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B.G. Pachpatte

ON ABSTRACT SECOND ORDER DIFFERENTIAL EQUATIONS

1. Introduction

In this paper we study the problem

(1) $$u''(t) + Au'(t) + Bu(t) = f(t,u(t)),$$

(2) $$u(0) = \phi, \quad u'(0) = \psi,$$

in an arbitrary Banach space E with norm $\|\cdot\|$, where A, B are usually unbounded linear operators in E and $f \in C[I\times E, E]$, $I = [0,a]$, $a > 0$. Equation (1) serves as an abstract model for the semilinear versions of the wave equation, the telegraph equation, and the equation of motion for a vibrating plate (see [1]). In [11] there was used a factoring technique and the method of successive approximations to study the existence, uniqueness and stability of solutions of (1), (2) (see also [1]-[4], [10]-[13], [15]).

In this paper we use the theory of infinitesimal generator of (C_o)-semigroup in a Banach space, a factoring technique and the general method of successive approximations based on the idea of Ważewski [14] (see also [6]-[8]). The results obtained here are nonlocal and the required conditions differ from those in [11].

2. Statement of results

The set of bounded linear operators $\{T(t);\ t \in R^+ := [0,\infty)\}$ is a (C_o)-semigroup on E, if

(i) $T(t+s) = T(t)T(s) = T(s)T(s), \quad t,s \geqslant 0,$

(ii) $T(0) = I_o$ (the identity operator),

(iii) $T(\cdot)$ is strongly continuous in $t \in R^+$,

(iv) $\|T(t)\| \leqslant Q\, e^{\mu t}$ for some $Q, \mu > 0$, $t \in R^+$.

The operator A is the generator of $T(\cdot)$, if

$$A\phi = \lim_{h \to 0^+} \left(\frac{(T(h) - T(0))}{h} \right) \phi$$

and $D(A)$, the domain of A, is the set of $\phi \in E$ for which the limit exists. Formally $T(t)\phi$ satisfies the Cauchy problem

$$u'(t) = Au(t), \quad u(0) = \phi. \tag{3}$$

If $\phi \in D(A)$, then $u(\cdot) \in C^1(R^+, E)$ and (3) holds. More generally, $u(t) = T(t)\phi$ is said to be a mild solution of (3) when $\phi \notin D(A)$. Following [11], we assume the existence of linear (possibly unbounded) operators A_1, A_2 such that $A_1 + A_2 = -A$, $A_2 A_1 = B$ and A_j generates the (C_o)-semigroup T_j, $j = 1,2$. The operators A_1, A_2 need not commute. For the elementary properties of strongly continuous semigroups we refer to [5], [9].

We say that a function u is a mild solution of (1), (2), if u is continuous and satisfies

$$u(t) = u_o(t) + \int_0^t \int_0^\tau T_1(t-\tau) T_2(\tau - s) f(s, u(s))\, ds\, d\tau, \tag{4}$$

where

$$u_o(t) = T_1(t)\phi + \int_0^t T_1(t-\tau) T_2(\tau)(\psi - A_1\phi)\, d\tau \tag{5}$$

and $\phi \in D(A_1)$.

By letting $z(t) = f(t,u(t))$, we rewrite (4) as

$$z(t) + f\left(t, u_o(t) + \int_0^t \int_0^\tau T_1(t-\tau)T_2(\tau-s)z(s)\,ds\,d\tau\right). \tag{6}$$

Now, by substituting $F(t,r) = f(t,u_o(t) + r)$ in (6), we get an integral equation of the form

$$z(t) = F\left(t, \int_0^t \int_0^\tau T_1(t-\tau)T_2(\tau-s)z(s)\,ds\,d\tau\right) =: Lz(t). \tag{7}$$

We make the following hypotheses used throughout this paper.

(H_1) Assume that:

(i) there exists a continuous function $g: I\times R^+ \to R^+$ nondecreasing with respect to r such that $g(t,0) \equiv 0$,

(ii) for (t,y_1), $(t,y_2) \in I\times E$ there is

$$\|F(t,y_1) - F(t,y_2)\| \leq g(t, \|y_1 - y_2\|).$$

(H_2) There exists a continuous function $\bar{r} : I \to R^+$ satisfying the inequality $M\bar{r}(t) + p(t) \leq \bar{r}(t)$, where

$$Mr(t) = g\left(t, N^2 \int_0^t \int_0^\tau r(s)\,ds\,d\tau\right), \tag{8}$$

$$p(t) = \sup_{0\leq\xi\leq t} \|F(\xi, 0)\| \tag{9}$$

and $N > 0$ is a constant.

(H_3) In the set of functions, satisfying the condition $0 \leq r(t) \leq \bar{r}(t)$ for $t \in I$, the function r such that $r(t) \equiv 0$ for $t \in I$ is the only measurable solution of the equation

$$r(t) = Mr(t), \quad t \in I. \tag{10}$$

Define the sequence $\{z_n\}$ by the relations

(11) $$z_o(t) \equiv 0, \qquad z_{n+1}(t) = Lz_n(t),$$

for $t \in I$, $n = 0,1,2,\dots$.

To prove the convergence of $\{z_n\}$ to the solution $\bar{z}$ of (7), we define the sequence $\{r_n\}$ by the relations

(12) $$r_o(t) = \bar{r}(t), \quad r_{n+1}(t) = Mr_n(t),$$

for $t \in I$, $n = 0,1,2,\dots$, where $\bar{r}(t)$ is defined in hypothesis (H_2).

Theorem 1. Let the hypotheses (H_1)-(H_3) hold. Then there exists a continuous solution $\bar{z}(t)$, $t \in I$, of equation (7). The sequence $\{z_n\}$ defined by (11) converges uniformly to $\bar{z}$ in I and the following estimates

(13) $$\|\bar{z}(t) - z_n(t)\| \leq r_n(t), \quad t \in I, \quad n = 0,1,2,\dots,$$

(14) $$\|\bar{z}(t)\| \leq \bar{r}(t), \quad t \in I,$$

hold. Moreover, the solution $\bar{z}$ of equation (7) is unique in the set of functions satisfying the condition (14).

The next theorem gives conditions under which equation (7) has at most one solution. These conditions do not guarantee the existence of a solution of equation (7).

Theorem 2. Let hypothesis (H_1) be fulfilled. If the function r, $r(t) \equiv 0$, $t \in I$, is the only nonnegative, finite and measurable solution of the inequality

(15) $$r(t) \leq Mr(t), \quad t \in I,$$

then equation (7) has at most one solution in I.

Consider now the equation

(16) $$v(t) = H\left(t, \int_0^t \int_0^\tau T_1(t-\tau)T_2(\tau-s)v(s)\,ds\,d\tau\right),$$

where $H \in C[I \times E, E]$.

T h e o r e m 3. Assume that the hypothesis (H_1) holds and

(i) $\bar{z}$ and $\bar{v}$ are solutions of equations (7) and (16), respectively,

(ii) the sequence $\{h_n(t)\}$, $t \in I$, defined by the relations

$$(17) \qquad \begin{cases} h_o(t) \geq \|\bar{z}(t)\| + \|\bar{v}(t)\|, \\ h_{n+1}(t) = Mh_k(t) + q(t). \end{cases}$$

for $t \in I$, $n = 0,1,2,\dots$, where

$$(18) \qquad q(t) = \|L\bar{v}(t) - \bar{v}(t)\|,$$

has the limit $\bar{h}(t)$ for $t \in I$. Then we have

$$(19) \qquad \|\bar{z}(t) - \bar{v}(t)\| \leq \bar{h}(t), \qquad t \in I.$$

R e m a r k 1. We note that Theorems 1-3 also yield the existence, uniqueness, error estimations and stability of the solutions of equivalent integral equation (4) and, consequently, of the mild solutions of the problem (1), (2). As shown in [11], Theorems 1-3 can be also extended to the initial value problems for a large class of partial differential equations.

3. Proofs of Theorems 1-3

L e m m a 1. If the condition (i) of hypothesis (H_1) and hypotheses (H_2)-(H_3) are satisfied, then

$$(20) \qquad 0 \leq r_{n+1}(t) \leq r_n(t) \leq \bar{r}(t),$$

for $t \in I$, $n = 0,1,2,\dots$ and $r_n \Rightarrow 0$ for $n \to \infty$, where $\Rightarrow$ denotes the uniform convergence in I.

The relation (20) follows by induction, in view of (12). But (20) implies the convergence of $\{r_n\}$ to some nonnegative measurable function $m(t)$ such that $0 < m(t) \leq \bar{r}(t)$ for $t \in I$. By the Lebesque theorem and the continuity of g, the function $m(t)$ satisfies (10) and, by (H_3), we have $m(t) \equiv 0$, $t \in I$. The uniform convergence of $\{r_n\}$ in I follows from the Dini theorem. This completes the proof of Lemma 1.

In order to prove Theorem 1, first we prove that the sequence $\{z_n(t)\}$ defined by (11) satisfies the conditions

(21) $$\|z_n(t)\| \leq \bar{r}(t), \quad t \in I, \quad n = 0,1,2,\dots .$$

Obviously, $\|z_o(t)\| \equiv 0 \leq \bar{r}(t)$, $t \in I$. Furthermore, if we suppose that (21) is true for $n \geq 0$, then, by (H_1), (H_2) and by the fact that for $t \in I$, $\|Tj(t)\| \leq N$, $j = 1,2$, $N > 0$ with constant, we have

$$\|z_{n+1}(t)\| \leq M\|z_n(t)\| + p(t) \leq M\bar{r}(t) + p(t) \leq \bar{r}(t),$$

for $t \in I$. The relation (21) follows by induction. Next we prove that

(22) $$\|z_{n+k}(t) - z_n(t)\| \leq r_n(t), \quad t \in I, \quad n, k = 0,1,2,\dots .$$

By (21), we have

$$\|z_k(t) - z_o(t)\| = \|z_k(t)\| \leq \bar{r}(t) = r_o(t),$$

for $t \in I$, $k = 0,1,2,\dots$. Suppose that (22) is true for $n, k \geq 0$; then, by the hypothesis (H_1) and by the fact that for $t \in I$, $\|T_j(t)\| \leq N$, $j = 1,2$; $N > 0$ with a constant, we have

$$\|z_{n+k+1}(t) - z_{n+1}(t)\| = \|Lz_{n+k}(t) - Lz_n(t)\| \leq$$

$$\leq M\|z_{n+k}(t) - z_n(t)\| < M\, r_n(t) = r_{n+1}(t),$$

and we obtain (22) by induction. By Lemma 1, $r_n(t) \Rightarrow 0$ in I and so we have from (22) $z_n \Rightarrow \bar{z}$ in I. The continuity of $\bar{z}$ follows from the uniform convergence of $\{z_n\}$ and from the continuity of all functions z_n. If $k \to \infty$, then (22) gives estimation (13) and estimation (14) is implied by (21). It is obvious that $\bar{z}$ is a solution of (7).

To prove that the solution $\bar{z}$ of (7) is unique, let us suppose that there exists another solution $\hat{z}$ of (7) such that $\bar{z}(t) \not\equiv \hat{z}(t)$ and $\|\hat{z}(t)\| \leq \bar{r}(t)$ for $t \in I$. By induction we get $\|\hat{z}(t) - z_n(t)\| \leq r_n(t)$, $t \in I$,

$n = 0,1,2,\dots$, and hence it follows that $\bar{z}(t) \equiv \hat{z}(t)$, $t \in I$. This contradiction proves the uniqueness of $\bar{z}$ in the set of functions satisfying relation (14). This completes the proof of Theorem 1.

To prove Theorem 2, let us suppose that there exist two solutions $\bar{z}$ and $\hat{z}$ of equation (7) in I, $\bar{z}(t) \not\equiv \hat{z}(t)$, $t \in I$. Now, by (H_1) and by the fact that for $t \in I$, $\|T_j(t)\| \leq N$, $j = 1,2$; $N > 0$ with a constant, we have

$$\|\bar{z}(t) - \hat{z}(t)\| \leq M\|\bar{z}(t) - \hat{z}(t)\|. \tag{23}$$

Taking $r(t) = \|\bar{z}(t) - \hat{z}(t)\|$, $t \in I$, in (23), we infer from (15) that $r(t) \equiv 0$ for $t \in I$, i.e. $\bar{z}(t) \equiv \hat{z}(t)$, $t \in I$. This contradiction completes the proof of Theorem 2.

To prove Theorem 3, let

$$h(t) = \|\bar{z}(t) - \bar{v}(t)\|, \quad t \in I, \tag{24}$$

then we have

$$\begin{aligned} h(t) &\leq \|L\bar{z}(t) - L\bar{v}(t)\| + \|L\bar{v}(t) - \bar{v}(t)\| \leq \\ &\leq M\|\bar{z}(t) - \bar{v}(t)\| + q(t) = Mh(t) + q(t). \end{aligned} \tag{25}$$

From (25) and (17) we obtain

$$h(t) \leq \|\bar{z}(t)\| + \|\bar{v}(t)\| \leq h_o(t), \quad t \in I. \tag{26}$$

Now, by induction, we get

$$h(t) \leq h_n(t), \quad t \in I, \quad n = 0,1,2,\dots. \tag{27}$$

The inequality (19) is implied by (27) as $n \to \infty$. This completes the proof of Theorem 3.

Remark 2. Theorems 1-3 can be easily extended to the Volterra integro-differential equation of the form

$$u''(t) + Au'(t) + Bu(t) = f\left(t, u(t), \int_0^t K[t,s,u(s)]ds\right), \tag{28}$$

with the given initial conditions (2), where A, B are as in (1) and $K \in C[I \times I \times E, E]$, $f \in C[I \times E \times E, E]$. In [13] were obtained some results on the existence and uniqueness of the solutions slightly different than (28) by using cosine families and fixed point theorems.

4. Further applications

Consider now the problem

$$u''(t) + Au'(t) = f(t,u(t),u'(t)), \tag{29}$$

$$u(0) = u_o, \quad u'(0) = u_1, \tag{30}$$

where A generates a strongly continuous semigroup $(T(t))$ on a Banach space E.

Recently, in [2] was used the Banach fixed point theorem to the mild solutions of (29), (30). Particular examples of (29) are, among others, the strongly damped nonlinear Klein-Gordon equation and the vibrating beam equation (see [2]).

To solve (29), (30), consider the system of integral equations

$$\tag{31} \begin{cases} u(t) = u_o + u_1 \int_0^t T(s)ds + \int_0^t \int_0^s T(s-\tau)f(\tau,u(\tau),v(\tau))d\tau ds, \\ v(t) = T(t)u_1 + \int_0^t T(t-s)f(s,u(s),v(s))ds. \end{cases}$$

If $(u(t), v(t))$ is a solution of (31), then $u(t) = u_o + \int_0^t v(s)ds$ is called a mild solution of (29), (30). By letting $z(t) = f(t,u(t),v(t))$, we rewrite (31) as

$$z(t) = f\left(t, u_o + u_1 \int_0^t T(s)ds + \int_0^t \int_0^s T(s-\tau)z(\tau)d\tau ds,\right. \tag{32}$$

$$\left. T(t)u_1 + \int_0^t T(t-s)z(s)ds\right).$$

Now, by substituting

$$F(t,r_1,r_2) = f\left(t,u_o + u_1\int_0^t T(s)ds + r_1,\ T(t)u_1 + r_2\right).$$

in (32) we get

$$(33)\qquad z(t) = F\left(t, \int_0^t\int_0^s T(s-\tau)z(\tau)d\tau ds, \int_0^t T(t-s)z(s)ds\right).$$

Theorems 1-3 can be naturally extended to (33) under some modificated hypotheses (H_1)-(H_3).

We also note that our method can be easily extended to the problem

$$(34)\qquad u''(t) + Au'(t) = f\left(t,u(t),u'(t), \int_0^t K[t,s,u(s),u'(s)]ds\right),$$

$$(35)\qquad u(0) = u_o, \quad u'(0) = u_1,$$

where A is from (29) and $K \in C[I{\times}I{\times}E{\times}E, E]$, $f \in C[I{\times}E{\times}E{\times}E, E]$. For the several interesting results on the existence and uniqueness of the solutions for slightly different versions of problems (29), (30) and (34), (35), by using different techniques, we refer to [12], [13].

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DEPARTMENT OF MATHEMATICS AND STATISTICS, MARATHWADA UNIVERSITY, AURANGABAD 431004, MAHARASHTRA, INDIA

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