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A SCHEME OF GENERATING CANONICAL FRAMES IN NORMAL BUNDLES OF IMMERSED MANIFOLDS IN EUCLIDEAN SPACES

1. Introduction

In this paper we define canonical frames in the normal bundle NM^n of a differentiable manifold M^n immersed in the Euclidean space E^{n+N}. The geometric idea of this construction was explained in [3]. More generally we repeat this concept as follows. Around a curve $x(I)\subset E^n$, $n\geqslant 3$, where E^n denotes the n-dimensional Euclidean space and

$$(1.1)\qquad x:I\longrightarrow E^n,\quad I=[0,1]$$

is an immersion, we take a tubular neighborhood consisting of a one-parameter family of (n-1)-dimensional disks of radius $\varepsilon>0$. There exists such an $\varepsilon_o>0$ that for every ε, $0<\varepsilon\leqslant\varepsilon_o$, the boundary $T^{n-1}(\varepsilon)$ of the tubular neighborhood is a regular surface. The Gauss curvature

$$(1.2)\qquad K:T^{n-1}(\varepsilon)\longrightarrow R,\quad \text{where } R \text{ denotes the real numbers,}$$

of $T^{n-1}(\varepsilon)$ has the following property. Let us suppose that the first curvature $k_1(s)$ of (1.1) is greater than zero for every $s\in[0,1]$. We consider the restriction of (1.2)

$$(1.3)\qquad K/S^{n-2}(s,\varepsilon),$$

where

$$(1.4)\qquad S^{n-2}(s,\varepsilon)\subset T^{n-1}(\varepsilon)$$

denotes the spherical "fiber" with radius ε and center $x(s)$. Then there exist exactly two antipodal points $y_1(s)$, $y_2(s)$ of the sphere (1.4) such that the function (1.3) attains its maximal value at $y_1(s)$ and its minimal value at $y_2(s)$ and the direction in E^n defined by the pair $(y_1(s), y_2(s))$ is that defined by the principal normal $\tilde{e}_2(s)$ of (1.1). In the following such a direction is called canonical.

We replace $x(I)$ by a surface $x(M^n) \subset E^{n+N}$, the tubular neighborhood by the normal bundle $NM^n = NM^n(x)$ of $x(M^n)$ and the Gauss curvature K by an elementary, symmetric function (called in the following also a scalar function) of the characteristic polynomial of the second quadratic form of $x(M^n)$. The stationary values of such a function restricted to the unit vectors of a fiber $N_pM^n(x)$, $p \in M^n$, of $NM^n(x)$ define directions in E^{n+N} which are called canonical.

Let

$$(1.5) \qquad e_2 : I \longrightarrow S^2, \quad I = [0,1], \quad e_2(0) = e_1(1),$$

denotes an immersion such that the direction of the principal normal $\tilde{e}_2$ of e_2 is identical with that of e_2. Then the second curvature of e_2 is zero and therefore $e_2(I) = S^1$, where $S^1 \subset S^2$ is the 1-dimensional unit sphere.

As an application of the method explained in Section 7 we define canonical immersions of manifolds in spheres as a generalization of (1.5) and prove that the image of a closed and connected manifold M^n, $\dim M^n = n$, $n \geqslant 2$, in the $(n+N-1)$-dimensional, unit sphere S^{n+N-1} by means of a canonical immersion is a unit sphere $S^n \subset S^{n+N-1}$ and M^n is diffeomorphic with S^n.

An outline of the content. In Section 2 we describe the scheme of generating canonical frames in an arbitrary vector bundle. In Section 3 basic notations are introduced. In Section 4 scalar functions in the normal bundle are defined and the construction of canonical frames is carried out. In Section 5 necessary conditions which define canonical vectors are rewritten as exterior form equations. In Section 6 we define sufficient conditions to get the uniqueness of the construction. In Section 7 we consider canonical cross sections of immersions of manifolds into spheres.

Cartan's method of moving frames and special related problems are considered by Ph. Griffiths [1] and G.R. Jenssen [2].

2. A scheme of generating canonical frames in vector bundles

Let

$$\mathcal{B} = (B, \pi, M, V)$$

denotes a vector bundle with total space B, base space M, the projection π and a standard fiber V. B and M are differentiable manifolds and V is a finite dimensional vector space. We denote

$$\dim M = n, \quad \dim V = N, \quad \dim B = n+N.$$

A metric g on the bundle $\mathcal{B}$ is a function which assigns to every $p \in M$ a positive definite scalar product g_p in the fiber V_p over $p \in M$, such that for every open set $U \subset M$ and every differentiable, local cross sections $s_1, s_2 : U \to B$, the function $g(s_1, s_2) : U \to R$ defined by $g(s_1,s_2)(p) = g_p(s_1(p), s_2(p))$ is differentiable. Differentiability means differentiability of class C^∞. We suppose throughout the paper that considered manifolds and functions are differentiable. By

$$\mathcal{N}_1 = (N_1, \pi_2, M, O(N)), \tag{2.1}$$

where O(N) denotes the orthogonal group acting on E^N from the right, we denote the principal bundle of orthonormal frames such that $\mathcal{B}$ is associated with $\mathcal{N}_1$. Let $L : B \to R$ denote a differentiable function, called a scalar function on $\mathcal{B}$. With the aid of L we define locall cross sections in $\mathcal{N}_1$, called canonical frames (with respect to L) as follows. Let $U \subset M$ denotes an open set such that there exists a vector bundle chart $H : \pi^{-1}(U) \to U \times V$. This implies that $\mathcal{B}$ restricted to U admits N linearly independent cross sections $(p,\bar{e}_r(p))$, $p \in U$, $\bar{e}_r(p) \in V_p = \pi^{-1}(p)$. These cross sections we denote in the following simply by $\bar{e}_r$. We denote

$$S_p^{N-1} = \left\{ e \in V_p : g_p(e,e) = 1 \right\}. \tag{2.2}$$

The function L restricted to the sphere (2.2) admits a maximal value at a point $\tilde{e}_{n+1}(p) \in S_p^{N-1}$ called the first canonical vector. Thus, $\tilde{e}_{n+1}(p)$ is a stationary point of the restriction L/S_p^{N-1}. If $\tilde{e}_{n+1}(p)$ is a non degenerate stationary point, i.e., if the Hessian of the function L/S_p^{N-1} is different from zero at $\tilde{e}_{n+1}(p)$, then $\tilde{e}_{n+1}(p)$ is isolated in the sphere S_p^{N-1}.

We have the following

Proposition 1. If $\tilde{e}_{n+1}(p) \in S_p^{N-1} \subset V_p$ is a non degenerate stationary point of the restriction L/S_p^{N-1} such that this function attains its maximal (or minimal) value at $\tilde{e}_{n+1}(p)$, then there exists a neighborhood $Q \subset M$ of $p \in M$ such that $\tilde{e}_{n+1}(q)$, $q \in Q$, is a non degenerate stationary point of L/S_q^{N-1} at which this function attains its maximal (or minimal) value, and $\tilde{e}_{n+1}(q)$, $q \in Q$, is a differentiable cross section in the bundle $\mathcal{B}$ restricted to $Q \subset M$.

Proof. Let $H^{-1}: U \times V \to B$ denotes a coordinate mapping, where U is an open set. We choose U such that (U,h) is a chart of M such that $h: U \to K^n$ is a diffeomorphism, where K^n denote the n-dimensional, unit and open disk in E^n referred to the coordinates $(u_1,\dots,u_n)$. By $H_1^{-1}: U \to B$ we denote the mapping defined by

$$H_1^{-1}(q,e) = H^{-1}(q,e), \quad q \in U, \quad e \in S^{N-1}. \tag{2.3}$$

By $\{(V,a),(W,b)\}$ we denote an atlas on S^{N-1} such that $U \cup W = S^{N-1}$ and $a(e) = (v_1,\dots,v_{N-1}) \in K^{N-1}$, $b(e) = (w_1,\dots,w_{N-1}) \in K^{N-1}$ are local coordinates of $e \in S^{N-1}$. The function $L: B \to B$ written in local coordinates has the form

(2.4) $f(u;v) = L(H_1^{-1}(h^{-1}(u),a^{-1}(v)))$ in the charts (U,H), (V,a)

and similarly

(2.5) $f(u;w) = L(H_1^{-1}(h^{-1}(u),b^{-1}(w)))$ in the charts (U,h), (W,b),

where $u = (u_1,\dots,u_n)$, $v = (v_1,\dots,v_{N-1})$, $w = (w_1,\dots,w_{N-1})$. If $(u^o;v^o) = (u_1^o,\dots,u_n^o;v_1^o,\dots,v_{N-1}^o)$ are coordinates of a non degenerate stationary point $\tilde{e}_{n+1}(p)$ of the function L/S_p^{N-1}, then

(2.6) $$\frac{\partial f}{\partial v_\alpha}(u^o;v^o) = 0,\ \det\left(\frac{\partial^2 f}{\partial v_\alpha \partial v_\beta}(u^o;v^o)\right) \neq 0,\ 1\leq\alpha,\beta\leq N-1.$$

By the implicit function theorem there exists a neighborhood $Q\subset U$ of $p\in U$, where $h(p) = (u_1^o,\dots,u_n^o)$, and uniquely defined differentiable functions $v_\alpha(u_1,\dots,u_n)$ such that $v_\alpha(u_1^o,\dots,u_n^o) = v_\alpha^o$, $1\leq\alpha\leq N-1$, and

(2.7) $$\frac{\partial f}{\partial v}(u_1,\dots,u_n;v_1(u),\dots,v_{N-1}(u)) = 0,$$

(2.8) $$\det \frac{\partial^2 f}{\partial v_\alpha \partial v_\beta}(u_1,\dots,u_n;v_1(u),\dots,v_{N-1}(u))) \neq 0,$$

where $u = (u_1,\dots,u_n)\in h(Q)\subset K^n$. The second differential of (2.4) with respect to the variables v_α, $1\leq\alpha\leq N-1$, is negative (or positive) definite at $(u_1^o,\dots,u_n^o;v_1^o,\dots,v_{N-1}^o)$. Hence from (2.8) we get that this differential is also negative (or positive) definite at every point $(u_1,\dots,u_n;\ v_1(u),\dots,v_{N-1}(u))$ for $u = (u_1,\dots,u_n)\in h(Q)\subset K^n$. This means that

(2.9) $$\tilde{e}_{n+1}(q) = H_1^{-1}(j^{-1}(u),a^{-1}(v(u)))\in S_q^{N-1},$$

where $h(q) = u$, $q\in Q$, $v(u) = (v_1(u),\dots,v_{N-1}(u))$, is a differentiable cross section of the bundle $\mathcal{B}$ restricted to $Q\subset U\subset M$ and the function L/S_q^{N-1} attains its maximal (or minimal) value at $\tilde{e}_{n+1}(q)$. In the case (2.5) the proof is the same.

As an immediate consequence of Proposition 1 we get the

Corollary 1. If for every $q \in U$, $U = h^{-1}(K^n)$, the Hessian of L/S_q^{N-1} is different from zero at a stationary point $\tilde{e}_{n+1}(q)$ such that L/S_q^{N-1} attains its maximal (or minimal) value at $\tilde{e}_{n+1}(q)$, then there exists a uniquely defined cross section of the bundle $\mathcal{B}$ restricted to $U \subset M$ and represented by the formula (2.9) for $q \in U$.

If

$$\tilde{e}_{n+1}(p), \dots, \tilde{e}_{n+\varrho-1}(p), \quad 2 \leq \varrho < N, \; p \in U = h^{-1}(K^n), \tag{2.10}$$

are canonical mutually orthogonal vectors, then $\tilde{e}_{n+\varrho}(p) \in S_p^{N-1}$ is defined as a non degenerate stationary point of the function $L/S_p^{N-\varrho}$, where $S_p^{N-\varrho} \subset S_p^{N-1}$ is contained in the linear subspace of V_p which is orthogonal to the vectors (2.10) such that $L/S_p^{N-\varrho}$ attains its maximal value at $\tilde{e}_{n+\varrho}(p)$. The last canonical vector $\tilde{e}_{n+N}(p)$ is uniquely determined by the foregoing canonical vectors and the orientation of V_p.

In the following we shall identify the function L restricted to $\pi^{-1}(U)$, $U \subset M$, and LH^{-1}, hence we set

$$L(e(p)) = L(p,e). \tag{2.11}$$

The vector $e \in S^{N-1}$ at the right of (2.11), is therefore a "projection" of $e(p) \in S_p^{N-1}$ on S^{N-1} defined by $e(p) \to (p,e) \to e$. In the following we shall identify the "projection" $\tilde{e}_{n+1} \in S^{N-1}$ with the cross section (2.9) for $q \in U$. Hence we set

$$\tilde{e}_{n+1}(q) = \tilde{e}_{n+1} \quad \text{for} \quad q \in U. \tag{2.12}$$

From Corollary 1 with the use of the identification (2.12) we get

Proposition 2. If for every $p \in U$, $U = h^{-1}(K^n)$, the Hessian of $L/S_p^{N-\varrho}$, where $S_p^{N-\varrho}$ is the $(N-\varrho)$-dimensional, unit sphere contained in the subspace of V_p orthogonal to the vectors (2.10), $1 \leq \varrho < N$, is different from zero at a stationary point $\tilde{e}_{n+\varrho}$ such that

$L/S_p^{N-\varrho}$ attains its maximal value at $\tilde{e}_{n+\varrho}$, then the function which sends $p \in U$ to $\tilde{e}_{n+\varrho}$, $1 \leq \varrho < N$, is a cross section over U in $\mathcal{B}$ and

$$o(p)\tilde{e}_{n+1} \ldots \tilde{e}_{n+N} \in N_1 \tag{2.13}$$

is a uniquely defined orthonormal frame of V_p attached to the origin $o(p)$ of V_p. The function which sends $p \in U$ to (2.13) is called a canonical cross section over U in $\mathcal{N}_1$ (see (2.1)) and $o: M \longrightarrow B$ is the zero section in $\mathcal{B}$.

Finally we have

Proposition 3. If the function L/V_p is linear for every $p \in U$, then exactly one canonical vector $\tilde{e}_{n+1}$ can be defined by means of L.

Proof. Since the nullspace of L/V_p is an (N-1)-dimensional subspace $W_p \subset V_p$, which, as follows from the definition of $\tilde{e}_{n+1}$, is orthogonal to $\tilde{e}_{n+1}$ it follows that the function L/S_p^{N-2}, $S_p^{N-2} \subset W_p$, is identically zero.

Generally, if L is identically constant on a subspace of V_p orthogonal to the already defined vectors (2.10), then the definition of $\tilde{e}_{n+\varrho}$ with the aid of L is not possible. In such a case besides L we need further scalar functions $L_\varrho : B_\varrho \longrightarrow R$, where B_ϱ denotes the total space of the subbundle $\mathcal{B}_\varrho$ of the restriction $\mathcal{B}/U$, whose fiber over $p \in U$ is the subspace of V_p orthogonal to the vectors (2.10). In applications the scalar function L is a homogeneous polynomial of N variables and of degree less than or equal to dim M = n. Since the estimation of stationary points of a polynomial can be described as an algebraic procedure, it follows that the process of defining canonical cross sections (2.13) is rather an algebraic and not differential geometric procedure. The function L depend on the second derivatives of an immersion of M in another manifold and the polynomial L_ϱ depend on derivatives of this immersion of order greater than two.

We define

$$l_r(p) = L(p, \tilde{e}_r), \quad p \in U \subset M. \tag{2.14}$$

3. Preliminaries

Let

(3.1) $$x: M^n \longrightarrow E^{n+N}$$

denotes an immersion of a differentiable manifold M^n of dimension n in the Euclidean space E^{n+N}. We use the following notations. TM^n denote the tangent bundle of M^n, T_pM^n denote the tangent space to M^n at $p \in M^n$, $NM^n(x)$ denote the normal bundle of M^n determined by (3.1). The fiber $N_pM^n(x)$ over $p \in M^n$ is the N-dimensional subspace of E^{n+N} of such vectors $e \in E^{n+N}$ that

(3.2) $$e.dx(p) = 0.$$

In the following we frequently set

(3.3) $$\|e\| = 1.$$

By $\mathcal{T}(x)$ we denote the principal tangent bundle of orthonormal frames $x(p)e_1 \ldots e_n \in T(x)$, where $e_i \in x_*(T_pM^n)$, $1 \leqslant i \leqslant n$, denote vectors tangent to the surface $x(M^n)$ at the point $x(p)$. We denote

(3.4) $$\mathcal{T}(x) = (T(x), \pi_1, M^n, O(n)),$$

By $\mathcal{N}_1(x)$ we denote the principal normal bundle of orthonormal frames

(3.5) $$x(p)e_{n+1} \ldots e_{n+N} \in N_1(x) \quad e_r \in N_pM^n(x), \quad n+1 \leqslant r \leqslant n+N.$$

We denote

(3.6) $$\mathcal{N}_1(x) = (N_1(x), \pi_2, M^n, O(N)).$$

By $\mathcal{F}_1(x) = \mathcal{T}(x) \oplus \mathcal{N}_1(x)$ we denote the Whitney sum of bundles, i.e.

(3.7) $$\mathcal{F}_1(x) = (F_1(x), \pi, M^n, O(x) \times O(N)).$$

The elements of the total space $F_1(x)$ are orthonormal frames

(3.8) $$x(p)e_1 \ldots e_n e_{n+1} \ldots e_{n+N} \in F_1(x).$$

The orthogonal groups $O(n)$, $O(N)$ and $O(n+N)$ act from the left on the respective orthonormal frames. Every frame (3.8) defines an element of the Euclidean group $E(n+N)$ which acts on E^{n+N} from the right, i.e.

(3.9) $\qquad x \longrightarrow xA + a, \quad A \in O(n+N), \quad a \in T(n+N), \quad x \in E^{n+N},$

where $T(n+N)$ denotes the translation group, isomorphic with E^{n+N}. The group operations in $E(n+N)$ have the form

(3.10) $$(A,a)\cdot(B,b) = (AB, aB+b), \quad (A,a)^{-1} = (A^T, -aA^T),$$

where $(A,a),(B,b) \in E(n+N)$ and A^T denotes the matrix transposed to A. The equations of structure of $E(n+N)$ have the form

(3.11) $$d\omega' = \omega' \wedge \Omega', \quad d\Omega' = \Omega' \wedge \Omega',$$

where $\omega' = daA^T = (\omega'_A)_{1\leq A\leq n+N}$, $\Omega' = dA\cdot A^T = (\omega_{AB})_{1\leq A,B\leq n+N}$, and on the right sides of the equations (3.11) we have matrix multiplication. In the following we use the convention

$$1 \leq i,j,k,l \leq n, \quad n+1 \leq r,s,t,u \leq n+N, \quad 1 \leq \rho, \sigma \leq N,$$

and repeated indices denote summation. The total space $F_1(x)$ of (3.7) can be identified with the set

(3.12) $$\left\{ (A,a) \in E(n+N) / e_i = (a_{i1}, \ldots, a_{i,n+N}) \in x_*(T_pM^n),\ a = x(p),\ p \in M^n \right\},$$

where $A = (a_{AB})_{1\leq A,B\leq n+N} \in O(n+N)$, $a = x(p) \in E^{n+N}$ and x_* denotes the differential of (3.1). The vectors $e_r = (a_{r1}, \ldots, a_{r,n+N})$ are orthogonal to $x(M^n)$ at $x(p)$. By means of the identification (3.12) we define the inclusion map

(3.13) $$F_1(x) \overset{\iota}{\subset} E(n+N).$$

We define linear forms

(3.14) $$\omega = \iota^*\omega' = (\omega_A), \quad \Omega = \iota^*\Omega' = (\omega_{AB}), \quad 1 \leq A,B \leq n+N,$$

on $F_1(x)$, where ι^* denotes the pull back of the components ω', Ω' of the moving frame of the Euclidean group $E(n+N)$, which corresponds to the inclusion map (3.13). On $F_1(x)$ we have

(3.15) $$\iota^*(da)\cdot e_r = dx(p)\cdot e_r = \omega_r = 0.$$

Hence

$$dx = \omega_i e_i. \tag{3.16}$$

Furthermore, we have

$$de_i = \omega_{ik} e_k + \omega_{ir} e_r \tag{3.17}$$

and the Weingarten equations have the form

$$de_r = \omega_{ri} e_i + \omega_{rs} e_s. \tag{3.18}$$

The equations of structure of the bundle $\mathcal{F}_1(x)$ have the form

$$d\omega_i = \omega_k \wedge \omega_{ki}, \tag{3.19}$$

$$d\omega_{ik} = \omega_{ij} \wedge \omega_{jk} + \omega_{ir} \wedge \omega_{rk}, \tag{3.20}$$

$$d\omega_{ir} = \omega_{ij} \wedge \omega_{jr} + \omega_{is} \wedge \omega_{sr}, \tag{3.21}$$

$$d\omega_{rs} = \omega_{rj} \wedge \omega_{js} + \omega_{rt} \wedge \omega_{ts}. \tag{3.22}$$

The formulas (3.20) are called the Gauss equations and (3.21) the Codazzi-Mainardi equations. From (3.11), (3.14) and (3.15) we get

$$\omega_{ri} \wedge \omega_i = 0 \tag{3.23}$$

and from Cartan's Lemma it follows

$$\omega_{ir} = A_{rij}\omega_j, \quad A_{rij} = A_{rji}. \tag{3.24}$$

We define the curvature forms by

$$\theta_{ik} = \omega_{ir} \wedge \omega_{rk} = -A_{rij}A_{rkl}\omega_j \wedge \omega_l = R_{ijkl}\omega_j \wedge \omega_l, \tag{3.25}$$

where R_{ijkl} are the components of the Riemann curvature tensor. From (3.16) it follows that the induced metric on M^n defined by (3.1) has the form

$$ds^2 = \sum_{i=1}^{n} \omega_i^2 \tag{3.26}$$

and the volume element of M^n has the form

(3.27) $$dV_n = \omega_1 \wedge \ldots \wedge \omega_n .$$

From (3.16), (3.18) and (3.24) we get

(3.28) $$d^2x \cdot e_r = -dx \cdot e_r = \omega_i \omega_{ri} = -A_{rij}\omega_i\omega_j .$$

From (3.28) we have that A_{rij} defined on $F_1(x)$ are coefficients of the second quadratic form of (3.1) in the direction defined by e_r. The coefficients of the characteristic polynomial

(3.29) $$\det(\lambda\delta_{ij} - A_{rij})$$

are scalar functions defined on $NM^n(x)$. With the use of the identifications (2.11) and (2.12) we denote by $L_k(p,e_r)$ the coefficient by λ^{n-k} in (3.29).

We have

Proposition 4. For every $p \in M^n$ there exist at most $\frac{n(n+1)}{2}$ canonical vectors which can be defined by means of L_k/S_p^{N-1}, $2 \leq k \leq n$.

Proof. We denote

(3.30) $$f_{ij} = (A_{n+1,ij}, \ldots, A_{n+N,ij}).$$

We prove that f_{ij} is a vector of $N_pM^n(x)$ in the base (3.5). Let

(3.31) $$e_r = a_{rs}\bar{e}_s, \quad (a_{rs}) \in O(N),$$

denotes a change of bases in $N_pM^n(x)$. We have

(3.32) $$\omega_{ir} = de_i \cdot e_r = a_{rs}\bar{\omega}_{is} = a_{rs} de_i \cdot \bar{e}_s .$$

From (3.24) and (3.32) we get

(3.33) $$A_{rij}\omega_j = a_{rs}\bar{A}_{sij}\omega_j .$$

Since ω_j are linearly independent we get

(3.34) $$A_{rij} = a_{rs}\bar{A}_{sij} .$$

From (3.31) and (3.34) it follows that A_{rij} undergo a change as a vector of $N_pM^n(x)$. Since $A_{rij} = A_{rji}$, we have $f_{ij} = f_{ji}$. Therefore the vectors f_{ij} with coordinates (3.30) define a subspace $N_p^1M^n(s) \subset \subset N_pM^n(x)$ and

(3.35) $$\dim N_p^1M^n(x) \leqslant \frac{n(n+1)}{2}.$$

The subspace $N_p^1M^n(x)$ remains invariant by change of bases in T^pM^n. Indeed, we have

(3.36) $$A_{rij} = a_{ik}a_{jl}\bar{A}_{rkl},$$

where

(3.37) $$e_i = a_{ij}\bar{e}_j, \qquad (a_{ij}) \in O(n),$$

is a change of bases in T_pM^n. From (3.30) and (3.36) we get $f_{ij} = a_{ik}a_{jl}\bar{f}_{kl}$, i.e. f_{ij} depend linearly on $\bar{f}_{ij}$.

For $p \in M^n$ fixed L_1 is a linear function and therefore from Proposition 3 it follows that by means of L_1 at most one canonical vector can be defined.

4. Canonical frames in $NM^n(x)$

The coefficient by λ^{n-k} in (3.29) has the form

(4.1) $$L_k(p,e_r) = (-1)^k \sum_{i_1<\ldots<i_k} \det(A_{rpq})_{p,q\in\{i_1,\ldots,i_k\}},$$

where $1 \leqslant i_l \leqslant n$, $1 \leqslant l \leqslant k$, $1 \leqslant k \leqslant n$. In particular we get

(4.2) $$L_1(p,e_r) = -\sum_{i=1}^{n} A_{rii},$$

(4.3) $$L_2(p,e_r) = \sum_{i<j} \det(A_{rpq})_{p,q\in\{i,j\}},$$

(4.4) $$L_n(p,e_r) = L(p,e_r) = -1)^n \det(A_{rij}).$$

The function (4.4) is the Killing-Lipschitz curvature. The functions (4.2), and (4.3) are considered more detailed below in a) and b) respectively. The general case is considered in c). Let

(4.5) $$x(p)\bar{e}_{n+1}\dots\bar{e}_{n+N} \in N_1(x), \quad p \in U,$$

denotes a cross section from an open set $U \subset M^n$ in $\mathcal{N}_1(x)$ (see (3.6)).

a) From (3.34) and (4.2) we get

(4.6) $$L_1(p,e_{n+1}) = -a_{n+1,s} \sum_{i=1}^{n} \bar{A}_{sii}.$$

We apply the method of Lagrange to determine stationary points of the function (4.6) restricted to S_p^{N-1}. We have

(4.7) $$\sum_{s=n+1}^{n+N} a_{n+1,s}^2 = 1.$$

We get

(4.8) $$\sum_{i=1}^{n} \bar{A}_{sii} - 2I_{1,n+1}a_{n+1,s} = 0,$$

where $I_{1,n+1}$ is a constant for $p \in U$ fixed. From (4.6), (4.7) and the formula

(4.9) $$\sum_{r=n+1}^{n+N} A_{rij}^2 = \sum_{r=n+1}^{n+N} (a_{rs}\bar{A}_{sij})^2 = a_{rs}a_{rt}\bar{A}_{sij}\bar{A}_{tij} = \sum_{r=n+1}^{n+N} \bar{A}_{rij}^2$$

we get

(4.10) $$4I_{1,n+1}^2 = \sum_{s=n+1}^{n+N}\left(\sum_{i=1}^{n} A_{sii}\right)^2 = \sum_{s=n+1}^{n+N}\left(\sum_{i=1}^{n} A_{sii}\right)^2.$$

The function $I_{1,n+1} = I_{1,n+1}(p)$, $p \in U$, is an invariant of $NM^n(x)$. This is the maximal (or minimal) value of $L_1(p,\tilde{e}_{n+1})$ restricted to

S_p^{N-1} (see (2.14)). We have $l_{1,n+1}(p) = L_1(p,\tilde{e}_{n+1})$. In the following an invariant $l_{kr}(p) = L_k(p,\tilde{e}_r)$ of $NM^n(x)$ is called simply an invariant of (3.1) or of the surface $x(M^n)$. Denoting by $\tilde{A}_{n+1,ij}$ the coefficients of the second quadratic form (3.28) evaluated in the direction of $\tilde{e}_{n+1}$ we get

$$2l_{1,n+1} = -\sum_{i=1}^{n} \tilde{A}_{n+1,ii}. \tag{4.11}$$

If $l_{1,n+1}(p) \neq 0$ for every $p \in U \subset M^n$, then from (3.31) and (4.8) we get

$$\tilde{e}_{n+1} = -\frac{1}{2l_{1,n+1}} \sum_{i=1}^{n} \bar{A}_{sii}\bar{e}_s. \tag{4.12}$$

From (4.12) it follows that $\tilde{e}_{n+1}$ is a differentiable cross section from U to $NM^n(x)$. This is the mean curvature vector and $l_{1,n+1}$ defined by (4.11) is the mean curvature of (3.1) in the direction of $\tilde{e}_{n+1}$. If we choose the cross section (4.5) such that $\bar{e}_{n+1} = \tilde{e}_{n+1}$ and $\bar{e}_s$, $n+2 \leqslant s \leqslant n+N$, remains arbitrary, then from (4.10) and (4.11) we get

$$\sum_{i=1}^{n} \bar{A}_{sii} = 0, \qquad n+2 \leqslant s \leqslant n+N. \tag{4.13}$$

Proposition 5. If $l_{1,n+1}(p) \neq 0$ for every $p \in M^n$, then there exists a cross section $\tilde{e}_{n+1}: M^n \longrightarrow NM^n(x)$ and for every open set $U \subset M^n$ which admit a cross section (4.5) $2l_{1,n+1}$ has the form (4.11) and $\tilde{e}_{n+1}$ the form (4.12).

Proof. It suffices to prove that the definition (4.12) of $\tilde{e}_{n+1}$ does not depend on the cross section (4.5). Let $x(p)\hat{e}_{n+1}\ldots\hat{e}_{n+N} \in N_1(x)$ denotes another cross section defined on an open set V such that $p \in U \cap V$. There exists a matrix $(a_{rs}) \in O(N)$ such that

$$\bar{e}_r = a_{rs}\hat{e}_s, \qquad \bar{A}_{rii} = a_{rs}\hat{A}_{sii}$$

for $p \in U \cap V$. We have

$$\bar{A}_{rii}\bar{e}_r = a_{rs}\hat{A}_{sii}a_{rt}\hat{e}_t = \delta_{st}\hat{A}_{sii}\hat{e}_t = \hat{A}_{tii}\hat{e}_t.$$

b) The function L_2 defined by (4.3) satisfies $L_2(p,e_{n+1}) = L_2(p,-e_{n+1})$ and therefore by means of L_2 we define canonical directions only. Let us suppose inductively that by means of L_2 we have defined canonical vectors (2.10) and invariants $I_{2,n+\sigma}$, $1\leq\sigma\leq\rho-1$, of $x(M^n)$ at $p\in U\subset M^n$. The vectors (2.10) are defined up to orientation. We define the canonical vector $\tilde{e}_{n+\rho}$ as follows. In (3.34) we change the range of indices such that $n+\rho\leq r,s\leq n+N$ and substitute

(4.14) $$A_{n+\rho,ij} = a_{n+\rho,s}\bar{A}_{sij}, \quad n+\rho\leq s\leq n+N,$$

in (4 3). The seeked unit vector $\tilde{e}_{n+\rho}$ is orthogonal to the vectors (2.10) and therefore can be written in the form

(4.15) $$\tilde{e}_{n+\rho} = a_{n+\rho,s}\bar{e}_s, \quad n+\rho\leq s\leq n+N,$$

where $\bar{e}_s$ together with $\tilde{e}_{n+\sigma}$, $1\leq\sigma\leq\rho-1$, define a base (4.5) of $N_pM^n(x)$. The coordinates $(a_{n+\rho,n+\rho},\ldots,a_{n+\rho,n+N})$ of a stationary point $\tilde{e}_{n+\rho}$ of $L_2/S_p^{N-\rho}$ satisfy the condition

(4.16) $$\sum_{r=n+\rho}^{n+N} a^2_{n+\rho,r} = 1$$

and the system of equations

(4.17) $$a_{n+\rho,s}\sum_{i<j}(\bar{A}_{rii}\bar{A}_{sjj}+\bar{A}_{rjj}\bar{A}_{sii}-2\bar{A}_{rij}\bar{A}_{sij}) - 2I_{2,n+\rho}a_{n+\rho,r} = 0,$$

where $1\leq\rho<N$, $n+\rho\leq r,s\leq n+N$. If we choose in (4.5) $\bar{e}_{n+\sigma}$ as canonical vectors $\tilde{e}_{n+\sigma}$, $1\leq\sigma\leq\rho$, and $\bar{e}_s$, $n+\rho+1\leq s\leq n+N$, remains arbitrary, then $\tilde{e}_{n+\rho}$ has coordinates $a_{n+\rho,n+\rho} = 1$, $a_{n+\rho,r} = 0$, where $n+1\leq r\leq n+N$, $r\neq n+\rho$, and the system of equations (4.17) takes the form

(4.18) $$\sum_{i<j}(\tilde{A}_{n+\rho,ii}\tilde{A}_{n+\rho,jj} - \tilde{A}^2_{n+\rho,ij}) = I_{2,n+\rho},$$

$$(4.19)\qquad \sum_{i<j} (\tilde{A}_{n+\rho,ii}\bar{A}_{rjj} + \tilde{A}_{n+\rho,jj}\bar{A}_{rii} - 2\tilde{A}_{n+\rho,ij}\bar{A}_{rij}) = 0,$$

where $1 \leq \rho \leq N-1$, $n+\rho+1 \leq r \leq n+N$. Because of our inductive method the identities (4.19) are valid also for the range of indices $1 \leq \sigma \leq \rho$, $n+\sigma+1 \leq r \leq n+N$, and therefore they are valid also for the range of indices

$$(4.20)\qquad 1 \leq \sigma \leq \rho, \quad n+\rho+1 \leq r \leq n+N.$$

Hence the coefficients of the quadratic form $\bar{A}_{rij}$, $n+\rho+1 \leq r \leq n+N$, cannot be arbitrary numbers, since they satisfy ρ equations of the form (4.19), where in (4.19) the index ρ is replaced by σ and the range of indices is defined by (4.20). The same remark concerning conditions imposed on $\bar{A}_{rij}$, $n+\rho+1 \leq r \leq n+N$, by the foregoing steps of the construction is valid also in the general case c) considered below. Direct calculations by means of the formula (4.9) yield to the result that besides I_{2r}, $n+1 \leq r \leq n+N$, also

$$(4.21)\qquad \sum_{r=n+1}^{n+N} \sum_{i<j} (A_{rii}A_{rjj} - A_{rij}^2)$$

is an invariant of $x(M^n)$. For $n = 2$ (4.21) is the Gauss curvature of $x(M^2) \subset E^3$. Since $L_2/S_p^{N-\rho}$, $p \in U$, is a quadratic form of the variables $a_{n+\rho,r}$, $n+\rho \leq r \leq n+N$, which satisfy (4.16), it follows that $I_{2,n+\rho}$, $1 \leq \rho \leq N$, defined by (4.18) are eigenvalues of this quadratic form.

c) In general case (4.1) we have $L_k(p,-e_{n+1}) = (-1)^k L_k(p,e_{n+1})$. Hence for k even we determine directions only. We substitute (4.14) into (4.1). We get

$$(4.22)\ L_k(p,e_{n+\rho}) = (-1)^k \overline{\sum_{i_1<\ldots<i_k}} \det(a_{n+\rho,s_1}\bar{A}_{s_1 i_1 q}, \ldots, a_{n+\rho,s_k}\bar{A}_{s_k i_k q}),$$

where $1 \leq \rho < N$, $1 \leq i_l \leq n$, $n+\rho \leq s_l \leq n+N$, $1 \leq l \leq k$, $q \in \{i_1,\dots,i_k\}$. The symbol $a_{n+\rho,s_l}\bar{A}_{s_l i_l q}$ denotes the l-th row of the determinant at the right of (4.22). The function $L_k/S_p^{N-\rho}$ is defined by the formula (4.22), where the variables $a_{n+\rho,s}$, $n+\rho \leq s \leq n+N$, in (4.22) satisfy the condition (4.16). The coordinates $(a_{n+\rho,n+\rho},\dots,a_{n+\rho,n+N})$ of a stationary point $\tilde{e}_{n+\rho}$ of $L_k/S_p^{N-\rho}$ satisfies besides (4.16) the system of equations

$$(4.23)\quad (-1)^k \sum_{l=1}^{k} \sum_{i_1<\dots<i_k} \det(A_{n+\rho,i_1q},\dots,\bar{A}_{ri_lq},\dots,A_{n+\rho,i_kq}) = 2I_{k,n+\rho}a_{n+\rho,r},$$

where $1 \leq \rho < N$, $n+\rho \leq r \leq n+N$, and $A_{n+\rho,i_mq}$, $m \neq l$, $1 \leq m \leq k$, is an abbreviation for $a_{n+\rho,s_m}\bar{A}_{s_m i_m q}$, $n+\rho \leq s_m \leq n+N$, in (4.23). If we choose (4.5) such that $\bar{e}_{n+\sigma}$ are canonical vectors $\tilde{e}_{n+\sigma}$, $1 \leq \sigma \leq \rho$, then in such a base $\tilde{e}_{n+\rho}$ has coordinates $a_{n+\rho,n+\rho} = 1$, $a_{n+\rho,r} = 0$, $n+1 \leq r \leq n+N$, $r \neq n+\rho$, and the system of equations (4.23) takes the form

$$(4.24)\quad (-1)^k\, k \sum_{i_1<\dots<i_k} \det(\tilde{A}_{n+\rho,pq})_{p,q\in\{i_1\dots i_k\}} = 2I_{k,n+\rho},$$

$$(4.25)\quad \sum_{l=1}^{k} \sum_{i_1<\dots<i_k} \det(\tilde{A}_{n+\rho,i_1q},\dots,\bar{A}_{ri_lq},\dots,\tilde{A}_{n+\rho,i_kq}) = 0,$$

where $1 \leq \rho < N$, $n+\rho+1 \leq r \leq n+N$. For $k = n$ the equations (4.24) and (4.25) have the form

$$(4.26)\quad (-1)^n\, n \det(\tilde{A}_{n+\rho,ik}) - 2I_{n,n+\rho} = 0,$$

$$(4.27)\quad \sum_{i=1}^{n} \det(\tilde{A}_{n+\rho,1k},\dots,\bar{A}_{rik},\dots,\tilde{A}_{n+\rho,nk}) = 0.$$

5. Canonical frames defined by systems of exterior equations

From (3.24) and (3.27) we get the identity

$$(5.1) \qquad \sum_{i=1}^{n} \omega_1 \wedge \dots \wedge \omega_{i-1} \wedge \omega_{ir} \wedge \omega_{i+1} \wedge \dots \wedge \omega_n = \sum_{i=1}^{n} A_{rii} dV_n .$$

From (3.24) we get

$$(5.2) \qquad \tilde{\omega}_{i,n+1} = de_i \cdot \tilde{e}_{n+1} = \tilde{A}_{n+1,ij} \omega_j ,$$

where $\tilde{e}_{n+1}$ in (5.2) is defined by (4.12). Denoting by $l_{1,n+1}$ the maximal value of the function (4.6) restricted to the sphere (4.7) we get from (4.11) and $L_1(p,-e_{n+1}) = -L_1(p,e_{n+1})$ the inequality

$$(5.3) \qquad -\sum_{i=1}^{n} \tilde{A}_{n+1,ii} \geq 0 .$$

From (4.10), (4.11) and (5.3) we get

$$(5.4) \qquad -\sum_{i=1}^{n} \tilde{A}_{n+1,ii} = \left(\sum_{s=n+1}^{n+N} \left(\sum_{i=1}^{n} A_{sii} \right)^2 \right)^{\frac{1}{2}} .$$

From (5.1), (5.2) and (5.4) we get

$$(5.5) \qquad -\sum_{i=1}^{n} \omega_1 \wedge \dots \wedge \omega_{i-1} \wedge \tilde{\omega}_{i,n+1} \wedge \omega_{i+1} \wedge \dots \wedge \omega_n = \left(\sum_{s=n+1}^{n+N} \left(\sum_{i=1}^{n} \omega_1 \wedge \dots \wedge \omega_{i-1} \wedge \omega_{is} \wedge \omega_{i+1} \wedge \dots \wedge \omega_n \right)^2 \right)^{\frac{1}{2}} .$$

The equations (4.13) can be written in the equivalent form

$$(5.6) \qquad \sum_{i=1}^{n} \omega_1 \wedge \dots \wedge \omega_{i-1} \wedge \bar{\omega}_{is} \wedge \omega_{i+1} \wedge \dots \wedge \omega_n = 0, \quad n+2 \leq s \leq n+N,$$

where

$$(5.7) \qquad \bar{\omega}_{is} = de_i \cdot \bar{e}_s = \bar{A}_{sij} \omega_j .$$

In the case b) from (3.24) and (3.27) we get the identity

$$(5.8)\qquad \sum_{i<j} \omega_1\wedge\ldots\wedge\omega_{i-1}\wedge\omega_{ir}\wedge\omega_{i+1}\wedge\ldots\wedge\omega_{j-1}\wedge\omega_{jr}\wedge\omega_{j+1}\wedge\ldots\wedge\omega_n =$$

$$= \sum_{i<j} (A_{rii}A_{rjj} - A^2_{rij})dV_n.$$

The equations (4.19) can be written in the equivalent form

$$(5.9)\qquad \sum_{i<j} (\omega_1\wedge\ldots\wedge w_{i-1}\wedge\bar{\omega}_{ir}\wedge\omega_{i+1}\wedge\ldots\wedge\omega_{j-1}\wedge\tilde{\omega}_{j,n+\rho}\wedge\omega_{j+1}\wedge\ldots\wedge\omega_n +$$

$$+ \omega_1\wedge\ldots\wedge\omega_{i-1}\wedge\tilde{\omega}_{i,n+\rho}\wedge\omega_{i+1}\wedge\ldots\wedge\omega_{j-1}\wedge\bar{\omega}_{jr}\wedge\omega_{j+1}\wedge\ldots\wedge\omega_n) = 0,$$

where $1\leqslant\rho<N$, $n+\rho+1\leqslant r\leqslant n+N$, and

$$(5.10)\qquad \tilde{\omega}_{i,n+\rho} = de_i\cdot\tilde{e}_{n+\rho}.$$

The invariant (4.21) is defined by means of the invariant form

$$(5.11)\qquad \sum_{r=n+1}^{n+N} \sum_{i<j} \omega_1\wedge\ldots\wedge\omega_{i-1}\wedge\omega_{ir}\wedge\omega_{i+1}\wedge\ldots\wedge\omega_{j-1}\wedge\omega_{jr}\wedge$$

$$\wedge\omega_{j+1}\wedge\ldots\wedge\omega_n.$$

In the general case c) we get from (3.24) and (3.27) the identity

$$(5.12)\qquad \sum_{i_1<\ldots<i_k} \omega_1\wedge\ldots\wedge\omega_{i_1-1}\wedge\omega_{i_1r}\wedge\omega_{i_1+1}\wedge\ldots\wedge\omega_{i_k-1}\wedge\omega_{i_kr}\wedge$$

$$\wedge\omega_{i_k+1}\wedge\ldots\wedge\omega_n = \sum_{i_1<\ldots<i_k} \det(A_{rpq})_{p,q\in\{i_1,\ldots,i_k\}}dV_n.$$

The equations (4.25) can be written in the equivalent form

$$(5.13)\qquad \sum_{l=1}^{k}\sum_{i_1<\ldots<i_k} \omega_1\wedge\ldots\wedge\omega_{i_1-1}\wedge\tilde{\omega}_{i_1,n+\rho}\wedge\omega_{i_1+1}\wedge\ldots\wedge\omega_{i_l-1}\wedge$$

$$\wedge\bar{\omega}_{i_lr}\wedge\omega_{i_l+1}\wedge\ldots\wedge\omega_{i_k-1}\wedge\tilde{\omega}_{i_k,n+\rho}\wedge\omega_{i_k+1}\wedge\ldots\wedge\omega_n = 0,$$

where $1 \leq \varrho < N$, $n+\varrho+1 \leq r \leq n+N$. In particular (4.27) can be written in the form

$$(5.14) \qquad \sum_{i=1}^{n} \tilde{\omega}_{1,n+\varrho} \wedge \ldots \wedge \bar{\omega}_{ir} \wedge \ldots \wedge \tilde{\omega}_{n,n+\varrho} = 0.$$

6. Local canonical cross sections in $NM^n(x)$

In Section 4 we have defined canonical frames

$$(6.1) \qquad x(p)\tilde{e}_{n+1} \ldots \tilde{e}_{n+N} \in N_1(x)$$

at a point $p \in M^n$ by means of the functions (4.1). From Proposition 1 and Corollary 1 it follows that if the Hessian

$$(6.2) \qquad \det(h^{\varrho}_{kru})_{n+\varrho+1 \leq r,u \leq n+N}, \quad 1 \leq \varrho < N, \quad 2 \leq k \leq n,$$

of $L_k/S_p^{N-\varrho}$ is different from zero at a stationary point $\tilde{e}_{n+\varrho}$, then there exists a neighborhood $U \subset M^n$, $p \in U$, diffeomorphic with the unit disk K^n, such that for every $q \in U$ the Hessian (6.2) is different from zero and

$$(6.3) \qquad \tilde{e}_{n+\varrho} : U \longrightarrow NM^n(x)$$

is a differentiable cross section. Let

$$(6.4) \qquad x(p)\tilde{e}_{n+1} \ldots \tilde{e}_{n+\varrho}\bar{e}_{n+\varrho+1} \ldots \bar{e}_{n+N} \in N_1(x), \quad p \in U.$$

We suppose that $\tilde{e}_{n+\sigma}$, $1 \leq \sigma < \varrho$, are cross sections defined by means of the functions L_j, $1 \leq j \leq n$, on $U \subset M^n$ and $\bar{e}_r$, $n+\varrho+1 \leq r \leq n+N$, are arbitrary cross sections of $NM^n(x)$ defined on $U \subset M^n$. From (4.1) and (6.4) it follows that the function $L_k/S_p^{N-\varrho}$ can be written in the form

$$(6.5) \quad L_k(p,e_{n+\varrho}) = (-1)^k \sum_{i_1 < \ldots < i_k} \det(A_{n+\varrho,pq})_{p,q \in \{i_1,\ldots,i_k\}},$$

where

$$(6.6) \qquad e_{n+\varrho} = a_{n+\varrho,n+\varrho}\tilde{e}_{n+\varrho} + \sum_{r=n+\varrho+1}^{n+N} a_{n+\varrho,r}\bar{e}_r$$

and $(a_{n+\varrho,n+\varrho},\dots,a_{n+\varrho,n+N})$ satisfies (4.16). The second derivative of (6.5) with respect to $a_{n+\varrho,r}$, $a_{n+\varrho,u}$, $n+\varrho+1\leq r,u\leq n+N$, at $a_{n+\varrho,r}=0$, where

(6.7) $$A_{n+\varrho,pq}=a_{n+\varrho,n+\varrho}\tilde{A}_{n+\varrho,pq}+\sum_{r=n+\varrho+1}^{n+N}a_{n+\varrho,r}\bar{A}_{rpq}$$

and

(6.8) $$a_{n+\varrho,n+\varrho}=\left(1-\sum_{r=n+\varrho+1}^{n+N}a^2_{n+\varrho,r}\right)^{\frac{1}{2}},$$

has the form

(6.9) $$h^{\varrho}_{kru}=(-1)^{k+1}\delta_{ru}k\sum_{i_1<\dots<i_k}\det(\tilde{A}_{n+\varrho,pq})_{p,q\in\{i_1,\dots,i_k\}}+$$
$$+(-1)^k\sum_{i_1<\dots<i_k}\sum_{l<m}(\det(\tilde{A}_{n+\varrho,i_1q},\dots,\bar{A}_{ri_lq},\dots,\bar{A}_{ui_mq},\dots,\tilde{A}_{n+\varrho,i_kq})$$
$$+\det(\tilde{A}_{n+\varrho,i_1q},\dots,\bar{A}_{ui_lq},\dots,\bar{A}_{ri_mq},\dots,\tilde{A}_{n+\varrho,i_kq})),$$

where $2\leq k\leq n$, $1\leq l<m\leq k$, $1\leq\varrho<N$, $n+\varrho+1\leq r,u\leq n+N$, and $q\in\{i_1,\dots,i_k\}$ denote the number of a column in the determinants of (6.9).

From (6.9) we get

(6.10) $$h^{\varrho}_{2ru}=-2\delta_{ru}\sum_{i<j}(\det(\tilde{A}_{n+\varrho,pq})_{p,q\in\{i,j\}}+$$
$$+\bar{A}_{rii}\bar{A}_{ujj}+\bar{A}_{rjj}\bar{A}_{uii}-2\bar{A}_{rij}\bar{A}_{uij}).$$

If in (6.1) $\tilde{e}_{n+\varrho}$, $\bar{e}_r$ are eigenvectors $\tilde{e}_{n+\varrho}$, $\tilde{e}_r$, $n+\varrho+1\leq r\leq n+N$, of the quadratic form $L_2/S^{N-\varrho}_p$ (not necessarily uniquely defined), then from (4.19) it follows

(6.11) $$\bar{A}_{rii}\bar{A}_{ujj}+\bar{A}_{rjj}\bar{A}_{uii}-2\bar{A}_{rij}\bar{A}_{uij}=\tilde{A}_{rii}\tilde{A}_{ujj}+\tilde{A}_{rjj}\tilde{A}_{uii}-2\tilde{A}_{rij}\tilde{A}_{uij}=0$$

and (6.10) get by (6.11) a diagonal matrix

(6.12) $$h^{\varrho}_{2rr} = -2(l_{2,n+\varrho} - l_{2r}), \quad n+\varrho+1 \leq r \leq n+N,$$

(6.13) $$h^{\varrho}_{2ru} = 0 \quad \text{for} \quad r \neq u, \quad n+\varrho+1 \leq r,u \leq n+N,$$

where l_{2r} are defined by (4.18).

We have

Proposition 6. Let

(6.14) $$x(p)\tilde{e}_{n+1}\ldots\tilde{e}_{n+\varrho}\ldots\tilde{e}_{n+N} \in N_1(x),$$

where $\tilde{e}_r$, $n+1 \leq r \leq n+N$, in (6.14) are eigenvectors of L_2/S^{N-1}_p. If the eigenvalue $l_{2,n+\varrho}(p)$ is different from $l_{2r}(p)$ for every r, $r \neq n+\varrho$, $n+1 \leq r \leq n+N$, then there exists a neighborhood $U \subset M^n$, such that $l_{2,n+\varrho}(q) \neq l_{2r}(q)$ for every $q \in U$ and a 1-dimensional, trivial vector bundle with fiber $\varepsilon_{n+\varrho}(q)$, $q \in U$, such that $\varepsilon_{n+\varrho}(q)$ is determined by a uniquely defined eigenvector $\tilde{e}_{n+\varrho}$.

Proof. From (6.12) it follows that the definition

(6.15) $$h^{\sigma}_{2,n+\varrho,n+\varrho} = -h^{\varrho}_{2,n+\sigma,n+\sigma},$$

where $1 \leq \varrho < N$, $\varrho+1 \leq \sigma < N$, is correct. From (6.15) we get

(6.16) $$h^{\varrho}_{2rr} = 2(l_{2n+\varrho} - l_{2r}),$$

where $n+1 \leq r \leq n+\varrho-1$, and similarly as (6.13) we get

(6.17) $$h^{\varrho}_{2ru} = 0 \quad \text{for} \quad r \neq u, \ n+1 \leq r,u \leq n+N, \ r,u \neq n+\varrho.$$

By assumption we have

(6.18) $$h^{\varrho}_{2rr} \neq 0, \quad r \neq n+\varrho, \ n+1 \leq r \leq n+N,$$

at $p \in M^n$. From (6.12), (6.16), (6.17) and (6.18) it follows that the Hessian

(6.19) $$\det(h^{\varrho}_{2ru}) \neq 0, \quad r,u \neq n+\varrho, \ n+1 \leq r,u \leq n+N,$$

at $p \in M^n$. Hence there exists a neighborhood $U \subset M^n$, $p \in U$, such that (6.19) is satisfied for every $q \in U$ and therefore from Proposition 1 and Corollary 1 it follows that there exists a cross section

$$\tilde{e}_{n+\rho} : U \longrightarrow NM^n(x) \tag{6.20}$$

which for every $q \in U$ defines the canonical direction $\mathcal{E}_{n+\rho}(q)$.

The extension of (6.12), (6.13) by (6.16), (6.17) is necessary, since we do not assume that $\tilde{e}_{n+\sigma}$, $1 \leq \sigma \leq \rho$, in (6.14) is a non degenerate stationary point of $L_2/S_p^{N-\sigma}$ and therefore that it is uniquely defined as the vector $\tilde{e}_{n+\sigma}$ in (6.4) is.

7. Immersion of manifolds in spheres

Let

$$\bar{e}_{n+1} : M^n \longrightarrow E^{n+N} \tag{7.1}$$

denotes an immersion such that

$$\bar{e}_{n+1}(M^n) \subset S^{n+N-1}, \tag{7.2}$$

where $S^{n+N-1} \subset E^{n+N}$ denotes the unit sphere.

We have

Proposition 7. If $\tilde{e}_{n+1}$ denotes the first canonical vector defined by means of L_k/S_p^{N-1}, then

$$L_k(p, \tilde{e}_{n+1}) \geq \binom{n}{k} \tag{7.3}$$

for every $p \in M^n$.

Proof. Let us take

$$\bar{e}_{n+1}(p) e_1 \ldots e_n \bar{e}_{n+1} e_{n+2} \ldots e_{n+N} \in F_1(\bar{e}_{n+1}), \tag{7.4}$$

where $F_1(x)$ denotes the total space of (3.7). From (3.15) it follows $d\bar{e}_{n+1} \cdot e_r = \omega_{n+1,r} = 0$ and therefore

$$d\bar{e}_{n+1} = \bar{\omega}_{n+1,i} e_i. \tag{7.5}$$

From (3.24) and (7.5) it follows

(7.6) $$\bar{\omega}_{i,n+1} = \bar{A}_{n+1,ij}\bar{\omega}_{n+1,j}$$

and therefore

(7.7) $$\bar{A}_{n+1,ij} = -\delta_{ij}.$$

From (7.4) it follows $\bar{e}_{n+1} \in S_p^{N-1} \subset N_pM^n(\bar{e}_{n+1})$. Hence setting (7.7) in (4.1) we get $L_k(p,\bar{e}_{n+1}) = \binom{n}{k}$. Since $L_k(p,\tilde{e}_{n+1})$ is the maximal value of $L_k(p,e_{n+1})$, $e_{n+1} \in S_p^{N-1}$, we have

(7.8) $$L_k(p,\tilde{e}_{n+1}) \geq L_k(p,\bar{e}_{n+1}) = \binom{n}{k}.$$

We have

Proposition 8. Let $\tilde{e}_{n+1}$ denotes the canonical vector of (7.1) defined by means of L_1/S_p^{N-1}. There exists a cross section

(7.9) $$\tilde{e}_{n+1}: M^n \longrightarrow NM^n(\bar{e}_{n+1}).$$

and for every $p \in M^n$ the vector $\tilde{e}_{n+1}$ is transversal to S^{n+N-1}. For every $e_{n+2} \in S_p^{N-1} \subset N_pM^n(\bar{e}_{n+1})$ such that

(7.10) $$e_{n+2} \cdot \tilde{e}_{n+1} = 0$$

the function (4.3) takes the form

(7.11) $$L_2(p,e_{n+2}) = -\frac{1}{2}\sum_{i,j=1}^{n} A^2_{n+2,ij}.$$

Proof. Fron (7.3) we have

(7.12) $$L_1(p,\tilde{e}_{n+1}) \geq n.$$

From (4.2), (4.11) and (7.12) we get

(7.13) $$2I_{1,n+1} = -\sum_{i=1}^{n} \tilde{A}_{n+1,ii} \geq n$$

for every $p \in M^n$. Hence, from Proposition 5 we get (7.9). From (7.4) we have

$$(7.14) \qquad \tilde{e}_{n+1} = a_{n+1,n+1}\bar{e}_{n+1} + \sum_{r=n+2}^{n+N} a_{n+1,r}e_r .$$

From (4.12) and (7.14) we get

$$(7.15) \qquad a_{n+1,n+1} = -\frac{1}{2l_{1,n+1}}\left(\sum_{i=1}^{n} \bar{A}_{n+1,ii}\right) .$$

From (4.2) and (7.13) it follows that (7.15) can be written in the form

$$(7.16) \qquad a_{n+1,n+1} = \frac{L_1(p,\bar{e}_{n+1})}{L_1(p,\tilde{e}_{n+1})} .$$

From (7.8) and (7.16) we get $0 < a_{n+1,n+1} \leq 1$, hence

$$(7.17) \qquad 0 < \tilde{e}_{n+1} \cdot \bar{e}_{n+1} \leq 1$$

for every $p \in M^n$. Denoting by $\alpha(p)$, $p \in M^n$, the angle between $\tilde{e}_{n+1}$ and $\bar{e}_{n+1}$, (7.17) can be written in the form

$$(7.18) \qquad 0 < \cos\alpha(p) \leq 1 \quad \text{or equivalently} \quad 0 \leq \alpha(p) < \frac{\pi}{2} .$$

From (7.18) it follows that $\tilde{e}_{n+1}$ is transversal to S^{n+N-1} at every point $\bar{e}_{n+1}(p) \in S^{n+N-1}$, $p \in M^n$.

For $r = n+2$ we get from (3.34)

$$(7.19) \qquad A_{n+2,ij} = a_{n+2,s}\bar{A}_{sij} .$$

From (4.2), (4.13) and (7.19) we get

$$(7.20) \quad L_1(p,e_{n+2}) = -\sum_{i=1}^{n} A_{n+2,ii} = -\sum_{s=n+2}^{n+N} a_{n+2,s} \sum_{i=1}^{n} \bar{A}_{sii} = 0,$$

where e_{n+2} in (7.20) satisfies (7.10). From (7.20) it follows that the function $L_2(p,e_{n+2})$ defined by (4.3) take the form (7.11).

By $\tilde{e}_{k,n+\rho}$, $1 \leq k \leq n$, $1 \leq \rho < N$, we denote the ρ-th canonical vector defined by means of L_k, i.e. $L_k(p, \tilde{e}_{k,n+\rho})$ is the maximal value of $L_k/S_p^{N-\rho}$.

An immersion (7.1) such that (7.2) is satisfied is called canonical, if there exists a k, $1 \leq k \leq n$, such that for every $p \in M^n$

$$\bar{e}_{n+1} = \tilde{e}_{k,n+1} \quad \text{or} \quad \bar{e}_{n+1} = -\tilde{e}_{k,n+1} \tag{7.21}$$

and

$$L_{2j}(p, \tilde{e}_{2j,n+2}) = 0 \quad \text{for every} \quad j, \ 1 < 2j \leq n. \tag{7.22}$$

Finally we prove the following

Theorem. If (7.1) is a canonical immersion of a closed and connected manifold M^n, $n \geq 2$, then

$$\bar{e}_{n+1}(M^n) = S^n \subset S^{n+N-1}, \tag{7.23}$$

where S^n denotes the unit sphere and M^n is diffeomorphic with S^n.

Proof. From (7.7) and (7.21) it follows

$$\bar{A}_{n+1,ij} = \pm\tilde{A}_{n+1,ij} = -\delta_{ij}, \tag{7.24}$$

where $\tilde{A}_{n+1,ij}$ are coefficients of the second quadratic form evaluated in the direction of $\tilde{e}_{k,n+1}$. From (7.24) it follows that the equations (4.25) for $\rho = 1$ take the form

$$\sum_{l=1}^{k} \sum_{i_1<\ldots<i_k} \det(-\delta_{i_1 q}, \ldots, \bar{A}_{r i_l q}, \ldots, -\delta_{i_k q}) = 0, \tag{7.25}$$

where $n+2 \leq r \leq n+N$, $1 \leq k \leq n$, $1 \leq i_m \leq n$, $1 \leq m \leq k$, $q \in \{i_1, \ldots, i_k\}$, and $\bar{A}_{rij}$ are coefficients of the second quadratic form evaluated in the direction of $\bar{e}_r$. The vector $\bar{e}_r$ is an element of a base in $N_p M^n(\bar{e}_{n+1})$ of the form

$$\bar{e}_{n+1}(p)\tilde{e}_{k,n+1}\bar{e}_{n+2}\ldots\bar{e}_{n+N} \in N_1(\bar{e}_{n+1}). \tag{7.26}$$

We have

$$\det(-\delta_{i_1q},\ldots,\bar{A}_{ri_1q},\ldots,-\delta_{i_kq}) = (-1)^{k-1}A_{ri_1i_1}. \tag{7.27}$$

Hence

$$\sum_{l=1}^{k}\det(-\delta_{i_1q},\ldots,\bar{A}_{ri_lq},\ldots,-\delta_{i_kq}) = \tag{7.28}$$

$$= (-1)^{k-1}(\bar{A}_{ri_1i_1} + \ldots + \bar{A}_{ri_ki_k}) = 0,$$

where $n+2\leq r\leq n+N$. From (7.25) and (7.28) we get

$$\sum_{i_1<\ldots<i_k}(\bar{A}_{ri_1i_1} + \ldots + \bar{A}_{ri_ki_k}) = 0, \quad n+2\leq r\leq n+N. \tag{7.29}$$

The equations (7.29) are exactly the equations (4.13). Hence we have proved that for every k, $1\leq k\leq n$, if $\tilde{e}_{k,n+1}$ satisfies (7.21), then the equations (4.13) are satisfied.

Let $\tilde{e}_{j,n+2}\in N_pM^n(\bar{e}_{n+1})$, $1\leq j\leq n$, denotes a second canonical vector. We choose a base

$$\bar{e}_{n+1}(p)\tilde{e}_{k,n+1}\tilde{e}_{j,n+2}\bar{e}_{n+3}\ldots\bar{e}_{n+N}\in N_1(\bar{e}_{n+1}). \tag{7.30}$$

The equations (4.25) now take the form

$$\sum_{l=1}^{j}\sum_{i_1<\ldots<i_j}\det(\tilde{A}_{n+2,i_1q},\ldots,\bar{A}_{ri_lq},\ldots,\tilde{A}_{n+2,i_jq}) = 0, \tag{7.31}$$

where $n+3\leq r\leq n+N$ and $\tilde{A}_{n+2,ij}$ are coefficients of the second quadratic form evaluated in the direction of $\tilde{e}_{j,n+2}$. Let

$$\bar{e}_{n+1}(p)\bar{e}_1\ldots\bar{e}_n\in T(\bar{e}_{n+1}), \tag{7.32}$$

where T(x) is the total space of (3.4), denotes a base in T_pM^n such that $\bar{e}_i$ define principal axes of the matrix $(\tilde{A}_{n+2,ij})$. Then $(\tilde{A}_{n+2,ij})$ is a diagonal matrix and (7.31) takes the form

$$\sum_{l=1}^{i}\sum_{i_1<\ldots<i_j}\det(\tilde{A}_{n+2,i_1i_1},\ldots,\bar{A}_{ri_lq},\ldots,\tilde{A}_{n+2,i_ji_j}) = 0. \tag{7.33}$$

We have

$$(7.34)\qquad \det(\tilde{A}_{n+2,i_1i_1},\ldots,\bar{A}_{ri_lq},\ldots,\tilde{A}_{n+2,i_ji_j}) = \\ = \tilde{A}_{n+2,i_1i_1}\cdot\ldots\cdot\bar{A}_{ri_li_l}\cdot\ldots\cdot\tilde{A}_{n+2,i_ji_j}.$$

From (7.33) and (7.34) it follows that (7.31) takes the form

$$(7.35)\qquad \sum_{l=1}^{i}\sum_{i_1<\ldots<i_j}\tilde{A}_{n+2,i_1i_1}\cdot\ldots\cdot\bar{A}_{ri_li_l}\cdot\ldots\cdot\tilde{A}_{n+2,i_ji_j} = 0,$$

where $n+3\leqslant r\leqslant n+N$. We have

$$\sum_{i_1<\ldots<i_j}\tilde{A}_{n+2,i_1i_1}\cdot\ldots\cdot\bar{A}_{ri_li_l}\cdot\ldots\cdot\tilde{A}_{n+2,i_ji_j} = \\ = \frac{1}{j!}\sum_{i_1,\ldots,i_j=1}^{n}\tilde{A}_{n+2,i_1i_1}\cdot\ldots\cdot\bar{A}_{ri_li_l}\cdot\ldots\cdot\tilde{A}_{n+2,i_ji_j}.$$

Hence, we get

$$\sum_{l=1}^{j}\sum_{i_1<\ldots<i_j}\tilde{A}_{n+2,i_1i_1}\cdot\ldots\cdot\bar{A}_{ri_li_l}\cdot\ldots\cdot\tilde{A}_{n+2,i_ji_j} = \\ = \frac{1}{(j-1)!}\sum_{i=1}^{n}\bar{A}_{rii}\sum_{i_2,\ldots,i_j=1}^{n}\tilde{A}_{n+2,i_2i_2}\cdot\ldots\cdot\tilde{A}_{n+2,i_ji_j}.$$

Therefore the equations (7.33) can be written in the form

$$(7.36)\qquad \sum_{i=1}^{n}\bar{A}_{rii}\sum_{i_2,\ldots,i_j=1}^{n}\tilde{A}_{n+2,i_2i_2}\cdot\ldots\cdot\tilde{A}_{n+2,i_ji_j} = 0,$$

Therefore the equations (7.33) can be written in the form

$$(7.36)\qquad \sum_{i=1}^{n}\bar{A}_{rii}\sum_{i_2,\ldots,i_j=1}^{n}\tilde{A}_{n+2,i_2i_2}\cdot\ldots\cdot\tilde{A}_{n+2,i_ji_j} = 0,$$

where $n+3 \leqslant r \leqslant n+N$. From (4.13) it follows that (7.36) are identities. This means that for every choice of

$$(7.37) \qquad e_{n+2} = a_{n+2,s}\bar{e}_s, \quad \sum_{s=n+2}^{n+N} a_{n+2,s}^2 = 1,$$

where $\bar{e}_s$ are the vectors of (7.24), the function $L_j(p,e_{n+2})$ has the same maximal value. Therefore

$$(7.38) \qquad L_j(p,e_{n+2}) = B_j = \text{const}$$

for every j, $1 \leqslant j \leqslant n$. From $L_{2m+1}(p,-e_{n+2}) = -L_{2m+1}(p,e_{n+2})$ and (7.38) we get

$$(7.39) \qquad B_{2m+1} = 0 \quad \text{for every } m, \ 1 \leqslant 2m+1 \leqslant n.$$

From (7.22) and (7.38) we get

$$(7.40) \qquad B_{2m} = 0 \quad \text{for every } m, \ 1 < 2m \leqslant n.$$

From (4.1), (7.38), (7.39) and (7.40) we get

$$(7.41) \qquad \sum_{i_1<\ldots<i_j} \det(A_{n+2,pq})_{p,q\in\{i_1,\ldots,i_k\}} = 0,$$

where $A_{n+2,ij}$ are coefficients of the second quadratic form evaluated in the direction of the vector (7.37). Let

$$(7.42) \qquad \bar{e}_{n+1}(p)e_1\ldots e_n \in T(\bar{e}_{n+1})$$

denotes a base in T_pM^n such that e_i define principal axes of $(A_{n+2,ij})$. Then (7.41) has the form

$$(7.43) \qquad \sum_{i_1<\ldots<i_j} A_{n+2,i_1i_1},\ldots,A_{n+2,i_ji_j} = 0$$

for every j, $1 \leqslant j \leqslant n$. From (7.43) it follows that the eigenvalues $A_{n+2,ii} = 0$, $1 \leqslant i \leqslant n$. Hence in an arbitrary base of the form (7.42) of T_pM^n we get $A_{n+2,ij} = 0$, $1 \leqslant i,j \leqslant n$. This proves (7.23).

Since M^n is by (7.23) locally diffeomorphic with S^n and compact, it follows that M^n is a finite covering of S^n. Such a connected covering is for $n \geq 2$ diffeomorphic with S^n.

REFERENCES

[1] Ph. Griffiths: On Cartan's method of Lie groups and moving frames as applied to uniqueness and existence questions in differential geometry, Duke Math. J. 41 (1974), 775-814.

[2] G.R. Jenssen: Higher order contact of submanifolds of homogeneous spaces, Lecture Notes in Mathematics 610 (1977).

[3] M. Rochowski: The Frenet frame of an immersion, J. Diff. Geom., 10, (1975), 181-200.

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