

Let $f \in A$ satisfying (a), (b) and (c) above and choose a family of positive real numbers $(r_i)_{i\in I}$ which are taken to be $\mathbf{Q}$-linearly independent and with I having the cardinal of the reals. We show now that the functions $(f^{r_i})_{i\in I}$ are $\mathbf{R}$-algebraically independent and this completes the proof by using proposition (6).

Let P be a real polynomial in the variables $(X_i)_{i\in I}$ and suppose $P \in \mathbf{R}[X_1,\dots,X_q]$. Assume that P has m distinct monomials with nonzero coefficients $(m \geq 1)$:

$$P = \sum_{k=1}^{m} a_k X_1^{n_{k1}} \dots X_q^{n_{k1}} \quad \text{with } a_k \neq 0 \quad (k=1,\dots,m) \text{ and}$$

$(n_{k1},\dots,n_{kq}) = (n_{k'1},\dots,n_{k'q}) \Leftrightarrow k = k'$.

We want to show that $P(f^{r_1},\dots,f^{r_q}) \neq 0$. Let $x \in X$, we have then

$$P(f^{r_1},\dots,f^{r_q})(x) = \sum_{k=1}^{m} a_k f(x)^{r_1 n_{k1}+\dots+r_q n_{kq}}.$$

Set $s_k = r_1 n_{k1} + \dots + r_q n_{kq}$ $(k = 1,\dots,m)$.

The numbers s_k are clearly nonnegative and distinct because whenever $s_k = s_{k'}$, we have $\sum_{i=1}^{q} r_i(n_{ki}-n_{k'i}) = 0$ and so, the r_i being $\mathbf{Q}$-linearly independent, $n_{ki} = n_{k'i}$ $(i=1,\dots,q)$. Thus $k = k'$.

Let s_j be the minimum of $\{s_1,\dots,s_m\}$ and write

$$\sum_{k=1}^{m} a_k f(x)^{s_k} = f(x)^{s_j}\Big(a_j + \sum_{k\neq j} a_k f(x)^{s_k - s_j}\Big) \text{ with } s_k - s_j > 0 \text{ for all } k\neq j.$$

But now, since f takes arbitrarily small positive values, there exists $x \in X$ such that $f(x) > 0$ and

$$\Big|\sum_{k\neq j} a_k f(x)^{s_k-s_j}\Big| < |a_j|.$$

Therefore, for this x, we have that $a_j + \sum_{k\neq j} a_k f(x)^{s_k-s_j}$ has the same sign as a_j. In particular it is nonzero and therefore $\sum_{k=1}^{m} a_k f(x)^{s_k} \neq 0$. Q.E.D.

(11) Theorem. If we replace condition (c) of theorem (10) by condition (d) below, the same result holds.

(d) $f(X)$ has nonempty interior in $\mathbf{R}$.

Proof. We follow word for word the proof of (10) until we have $\sum_{k=1}^{m} a_k f(x)^{s_k}$ with all $s_k \geq 0$ and distinct.

Condition (d) implies the existence of an interval $[a,b]$ contained in $f(X)$ and such that $a \geq 0$. Choose then a real number λ such that $0<\lambda<1$ and $\lambda^m > \frac{a}{b}$. Therefore

$$a < b\lambda^m < b\lambda^{m-1} < \dots < b\lambda < b.$$

But now, by hypothesis, there exist points $x_1,\dots,x_m$ in X such that $f(x_k) = b\lambda^k$ $(k=1,\dots,m)$ and we have

$$\begin{vmatrix} f(x_1)^{s_1} & \dots & f(x_1)^{s_m} \\ \vdots & & \vdots \\ f(x_m)^{s_1} & & f(x_m)^{s_m} \end{vmatrix} = b^{s_1+\dots+s_m}\lambda^{s_1+\dots+s_m} \cdot \prod_{i>j} (\lambda^{s_i} - \lambda^{s_j}) \neq 0.$$

Thus, $\sum_{k=1}^{m} a_k f(x_i)^{s_k} \neq 0$ for some i, because if not we had $a_k = 0$ $(k=1,\dots,m)$ contrary to our hypothesis. This shows that $P\left(f^{r_1},\dots,f^{r_q}\right) \neq 0$.

(12) Remark. Condition (d) of theorem (11) above can be, of course, replaced by the following weaker condition, which is really what is used in the proof:

(d′) For each finite set of nonnegative distinct real numbers $s_1,\dots,s_m$, there exist real numbers $\lambda_1,\dots,\lambda_m$ in the image of f such that

$$\begin{vmatrix} \lambda_1^{s_1} & \dots & \lambda_1^{s_m} \\ \vdots & & \vdots \\ \lambda_m^{s_1} & \dots & \lambda_m^{s_m} \end{vmatrix} \neq 0.$$

(13) Definition. For each topological space X there exists a completely regular space X′ and a surjective continuous map $\tau : X \to X'$ such that $f \to f\circ\tau$ is an isomorphism from $C_{\mathbf{R}}X'$ onto $C_{\mathbf{R}}X$ (see Theorem 3.9 page 41 of [2]). The space X′ is the quotient of X by the equivalence relation $x \sim x' \Longleftrightarrow f(x) = f(x')$ for all $f \in C_{\mathbf{R}}X$ (i.e. it identifies points that cannot be separated by real continuous functions). X′ is endowed with the weak topology with respect to the family of maps $\left\{ f : X' \to \mathbf{R} \mid f\tau \in C_{\mathbf{R}}(X) \right\}$ where $\tau : X \to X'$ is the canonical map. We say that X′ is the completely regular space associated to X.

(14) Corollary (of Theorem 10). Let X be a topological space whose completely regular space associated have infinite by many points, then any set of generators of $\Omega_{\mathbf{R}}(C_{\mathbf{R}}(X))$ has at least the cardinal of the reals.

P r o o f . a) Suppose first that X is not pseudocompact, i.e. there exists a non bounded function $g \in C_{R}(X)$. Define then $f = \frac{1}{1+g^2}$. It is clear that f satisfies conditions (a), (b) and (c) of Theorem (10) and so the Corollary holds in this case.

b) Suppose now that X is pseudocompact, then its associate completely regular space X' is again pseudocompact and has an infinite number of points by hypothesis. Let $\beta X'$ be the Stone-Čech compactification of X' and we have isomorphisms

$$C_{R}(\beta X') \xrightarrow[i^*]{\cong} C_{R}(X') \xrightarrow[\tau^*]{\cong} C_{R}(X)$$

($i: X' \hookrightarrow \beta X'$ and $\tau: X \longrightarrow X'$ is the canonical projection). Let X_o be a connected component of $\beta X'$ having at least two points. It exists because if all components of $\beta X'$ were points, $\beta X'$ would have a finite number of points and the same would happen to X' contrary to our hypothesis. Let $f: \beta X' \longrightarrow R$ be continuous not vanishing identically on X_o, but having one zero in X_o. Clearly f^2 satisfies the hypothesis of Theorem (10) and so the Corollary also holds now.

(15) C o r o l l a r y (of Theorem 11). Let X be a C^{∞}-manifold of dimension $n \geq 1$. Then the cardinal of any set of generators of $\Omega_{R}(C^{\infty}(X))$ is at least that of the real numbers.

P r o o f . It is clear the existence of a function $f \in C^{\infty}(X)$ such that is positive and $f(X)$ has nonempty interior. Then we apply theorem (11) since f^r is also C^{∞} for all $r > 0$ since we are away from zero.

(16) T h e o r e m . Let X be a C^{∞} manifold of dimension $n \geq 0$ and let $\phi : \Omega_{R}(C^{\infty}(X)) \longrightarrow A^1(X)$ be the canonical epimorphism, where $A^1(X)$ denotes the C^{∞} 1-forms on X and ϕ is the unique homomorphism of $C^{\infty}(X)$-modules making commutative the diagram

$$\begin{array}{ccc} C^{\infty}(X) & \xrightarrow{d} & A^{1}(X) \\ \downarrow d & \nearrow \phi & \\ \Omega_{\mathbf{R}}(C^{\infty}(X)) & & \end{array}$$

Then ϕ is an isomorphism if and only if X consists of a finite number of points, and then both $\Omega_{\mathbf{R}}(C^{\infty}(X))$ and $A^{1}(X)$ are zero.

Proof. It is an easy consequence of (14) and (15) together with the fact that $A^{1}(X)$ is a finitely generated $C^{\infty}(X)$-module when X is connected, see Corollary on page 107 of [1]).

(17) Remark. If X is a C^{∞} manifold with an infinite number of components we have two differentials defined on $C^{\infty}(X)$: the algebraic differential $d: C^{\infty}(X) \to \Omega_{\mathbf{R}}(C^{\infty}(X))$ and the usual exterior derivative $d: C^{\infty}(X) \to A^{1}(X)$. We have shown that the functions with zero differential are not the same for both, since if f is constant on each component but having an infinite image the algebraic differential is nonzero but the usual exterior derivative is zero.

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