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MOMENT PROBLEM IN $\mathbb{R}^n$

By a counterexample from M. Sakai the classical moment problem in the complex plane $\mathbb{C}$ is known not to be uniquely solvable. The same construction is used here to show that this is true for the moment problem in $\mathbb{R}^n$ for any $n \geq 3$.

1. Moments in $\mathbb{R}^n$

The moments of a bounded domain D in the complex plane $\mathbb{C}$ defined by

$$(1) \qquad m_k := \int_D z^k \, dxdy \qquad (k \in \mathbb{N}_0)$$

appear as the coefficients of the Taylor expansion of

$$(2) \qquad \int_D \frac{d\xi d\eta}{z-\zeta} = \int_D \overline{\frac{z-\zeta}{|z-\zeta|^2}} \, d\xi d\eta \qquad (\zeta = \xi + i\eta)$$

at infinity. An analogue to this function in $\mathbb{R}^n$ is

$$(3) \qquad \int_D \frac{x-y}{|x-y|^n} dy = \begin{cases} \nabla \frac{1}{2-n} \int_D |x-y|^{2-n} dy, & 3 \leq n, \\ \nabla \int_D \log |x-y| \, dy, & 2 = n. \end{cases}$$

In the following only the case $3 \leq n$ will be considered. The coefficients of the power series development at infinity of the integral on the right-hand side of (3) are defined as the moments of D, see [1]. They are expressed by the Gegenbauer polynomials $C_k^{(\frac{n}{2}-1)}$ as

$$(4) \quad M_k(\xi) = \frac{n+k-2}{n-2} \int_D C_k^{(\frac{n}{2}-1)}\left(\xi \cdot \frac{y}{|y|}\right)|y|^k \, dy \quad (k \in \mathbb{N}_0, \ \xi \in \mathbb{R}^n, |\xi| = 1)$$

which is suggested by

$$(5) \qquad |x-y|^{2-n} = |x|^{2-n}\left(1 - 2\frac{x}{|x|} \cdot \frac{y}{|y|}\frac{|y|}{|x|} + \frac{|y|^2}{|x|^2}\right)^{1-\frac{n}{2}} =$$

$$= \sum_{k=0}^{\infty} C_k^{(\frac{n}{2}-1)}\left(\frac{x}{|x|} \cdot \frac{y}{|y|}\right)|y|^k |x|^{2-n-k}.$$

The infinite series is convergent for $|y|<|x|$.

Moment problem. For a given sequence $(M_k(\xi))$ of real-valued functions on $|\xi| = 1$ find a bounded domain D in $\mathbb{R}^n$ with the moments $(M_k(\xi))$.

In the complex plane this problem was posed by H.S. Shapiro (see [4], p.193) and for domains with analytic boundaries under additional assumptions shown to be uniquely solvable by e.g. S. Richardson [2]. Later M. Sakai [3] gave an example of two different domains with the same moments.

2. Counterexample

Theorem. The moment problem is general is not uniquely solvable.

Proof. Let e be any unit vector, $|e| = 1$. The moments of the circles $\{|x \mp e| < r\}$ are by the mean value property of harmonic functions

$$\frac{n+k-2}{n-2}\,\omega_n\, r^n\, C_k^{(\frac{n}{2}-1)}\,(\xi\cdot(\pm e)) \qquad (k\in \mathbf{N}_0). \tag{6}$$

Observing the homogeneity of $C_k^{(\frac{n}{2}-1)}$ it follows that the moments of the rings $\left\{3<|x\mp e|<\sqrt[n]{3^n+1}\right\}$ and for $\left\{|x\mp e|<1\right\}$ are

$$(\pm 1)^k\,\frac{n+k-2}{n-2}\,\omega_n\, C_k^{(\frac{n}{2}-1)}\,(\xi\cdot e) \qquad (k\in \mathbf{N}_0). \tag{7}$$

Hence $\left\{|x\mp e|<1\right\}\cup\left\{3<|x\pm e|<\sqrt[n]{3^n+1}\right\}$ both have the same moments

$$\left[(+1)^k + (-1)^k\right]\frac{n+k-2}{n-2}\,\omega_n\, C_k^{(\frac{n}{2}-1)}\,(\xi\cdot e) \qquad (0\leq k). \tag{8}$$

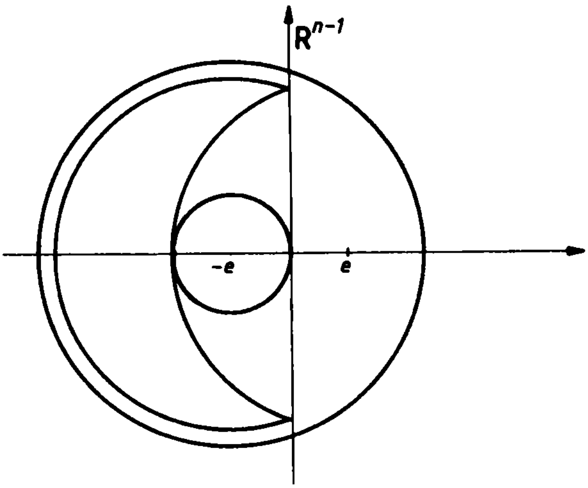
Because

$$\left\{1<|x-e|<3\right\}\cap\left\{1<|x+e|<3\right\} \tag{9}$$

is symmetric with respect to the (n-1)-dimensional plane through the origin with e as normal vector, the two domains

$$\left\{3<|x\mp e|<\sqrt[n]{3^n+1}\right\}\cup\left\{|x\pm e|<1\right\}\cup \tag{10}$$
$$\cup\left\{\left\{1<|x-e|<3\right\}\cap\left\{1<|x+e|<3\right\}\right\}\setminus\left\{0\right\}$$

have the same moments.



Figure

The previous figure shows a crosscut of one of the domains. The other one is the reflection image of this one on the mentioned plane $\mathbb{R}^{n-1}$.

3. Harmonic surfaces

A surface S in $\mathbb{R}^n$ is called a harmonic surface if it is representable by a function h harmonic in the neighbourhood of S such that

(12) $$x = \nabla h(x) \qquad (x \in S).$$

The sphere $\{|x| = R\}$ for example is harmonic because

(13) $$x = \nabla \frac{R^n}{2-n} |x|^{2-n} \quad \text{for} \quad |x| = R.$$

A harmonic surface is smooth while the domain constructed in Section 2 has a boundary which is not smooth everywhere. In [1] any smooth boundary of a bounded domain D in $\mathbb{R}^n$ is shown to be representable by a harmonic function h_i inside D and a harmonic function h_e outside D as

(14) $$x = \nabla h_i(x) + \nabla h_e(x) \qquad (x \in \partial D).$$

The boundary will be harmonic if h_i and h_e may be harmonically continued across ∂D.

The proof of (14) simply follows from the continuity of

(15) $$H(x) := \int_D \frac{x-y}{|x-y|^n} dy$$

in $\mathbb{R}^n$, and the formula (see [1])

(16) $$\int_{|y|<R} \frac{x-y}{|x-y|^n} dy = \frac{\omega_n}{n} x \qquad (|x|<R),$$

because if $\overline{D} \subset \{|x| < R\} =: B_R$

(17) $$H(x) = \nabla \frac{1}{2-n} \int_D |x-y|^{2-n} dy \qquad (x \notin D),$$

(18) $$H(x) = \int_{|y|<R} \frac{x-y}{|x-y|^n}\,dy - \nabla \frac{1}{2-n} \int_{B_R \setminus D} |x-y|^{2-n}\,dy \quad (x \in D).$$

An example for a representation of kind (13) is for the surface of the ball $\{|x-x_0| < R\}$

(19) $$x = \nabla x \cdot x_0 + \nabla \frac{R^n}{2-n} |x-x_0|^{2-n} \quad \text{for} \quad |x-x_0| = R.$$

Harmonic surfaces in $\mathbb{R}^n$ seem to have similar properties to analytic curves in $\mathbb{C}$. We therefore conjecture that the moment problem in the class of bounded domains with harmonic boundary is uniquely solvable under some additional conditions.

REFERENCES

[1] H. Begehr, R.P. Gilbert: Hele-Shaw type flows in $\mathbb{R}^n$, Nonlinear Anal. Theory, Methods and Appl. 10 (1986), 65-85.

[2] S. Richardson: Hele-Shaw flows with time-dependent free boundaries, J. Fluid Mech. 102 (1981), 263-278.

[3] M. Sakai: A moment problem on Jordan domains, Proc. Amer. Math. Soc. 70 (1978), 35-38.

[4] O.B. Bekken, B.K. Oksendal, A. Stray (Editors): Spaces of analytic functions. Lecture Notes in Math. 512, Springer-Verlag, Berlin, 1976.

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