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CR-SUBMANIFOLDS OF HYPERBOLICAL ALMOST HERMITIAN MANIFOLDS

Introduction

The aim of this paper is to study the class of CR-submanifolds of hyperbolical almost Hermitian manifolds, by following the same ideas of those used in the case of CR-submanifolds of almost Hermitian manifolds [5]. We mention that the class considered here is different of that studied in [2]. A corresponding notion of semi-invariant submanifolds of locally product Riemannian manifolds was given by A. Bejancu in [4], but the condition satisfied by the metric in our case lead to different results. Throughout the paper some examples are also included.

1. CR-submanifolds of hyperbolical almost Hermitian manifolds

We assume here $\widetilde{M}$ to be a hyperbolical almost Hermitian manifold, i.e. M is endowed with an almost product structure F (that is $F^2 = \mathrm{Id}$ and $F \neq \pm \mathrm{Id}$) and a semi-Riemannian metric g such that $g(FX,FY) = -g(X,Y)$ for $X,Y \in \Gamma(T\widetilde{M})$. It follows that $\dim \widetilde{M} = 2n$ and the index of g is n. We denote by $\widetilde{\nabla}$ the Levi-Civita connection on $(\widetilde{M},g)$.

Let M be a semi-Riemannian submanifold of $\widetilde{M}$, i.e. M is a submanifold of $\widetilde{M}$ on which g is nondegenerate and of constant index, [6]. Thus, for any $X \in \Gamma(TM)$ and $V \in \Gamma(TM^{\perp})$, we put:

(1.1) $$FX = fX + tX,$$

(1.2) $$FV = BV + CV,$$

where BV, $fX \in \Gamma(TM)$ and $tX, CV \in \Gamma(T\tilde{M})$.

Also, by taking $\{N_i\}_{i=\overline{1,s}}$ to be a local orthonormal basis of $TM^{\perp}$ have locally

(1.3) $$FX = fX + \sum_{i=1}^{s} \rho_i(X)\, N_i,$$

where ρ_i, $i = \overline{1,s}$ are some local 1-forms.

Let ∇, h, A and $\nabla^{\perp}$ be the induced covariant differentiation on M, the second fundamental form, the second fundamental tensor and the normal connection, respectively.

The Gauss and Weingarten formulas are given by

(1.4) $$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y),$$

(1.5) $$\tilde{\nabla}_X \xi = -A_{\xi} X + \nabla^{\perp}_X \xi, \quad \text{for } X,Y \in \Gamma(TM), \quad \xi \in \Gamma(TM^{\perp}),$$

The Codazzi equation is

(1.6) $$[\tilde{R}(X,Y)Z]^{\perp} = (\bar{\nabla}_X h)(Y,Z) - (\bar{\nabla}_Y h)(X,Z) \quad \text{for } X,Y,Z \in \Gamma(TM),$$

where $\perp$ denotes the normal component and $\bar{\nabla}$ is defined by

$$(\bar{\nabla}_X h)(Y,Z) = \nabla^{\perp}_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) \text{ for } X,Y,Z \in \Gamma(TM).$$

We say that a semi-Riemannian submanifold M of $\tilde{M}$ is a CR-submanifold if it carries a differential distribution D which is nondegenerate and of constant index with respect to g (if D is non null), satisfying:

(1.7) $$F(D_p) = D_p \quad \text{and}$$

(1.8) $$F(D_p^{\perp}) \subset (T_p M)^{\perp} \quad \text{for } p \in M,$$

where $D^{\perp}$ is the orthogonal complementary distribution of D with respect to g.

We denote by $\nu_p^{\perp} = F(D_p^{\perp})$ for $p \in M$ and we obtain that $\nu^{\perp}$ is a vector subbundle of $TM^{\perp}$. From the assumption made for D, it follows that if it is non null, then $D^{\perp}$ (resp. $\nu^{\perp}$, ν) is nondegenerate and of constant index with respect to g, where ν is the orthogonal complementary vector subbundle of $\nu^{\perp}$ in $TM^{\perp}$.

In particular, we say that M is:

1. invariant, when $D^{\perp} = \{0\}$,
2. anti-invariant, when $D = \{0\}$,
3. proper CR, when $D \neq \{0\}$ and $D^{\perp} \neq \{0\}$,
4. generic, when $\dim D_p^{\perp} = \dim(T_pM)^{\perp} \neq 0$ for $p \in M$.

By a semi-Riemannian hypersurface of $\widetilde{M}$, we mean a semi-Riemannian submanifold of codimension one.

In the case of locally product Riemannian manifolds, not all hypersurfaces are proper semi-invariant [4]. But similarly to the case of CR-submanifolds of almost Hermitian manifolds (see [5]), in our case, every semi-Riemannian hypersurface of $\widetilde{M}$ is an example of a generic proper CR-submanifold of $\widetilde{M}$ (for $n \geq 2$) and generic anti-invariant submanifold of $\widetilde{M}$ (for $n = 1$), since no normal vector at a point of a semi-Riemannian hypersurface of $\widetilde{M}$ can be an eigenvector of F.

We give now an example of a proper CR-submanifold which is not generic.

Let's take the 3-dimensional torous $T^3 = S^1 \times S^1 \times S^1$, endowed with the Riemannian metric g obtained as a product of the standard metric on S^1 and let $\{X_i\}_{i=\overline{1,3}}$ be an orthonormal basis giving a parallelization of T^3, with X_3 normal to T^2 and X_1 tangent to S^1, where $S^1 \times \{0\} \times \{0\} \hookrightarrow T^2 \; \{0\} \hookrightarrow T^3$. By taking $\overline{M} = T^3 \times T^3$ and $\overline{g} = \begin{pmatrix} -g & 0 \\ 0 & g \end{pmatrix}$, we define F pointwise by $F_{(p,q)} = \begin{pmatrix} 0 & I_3 \\ I_3 & 0 \end{pmatrix}$ with respect to

$\{(X_i,0),\ i=\overline{1,3};\ (0,X_k),\ k=\overline{1,3}\}$ for $(p,q)\in\bar{M}$. Thus $T^2\times S^1$ is a proper CR-submanifold of the hyperbolical almost Hermitian manifold $(\bar{M},F,\bar{g})$, which is not generic.

Next we shall give a way to construct some proper CR-submanifolds of $\tilde{M}$.

Proposition 1.1. Let (L_i,F_i,g_i), $i=\overline{1,2}$ be two hyperbolic almost Hermitian manifolds with $\dim L_1>2$. Then $L=L_1\times L_2$ endowed with $F=\begin{pmatrix}F_1 & 0\\ 0 & F_2\end{pmatrix}$ and $g=\begin{pmatrix}g_1 & 0\\ 0 & g_2\end{pmatrix}$ is a hyperbolical almost Hermitian manifold and therefore any semi-Riemannian hypersurface of L_1 provides an example of a proper CR-submanifold of L which is not generic.

Remark The previous example is not a particular case of Proposition 1.1.

Proposition 1.2. Let M be a semi-Riemannian hypersurface of $\tilde{M}$ having a spacelike unit normal vector field N (i.e. $g(N,N)=1$). Then M is a hyperbolical almost paracontact manifold (said also almost paracohermitian manifold, see [3]).

Proof. Let $X\in\Gamma(TM)$. From (1.3), we get $f^2X=X-\eta(X)FN$ and $\eta(FN)=1$. We also have $g(fX,fY)=g(FX,FY)+\eta(X)\eta(Y)=$ $=-g(X,Y)+\eta(X)\eta(Y)$ for $X,Y\in\Gamma(TM)$.

As a consequence of this proposition, we get the following

Example. Let $(R^m_k,<.>)$ be the semi-Euclidean space and let $S^{m-1}_k(r)=\{x\in R^m_k \mid <x,x>=r^2\}$ be the pseudosphere, where $<,>=\begin{pmatrix}-I_k & 0\\ 0 & I_{m-k}\end{pmatrix}$ with respect to the standard basis of R^m, $m\geq 2$, $0<k\leq m$, see [6]. We remark that $(R^{2n}_n, F, <,>)$ and $S^{2n-1}_n(r)$ satisfy the conditions assumed in Proposition 1.2 for $\tilde{M}$ and M respectively, where F is given by $F=\begin{pmatrix}0 & I_n\\ I_n & 0\end{pmatrix}$ with respect to the standard basis of R^{2n}, $n \geq 1$. Thus, we obtain that all (2n-1)-dimensional pseudospheres of index $n \geq 1$ are hyperbolical almost paracontact manifolds.

Now, we take M to be a CR-submanifold of $\tilde{M}$. For any $x \in \Gamma(TM)$, we put

$$(1.9) \qquad X = PX + QX,$$

where $PX \in \Gamma(D)$ and $QX \in \Gamma(D^{\perp})$.

Remark. From (1.9) it follows that f is an f-structure on M [7]. M is said to be D-geodesic, $D^{\perp}$-geodesic or $(D,D^{\perp})$-geodesic if $h(D,D) = \{0\}$, $h(D^{\perp},D^{\perp}) = \{0\}$ or $h(D,D^{\perp}) = \{0\}$, respectively.

2. CR-submanifolds of hyperbolical Kählerian manifolds

We assume in this section $\tilde{M}$ to be a hyperbolical Kählerian manifold, i.e. $(\tilde{M},F,g)$ is a hyperbolical almost Hermitian manifold such that

$$(2.1) \qquad \tilde{\nabla} F = 0,$$

where $\tilde{\nabla}$ is the Levi-Civita connection of g. Now, from §1 it follows that $(R^{2n}_n, F, <,>)$ is a hyperbolical Kählerian manifold, $n \geq 1$. As it is well known that a connected, simply connected, complete m-dimensional semi-Riemannian manifold of index k and zero sectional curvature is isometric to R^m_k and as it can be proved similarly to the Kählerian case that a q-dimensional hyperbolical Kählerian manifold $(q > 2)$ of constant sectional curvature c has $c = 0$, then it follows that $(R^{2n}_n, F, <,>)$ is the only one (up to isometry) among all connected, simply connected and complete 2n-dimensional hyperbolical Kählerian manifolds of constant sectional curvature, with $n > 1$. Let $H^{m-1}_{k-1}(r) = \left\{ x \in R^m_k \middle| <x,x> = -r^2 \right\}$ be the pseudohyperbolic space. Thus $S^{2n}_n(r)$ and $H^{2n}_n(r)$ can not be endowed with hyperbolical Kählerian structure for $n>1$. For $n = 1$, we get that the 2-dimensional de Sitter space $(S_1^2(r), F_1, <,>)$ and $(H_1^2(r), F_2, <,>)$ are hyperbolical Kählerian manifolds, where $F_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ with respect to the orthonormal basis $\left\{X_1 = (1/\sqrt{r^2+x_1^2})(-x_3e_2+x_2e_3);\ X_2=(1/r\sqrt{r^2+x_1^2})\left[(r^2+x_1^2)e_1 + x_1x_2e_2 + x_1x_3e_3\right]\right\}$ on $S_1^2(r)$ and $F_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ with respect to the orthonormal basis $\left\{Y_1 = (1/\sqrt{r^2+x_3^2})(x_2e_1-x_1e_2);\ Y_2 = (1/r\sqrt{r^2+x_3^2}) \times \left[x_1x_3e_1+x_2x_3e_2+(r^2+x_3^2)e_3\right]\right\}$ on $H_1^2(r)$, where $\{e_i\}_{i=\overline{1,3}}$ denotes the standard basis of R^3.

From now on, we assume in this section M to be a CR-submanifold of $\widetilde{M}$. If in Proposition 1.1 we take L_i, $i=\overline{1,2}$ to the hyperbolical Kählerian manifolds, then we get some examples of proper CR-submanifolds of hyperbolical Kählerian manifolds.

We deal here with the integrability of the distributions of M and we omit the proofs which are similar to those given in the Kählerian case, [5].

First, remark that from (1.9) it follows

$$\nabla_X FPY - A_{FQY}X = FP\nabla_X Y + Bh(X,Y), \tag{2.2}$$

$$h(X,FPY) + \nabla^\perp_X FQY = FQ\nabla_X Y + Ch(X,Y) \text{ for } X,Y \in \Gamma(TM). \tag{2.3}$$

From (2.2) and (2.3) we get the following

Proposition 2.1. a) D is integrable if and only if

$$B[h(X,FY) - h(Y,FX)] = 0 \quad \text{for} \quad X,Y \in \Gamma(D). \tag{2.4}$$

b) D is integrable and its leaves are totally geodesic in $\widetilde{M}$ if and only if M is D-geodesic.

c) D is integrable and its leaves are totally geodesic in M if and only if $h(D,D) \subset \Gamma(\nu)$.

Remark. The relation (2.4) is equivalent with

$$h(X,FY) - h(Y,FX) = 0 \quad \text{for} \quad X,Y \in \Gamma(D).$$

By using (2.2) we get

Lemma 2.1. If $X,Y \in \Gamma(D^{\perp})$, then

$$A_{FX}Y = A_{FY}X. \tag{2.5}$$

Proposition 2.2. a) $D^{\perp}$ is always integrable.

b) $h(D,D\ ) \subset \Gamma(\nu)$ if and only if the leaves of $D^{\perp}$ are totally geodesic in M.

In particular, when M is $(D,D^{\perp})$-geodesic, the leaves of $D^{\perp}$ are totally geodesic in M.

Proposition 2.3. The following assertions are equivalent:

a) D is parallel; b) $D^{\perp}$ is parallel; c) $h(X,Y) \in \Gamma(\nu)$ for $X \in \Gamma(TM)$ and $Y \in \Gamma(D)$; d) D and $D^{\perp}$ are integrable and their leaves are totally geodesic in M; e) $(\nabla_X f)Y = 0$ for $X \in \Gamma(TM)$ and $Y \in \Gamma(D)$.

Proof. We assume M to be a proper CR-submanifold, otherwise the assertion is trivial. We get a) $\Longleftrightarrow$ b) since ∇ is the Levi-Civita connection of M. To prove b) $\Longleftrightarrow$ c), we take $Z \in \Gamma(D^{\perp})$. From (2.2) we get $-A_{FZ}X = FP\nabla_X Z + Bh(X,Z)$. Since $g(A_{FZ}X,Y)=g(h(X,Y),FZ)$, then $h(X,Y) \in \Gamma(\nu) \Longleftrightarrow A_{FZ}X \in \Gamma(D^{\perp}) \Longleftrightarrow \nabla_X Z \in \Gamma(D^{\perp})$. Next, c) $\Longleftrightarrow$ d) follows from Proposition 2.1 and 2.2. To prove c) $\Longleftrightarrow$ e), we remark that for any $V,W \in \Gamma(TM)$, the relation (2.2) can be written as $(\nabla_V f)W - A_{FQW}V = Bh(V,W)$ from which we complete the proof.

Corollary 2.1. If $h(TM,D) \subset \Gamma(\nu)$, then

$$M = M_1 \times M_2 \quad \text{(locally)}, \tag{2.6}$$

where M_1 is a leaf of D and M_2 is a leaf of $D^{\perp}$.

3. Totally umbilical CR-submanifolds of hyperbolical Kählerian manifolds

In this section, we assume that M is a totally umbilical CR-submanifold of a hyperbolical Kählerian manifold $\tilde{M}$, i.e. M is a CR-submanifold of $\widetilde{M}$ such that $h(X,Y) = g(X,Y)H$ for $X,Y \in \Gamma(TM)$, where $H \in \Gamma(TM^{\perp})$. Remark that $S_k^{2k-1}(r)$ and $H_{k-1}^{2k-1}(r)$ are, for instance, totally umbilical CR-submanifolds of $(R_k^{2k}, F, <,>)$, $k \geq 1$.

Proposition 3.1. If $\dim D^{\perp} > 1$, then either M is totally geodesic or M is anti-invariant.

Proof. We suppose $H \neq 0$. From Lemma 2.1 we get $A_{FX}BH = A_{FBH}X$ for $X \in \Gamma(D^{\perp})$. Thus $g(A_{FX}BH,X) = g(A_{FBH}X,X) \Leftrightarrow$ $\Leftrightarrow g(h(BH,X),FX) = g(h(X,X),FBH)$ and we get

$$(3.1) \qquad [g(BH,X)]^2 = g(X,X)g(BH,BH) \quad \text{for} \quad X \in \Gamma(D^{\perp}).$$

We obtain that BH is isotropic, for if we suppose not, as $\dim D^{\perp} > 1$, we can take $Y \in \Gamma(D^{\perp})$ to be a unit vector field orthonormal to BH and from (3.1) we get that BH is isotropic again. Thus, from (3.1) it follows $g(BH,X) = 0$ for $X \in \Gamma(D^{\perp})$ and as $D^{\perp}$ is nondegenerate with respect to g, we get $BH = 0$. We have that M is anti-invariant, for if we suppose not, then $D \neq \{0\}$ and we take $Z \in \Gamma(D)$ to be a unit vector field (i.e. $|g(Z,Z)| = 1$). From (2.3) we get $g(Z,FZ)H = FQ\nabla_Z Z + g(Z,Z)CH$. We have $g(Z,FZ) = 0$, since F is sqew-symmetric with respect to g. As we have $FQ\nabla_Z Z \in \Gamma(\nu^{\perp})$, then $CH = 0$ and we complete the proof.

Corollary 3.1. If M is a proper CR-submanifold, with $\dim D^{\perp} > 1$, then M can be written as in (2.6).

Now, by using (1.6) and since $\widetilde{R}(FX,FY) = -\widetilde{R}(X,Y)$ for $X,Y \in \Gamma(T\widetilde{M})$, we get $\widetilde{K}(X \wedge Y) = 0$, where $X \in \Gamma(D)$ and $Y \in \Gamma(D^{\perp})$ span a nondegenerate plane with respect to g and $\widetilde{K}$ is the sectional curvature of $\widetilde{M}$. Thus we obtain

Proposition 3.2. There are no proper totally umbilical CR-submanifolds in any positively (or negatively) curved hyperbolical Kählerian manifold.

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