DEMONSTRATIO MATHEMATICA

Vol. XXIII No 2 1990

T.D. Narang*)

METRIC PROJECTIONS IN LINEAR METRIC SPACES

Various properties of the metric projection are known in normed linear spaces (see e.g. [7]). G. Pantelidis [5] discussed some of these properties in linear metric spaces. In this paper we also discuss some necessary and sufficient conditions for the metric projections to be continuous and Lipschitzian in linear metric spaces.

Let G be a subspace of a linear metric space (E,d) and $x \in E$. An element $g_o \in G$ is said to be a best approximation to x in G if

$$d(x,g_o) = d(x,G).$$

The set of all such $g_o \in G$ is denoted by $L_G(x)$ i.e.

$$L_G(x) = \{g_o \in G: d(x,g_o) = d(x,G)\}.$$

G is said to proximinal if $L_G(x)$ is non-empty for each $x \in E$ and it is said to be Chebyshev if $L_G(x)$ consists of exactly one element for each $x \in E$. The mapping π_G, which takes each element x of E to the set $L_G(x)$ is called the metric projection of E onto G. For Chebyshev sets G, π_G is single-valued.

The canonical mapping $W_G : E \to E/G$ defined by $W_G(x) = x + G$ is clearly related to best approximation, since

$$L_G(x) = \{g_o \in G: d(x,g_o) = \mathrm{dist}(W_G(x),0)\}.$$

*) The author is thankful to U.G.C., India for financial support.

We shall denote by $P_G^{-1}(o)$ the set

$$P_G^{-1}(o) = \left\{ x \in E : 0 \in L_G(x) \right\}.$$

This set is called kernel of the mapping π_G.

The first two theorems give necessary and sufficient conditions for the continuity of the metric projections in linear metric spaces. In normed linear spaces, the first theorem was proved by Holmes [2] and the second by Cheney and Wulbert [1] and Holmes [2].

Theorem 1. For a Chebyshev subspace G of a linear metric space (E,d), the metric projection π_G is continuous if and only if the restriction $W = W_G|_{P_G^{-1}(o)}$ of the canonical mapping $W_G: E \to E/G$ to the set $P_G^{-1}(o)$ is homeomorphism of $P_G^{-1}(o)$ onto E/G.

Proof. As G is a Chebyshev subspace, W is one-to-one (see [4], Theorem 2.3). Further, W is always continuous (see [6]) and a mapping onto E/G, since for any $x + G \in E/G$, we have $x - \pi_G(x) \in \in P_G^{-1}(o)$ and $W_G[x - \pi_G(x)] = x + G$. Thus the condition that $W = W_G|_{P_G^{-1}(o)}$ be a homeomorphism onto E/G is equivalent to the continuity of W^{-1}.

Assume that W^{-1} is continuous. We shall show that π_G is continuous. Let x_n, $x \in E$ and $x_n \to x$ i.e. $d(x_n,x) \to 0$ as $n \to \infty$. Then as above, $x_n - \pi_G(x_n) = W^{-1}(x_n+G)$ and $x - \pi_G(x) = W^{-1}(x+G)$, and so

$$d\left[\pi_G(x_n), \pi_G(x)\right] \leq d\left[\pi_G(x_n) - \pi_G(x),\ x_n - x\right] + d(x_n,x) =$$

$$= d[W^{-1}(x+G),\ W^{-1}(x_n+G)] + d(x_n,x) \to 0 \quad \text{as} \quad n \to \infty$$

and so π_G is continuous.

Now assume that π_G is continuous. We shall show that W^{-1} is continuous. Let x_n+G, $x+G \in E/G$, $\lim(x_n+G) = x + G$ and $\varepsilon > 0$ be given. Since π_G is continuous at the point $W^{-1}(x+G) \in P_G^{-1}(o)$, there exist a $\delta > 0$ such that

$$d[z, W^{-1}(x+G)] < \delta \Longrightarrow d[\pi_G(z), 0] = d[\pi_G(z), \pi_G(W^{-1}(x+G)] < \frac{\varepsilon}{2}.$$

Consider the open ball

$$V = \left\{ z \in E:\ d[z, W^{-1}(x+G)] < \min(\delta, \frac{\varepsilon}{2}) \right\}.$$

Since the canonical mapping W_G is open (see [6]), the set $W_G(v)$ is open and obviously, $x+G \in W_G(V)$. Hence $x_n+G \in W_G(V)$ for all $n > N = N(\varepsilon)$ and so there exist elements $z_n \in V$ such that

$$z_n+G = W_G(z_n) = x_n+G, \quad n > N,$$

i.e. $x_n - z_n \in G$ and hence by the quasi-additivity of π_G (see [5]), we have

$$\pi_G(x_n) = \pi_G(x_n - z_n + z_n) = x_n - z_n + \pi_G(z_n).$$

Consequently, since $z_n \in V$ $(n > N)$, we obtain

$$\begin{aligned} d\left[W^{-1}(x_n+G), W^{-1}(x+G)\right] &= d\left[x_n - \pi_G(x_n),\ W^{-1}(x+G)\right] \leq \\ &\leq d\left[x_n - \pi_G(x_n), z_n\right] + d\left[z_n, W^{-1}(x+G)\right] = \\ &= d\left[\pi_G(z_n), 0\right] + d\left[z_n, W^{-1}(x+G)\right] < \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}, \qquad n > N \end{aligned}$$

and thus W^{-1} is continuous.

Remark. The above proof is similar to one given in [7] (Theorem 4.2) for normed linear spaces.

Theorem 2. For a Chebyshev subspace G of a linear metric space (E,d), the following statements are equivalent:

(i) The metric projection π_G is continuous.

(ii) π_G is continuous at each point of $P_G^{-1}(o)$.

(iii) The direct sum decomposition $E = G \oplus P_G^{-1}(o)$ is topological (i.e. $\lim x_n = x$ if and only if $\lim \pi_G(x_n) = \pi_G(x)$ and $\lim[x_n - \pi_G(x_n)] = x - \pi_G(x)$).

(iv) The functional $\phi_G(x) = d(\pi_G(x).o)$, $x \in E$ is continuous.

Proof. We shall show that (i)$\Longleftrightarrow$(ii), (i)$\Longleftrightarrow$(iii) and (i)$\Longleftrightarrow$(iv). (i)$\Longrightarrow$(ii) is obvious.

(ii)$\Longrightarrow$ (i) Suppose (i) does not hold i.e. $x_n \longrightarrow x$ but $\pi_G(x_n) \longrightarrow\!\!\!\!\!\!/\;\; \pi_G(x)$, x_n, $x \in E$. Then $x_n - \pi_G(x) \longrightarrow x - \pi_G(x) \in P_G^{-1}(o)$ but $\pi_G(x_n - \pi_G(x)) = \pi_G(x_n) - \pi_G(x) \longrightarrow\!\!\!\!\!\!/\;\; 0$ contradicting (ii).

(i)$\Longleftrightarrow$ (iii) is obvious and so is (i)$\Longrightarrow$ (iv). Now we show that (iv)$\Longrightarrow$ (i).

Let $x_n \longrightarrow x \in P_G^{-1}(o)$, then $d\left[\pi_G(x_n),\ \pi_G(x)\right] = d\left[\pi_G(x_n),0\right] \longrightarrow d\left[\pi_G(x),0\right] = 0$ i.e. (iv)$\Longrightarrow$(ii) but (ii)$\Longrightarrow$(i). This completes the proof of Theorem 2.

Remark. The proof of Theorem 2 is a minor modification of the one given in [7] for normed linear spaces (Proposition 4.1).

The following theorem deals with the Lipschitzian metric projections in linear metric spaces. In normed linear spaces this theorem was proved by Holmes [2].

Theorem 3. For a Chebyshev subspace G of a linear metric space (E,d), the metric projection π_G is Lipschitzian if and only if the restriction $W = W_G\big|_{P_G^{-1}(o)}$ of the canonical mapping $W_G : E \longrightarrow E/G$ is a Lipschitzian homeomorphism of $P_G^{-1}(o)$ onto E/G.

Proof. For any $x+G \in E/G$, we have $x - \pi_G(x) \in P_G^{-1}(o)$ and $W\left[x-\pi_G(x)\right] = x+G$. Since

$$d\left[W_G(x),W_G(y)\right] = d(x+G,y+G) = d(x-y,G) \leq d(x-y,o) = d(x,y)$$

for all $x,y \in E$, the condition amounts to W^{-1} being Lipschitzian.

Let W^{-1} be Lipschitzian, consider

$$d\left[\pi_G(x),\pi_G(y)\right] = d\left[\pi_G(x)-x+x,\pi_G(y)-y+y\right] =$$

$$= d\left[-W^{-1}(x+G)+x,\ -W^{-1}(y+G)+y\right] \leq d\left[W^{-1}(y+G),W^{-1}(x+G)\right] + d(x,y) \leq$$

$$\leq d(y+G,x+G) + d(x,y) \leq 2d(x,y)$$

i.e. π_G is Lipschitzian. Suppose now that π_G is Lipschitzian. Consider

$$d\left[W^{-1}(x+G),\ W^{-1}(y+G)\right] = d\left[x-\pi_G(x),\ y-\pi_G(y)\right] =$$

$$= d\left[x-\pi_G(x+g),y-\pi_G(y-g)\right] \qquad \text{for all} \quad g\in G$$

$$= d\left[x-\pi_G(x)+g,y-\pi_G(y-g)\right] \qquad \text{for all} \quad g\in G$$

$$\leq d(x+g,y)+d\left[\pi_G(x),\pi_G(y-g)\right] \qquad \text{for all} \quad g\in G$$

$$\leq d(x,y-g)+d(x,y-g) \qquad \text{for all} \quad g\in G.$$

Therefore, $d\left[W^{-1}(x+G),W^{-1}(y+G)\right]\leq 2d(x,y+G) = 2d(x+G,y+G)$ i.e. W^{-1} is Lipschitzian.

The following theorem deals with the continuity of metric projections in quotient spaces of linear metric spaces. This result in normed linear spaces was proved by Cheney and Wulbert [1].

Theorem 4. Let P be a Chebyshev subspace of a linear metric space (E,d) with continuous π_P and let Q be a subspace of E which contains P. Then the following conditions are equivalent:

(i) Q is a Chebyshev subspace of E with continuous π_Q.

(ii) Q/P is a Chebyshev subspace of E/P with continuous $\pi_{Q/P}$.

Proof. (i) $\Longrightarrow$ (ii). Since Q is proximinal in E, Q/P is proximinal in E/P (Theorem 2 [3]). Since P is proximinal in E and Q is semi-Chebyshev in E, Q/P is semi-Chebyshev in E/P (Theorem 2 [3]). Hence Q/P is a Chebyshev subspace in E/P. Now we show that $\pi_{Q/P}$ is continuous.

Let $f_n+P \longrightarrow f+P$. Put $g_n = f_n - \pi_Q f$ and $g = f - \pi_Q f$. Then $g_n+P \longrightarrow g+P$ and $g\in\pi_Q^{-1}(o)$. It follows that dist $(g_n-g,P)\longrightarrow 0$ i.e. $d\left[g_n-g,\pi_P(g_n-g)\right] \longrightarrow 0$ and so $g_n-\pi_P(g_n-g) \longrightarrow g$. By the continuity of π_P (hypothesis), the quasi-additivity of π_P (see [5]) and idempotency of π_P (see [5]), it follows that

$$\pi_P(g_n)-\pi_P(g_n-g) \longrightarrow \pi_P(g) = 0 \quad \text{as} \quad g = f-\pi_Q f.$$

Thus $g_n - \pi_P(g_n - g) \longrightarrow g$. By the continuity of π_Q,

$$\pi_Q g_n - \pi_P(g_n - g) \longrightarrow \pi_Q(g) = 0.$$

Thus dist $(\pi_Q g_n, P) \longrightarrow 0$, dist $(\pi_Q f_n - \pi_Q f, P) \longrightarrow 0$, and $\pi_Q f_n + P \longrightarrow \pi_Q f + P$. By Theorem 2 [3], this implies that $\pi_{Q/P}(f_n + P) \longrightarrow \pi_{Q/P}(f + P)$.

(ii) $\Longrightarrow$ (i) Since Q/P is Chebyshev in E/P and P is Chebyshev in E, Q is Chebyshev in E (Theorem 2[3]). Now we show that π_Q is continuous. Since $E = Q \oplus P_Q^{-1}(o)$ (Theorem 2.3 [4]), it is sufficient to prove the continuity of π_Q at the points of $P_Q^{-1}(o)$. Let $f_n \longrightarrow f \in P_Q^{-1}(o)$. Then $f_n + P \longrightarrow f + P$, and by the continuity of $\pi_{Q/P}(f_n + P) \longrightarrow \pi_{Q/P}(f+P)$. By Theorem 2 [3], $\pi_Q f_n + P \longrightarrow \pi_Q f + P = P$. Hence $f - f_n + \pi_Q f_n + P \longrightarrow P$. It follows that

$$\mathrm{dist}(f - f_n + \pi_Q f_n, P) \longrightarrow 0,$$

$$d\left[f - f_n + \pi_Q f_n,\ \pi_P(f - f_n + \pi_Q f_n)\right] \longrightarrow 0$$

$$f_n - \pi_Q f_n + \pi_P(f - f_n + \pi_Q f_n) \longrightarrow f \longrightarrow (a)$$

By the continuity of π_P, we have

$$\pi_P(f_n - \pi_Q f_n) + \pi_P(f - f_n + \pi_Q f_n) \longrightarrow \pi_P f = 0.$$

Since $\pi_P(f_n - \pi_Q f_n) = 0$, we have $\pi_P(f - f_n + \pi_Q f_n) \longrightarrow 0$. It follows from (a) that $f_n - \pi_Q f_n \longrightarrow f$, whence $\pi_Q f_n \longrightarrow 0 = \pi_Q(f)$.

Remark. The converse of Theorem 4 is not true even in normed linear spaces (see [1]).

REFERENCES

[1] E.W. Cheney, D.E. Wulbert: The existence and unicity of best approximations, Math. Scand. 24 (1969), 113-140.

[2] R.B. Holmes: On the continuity of best approximation operators, Proc. Symp. Infinite Dimensional Topology, Ann. Math. Studies 69, Princeton Univ. Press, Princeton (1972), 137-157.

[3] T.D. Narang: Existence and unicity of best approximation and different types of continuity of proximity maps, Bull. Calcutta Math. Sco. 69 (1977), 39-43.

[4] T.D. Narang: Best approximation in metric linear spaces, Math. Today, 5 (1987), 21-28.

[5] G. Pantelidis: Approximationstheorie für metrische linear Räume, Math. Ann., 184 (1969), 30-48.

[6] W. Rudin: Functional analysis. Tata McGraw Hill, 1974.

[7] I. Singer: Best approximation in normed linear spaces, Constructive aspects of functional analysis, Part II, Edizioni Cremonese, Roma (1973).

DEPARTMENT OF MATHEMATICS, GURU NANAK DEV UNIVERSITY, AMRITSAR - 143005 (INDIA)

Received March 2nd, 1988.