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ON INTRINSIC ISOMETRIES AND RIGID SUBSETS OF EUCLIDEAN SPACES*)

0. Introduction

We investigate the notions of rigidity and weak rigidity of geometrically acceptable sets (comp. [1], [2], [6]). Our purpose is to examine which class of mappings and which geometrical or topological operations (as union, Cartesian product, cone over a set etc.) preserve rigidity and weak rigidity.

In any metric space (X,ϱ) in which every two distinct points can be joined by an arc L of the finite length $|L|$ the intrinsic metric ϱ^* can be introduced as follows

$$\varrho^*(x,y) = \inf\{|L| : L \text{ is an arc in } X \text{ and } x,y \in L\} .$$

The space (X,ϱ) is said to be geometrically acceptable if ϱ^* is topologically equivalent to ϱ. The class of geometrically acceptable spaces is denoted by GA (comp. [1], [2]).

A surjective map $f : X \to Y$ is an intrinsic isometry if and only if it is an isometry with respect to the intrinsic metrics. A map $f : X \to Y$ is an intrinsic embedding if and only if $f : X \to f(X)$ is an intrinsic isometry. By Theorem 2.1 [3] a map $f : X \to Y$ is an intrinsic isometry if and only if it is a homeomorphism preserving the lengths of arcs. Let

*) The paper contains most of the results of the Author's Ph.D. thesis written under the supervision of Maria Moszyńska.

(X, ρ_X), (Y, ρ_Y) be metric spaces, let $A \subset X$, $(A, \rho_X|A^2) \in GA$ and let $\mathcal{F}$ be a class of intrinsic embeddings of A in Y.

0.1. Definition (comp. [6]):

(i) A is weakly rigid with respect to $\mathcal{F}$ iff for every $h \in \mathcal{F}$ the set h(A) is isometric to A.

(ii) A is rigid with respect to $\mathcal{F}$ iff every $h \in \mathcal{F}$ is an isometric embedding of A in Y.

Let us consider the following particular case:

0.2. Definition. A is rigid (weakly rigid) in X iff it is rigid (weakly rigid) with respect to all intrinsic embeddings of A in X.

Using this terminology we can reformulate 3.1 [3] as follows:

0.3. Every open, connected subset of the Euclidean n-space E is rigid in E.

By 2.2 [6]

0.4. Every open, connected subset of the sphere S^n in the Euclidean n-space is rigid in S^n.

In the sequel we shall use the following terminology and notation:

R^n or (R^n, ρ_n) - the Cartesian n-space.

$\mathcal{E}^n$ - the class of Euclidean n-spaces (i.e. spaces isometric to R^n).

For a subset A of a metric space (X, ρ):

diam A - the diameter of A,

$Cl_X A$ - the closure of A in X,

$Int_X A$ - the interior of A in X,

$\Gamma_X(A) := \{C : C \text{ is a component of } Int_X A\}$,

$\Omega_X(A) := \{Cl_A C : C \in \Gamma_X(A)\}$.

If it does not lead to a confusion we omit the index X in this notation.

We shall frequently use the following immediate consequence of 0.3.

0.5. Let $E \in \mathcal{E}^n$ and $A \subset E$. If $A = \mathrm{Cl}_A \mathrm{Int}A$, $A \in GA$ and $\mathrm{Int}A$ is connected then A is rigid in E.

It is evident that if A is rigid in X then it is weakly rigid in X. The following simple example shows that the converse does not hold.

0.6. E x a m p l e . Let $A_i = \{(x,y) \in R^2 : (x+(-1)^i)^2 + y^2 \leq 1\}$ for $i = 1,2$, and let $A = A_1 \cup A_2$ (fig.1).

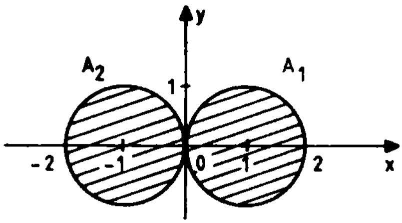


Fig.1

Obviously $A \in GA$ and A is weakly rigid in R^2. However the function $h : A \to R^2$ defined by the formula

$$h(x,y) = \begin{cases} (x,-y) & \text{for} \quad (x,y) \in A_1 \\ (x,y) & \text{for} \quad (x,y) \in A_2 \end{cases}$$

is an intrinsic embedding but not an isometric embedding. ∎

Let $E \in \mathcal{E}^n$ and let H be an affine subspace of E. If $\dim H = n-1$, then H is called a hyperplane.

We shall use the following notation:

$\pi_H : E \to H$ - the orthogonal projection of E onto H,

$H^{\perp}(a)$ - the affine subspace orthogonal to H and passing through a,

$\mathrm{Af}\, A = \cap \{H : H$ is an affine subspace of E and $A \subset H\}$,

$\mathrm{Conv}\, A$ - the convex hull of A,

$\Delta(a_0,\ldots,a_n) = \mathrm{Conv}\{a_0,\ldots,a_n\}$ for affine independent points $a_0,\ldots,a_n$,

$x \vee y$ - the line in E passing through x, y (for $x \neq y$).

For $A \subset H$ and $c_0 \in E - H$ the cone $C(c_0,A)$ over A with vertex c_0 is defined as follows

$$C(o_0,A) = \bigcup_{a\in A} \Delta(o_0,a).$$

A cylinder over A is the Cartesian product of A by an Euclidean segment.

Given a metric space (X,ρ), a point $x_0 \in X$ and a $\lambda > 0$, let

$$B_\rho(x_0,\lambda) := \{x \in X : \rho(x_0,x) < \lambda\}.$$

Let (X_i,ρ_i) be a metric space for $i = 1,2$. Then p_{X_i} is a projection of $X_1 \times X_2$ on X_i, $i = 1,2$.

1. Preliminary lemmas and theorems

Let us start with an easy lemma, proof of which will be omitted.

1.1. Lemma. Let $E_1 \in \mathcal{E}^k$, $E_2 \in \mathcal{E}^n$, $A \subset E_1$, $B \subset E_2$, $\mathrm{Af}\, A = E_1$, $\mathrm{Af}\, B = E_2$, and let $(a,b) \in A \times B$. Then

(i) for every isometric embedding $f : A \times B \to E_1 \times E_2$ there is exactly one pair (H_1,H_2) of orthogonal affine subspaces of $E_1 \times E_2$, such that $\dim H_1 = k$, $\dim H_2 = n$, $f(A\times\{b\}) \subset H_1$ and $f(\{a\}\times B) \subset H_2$.

(ii) for any isometric embeddings $f_1 : A \to E_1$, $f_2 : B \to E_2$ and $f : A\times B \to E_1\times E_2$ if

$$f|(A\times\{b\}) \cup (\{a\}\times B) = (f_1\times f_2)|(A\times\{b\}) \cup (\{a\} \times B)$$

then $f = f_1\times f_2$.

Now we shall prove some statements concerning GA-spaces.

1.2. Theorem. Let $f : X \to Y$ be an open Lipschitz function of (X,ρ_X) onto (Y,ρ_Y). If $X \in GA$ then $f(X) \in GA$.

Proof. Let λ be the constant of f. Take y_1, y_2 in Y an x_1, x_2 in X such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Let $L \subset X$ be an arc of the finite length joining x_1 and x_2. Since f(L) is arcwise connected, there is an arc L' in f(L) joining y_1 and y_2. Moreover, $|L'| < \lambda|L| < \infty$. Hence ρ_Y induces the intrinsic metric ρ_Y^*. To show that ρ_Y is topologically equivalent to ρ^*_Y it suffices to prove the following condition (comp. [5])

(1) $$\forall y \in Y \;\; \forall \varepsilon > 0 \;\; \exists \delta > 0 \;\; B_{\rho_Y}(y,\delta) \subset B_{\rho^*_Y}(y,\varepsilon).$$

Take $y \in Y$ and $x \in X$ such that $f(x) = y$. Let us take an $\varepsilon > 0$ and let $\delta_0 > 0$ satisfies the condition

(2) $$B_{\rho_X}(x,\delta_0) \subset B_{\rho^*_X}(x,\tfrac{\varepsilon}{\lambda}).$$

Since f is open, the set $f(B_{\rho_X}(x,\delta_0))$ is open in (Y, ρ_Y) whence there is $\delta > 0$ such that $B_{\rho_Y}(y,\delta) \subset f(B_{\rho_X}(x,\delta_0))$. Take $y' \in B_{\rho_y}(y,\delta)$ and $x' \in B_{\rho_X}(x,\delta_0)$ such that $f(x') = y'$. By (2) the points x' and x can be joined in X by an arc L_0 with $|L_0| < \frac{\varepsilon}{\lambda}$. Since $f(L_0)$ is arcwise connected, there is an arc L'_0 in $f(L)$ joining y' and y. Then $y' \in B_{\rho^*}(y,\varepsilon)$ because $|L'_0| < \lambda |L_0| < \varepsilon$. Hence (1) is proved. ∎

From Theorem 1.2 we obtain immediately

1.3. Corollary. Let $E \in \mathcal{E}^n$, $X \subset E$, and let $f : E \to E$ be an affine automorphism. If $X \in GA$ then $f(X) \in GA$.

1.4. Theorem. Let $(E,\rho) \in \mathcal{E}^n$, let H be a hyperplane in E, and let $X \subset H$ and $c_0 \in E-H$. If $X \in GA$ then $C(c_0,X) \in GA$.

Proof. Let $A = C(c_0,X)$. Let $\{x'\} = (x \vee c_0) \cap H$ for $x \in A - \{c_0\}$. First we shall prove that every two distinct points in A can be joined by an arc of a finite length. Take $x_1, x_2 \in A$, $x_1 \neq x_2$. If $x_1 = c_0$ or $x_2 = c_0$ or $x'_1 = x'_2$ then evidently $\Delta(x_1,x_2) \subset A$. Assume now, that $x_1, x_2 \in A - \{c_0\}$ and $x'_1 \neq x'_2$. Let L' be an arc of a finite length joining x'_1 and x'_2. Then $L = \Delta(x_1,x'_1) \cup L' \cup \Delta(x'_2,x_2)$ is an arc in A joining x_1 and x_2 and $|L| < \infty$.

It remains to prove that $\rho|A\times A$ is topologically equivalent to $(\rho|A\times A)^*$. To this aim we shall prove

(1) $$\forall x \in A \;\; \forall \varepsilon > 0 \;\; \exists \delta > 0 \;\; B_{\rho|A\times A}(x,\delta) \subset B_{(\rho|A\times A)^*}(x,\varepsilon).$$

For $x = c_0$ the condition (1) is evidently satisfied. Take $x \in A - c_0$ and let $\varepsilon > 0$. Since $x' \in X$ and $X \in GA$, there is $\delta_0 > 0$ such that

(2) $$B_{\varrho|X\times X}(x', \delta_0) \subset B_{(\varrho|X\times X)^*}(x', \tfrac{\varepsilon}{2}).$$

Denote by H_x the hyperplane in E passing through x and parallel to H. Let $y'' = (y \vee c_0) \cap H_x$ for $y \in A - \{c_0\}$. For any $\alpha > 0$ let

$$U_\alpha = \{y \in E : \varrho(y, H_x) < \alpha\}.$$

Take $\alpha_1 > 0$ such that $c_0 \notin U_\alpha$ for $\alpha < \alpha_1$. Since $Cl_H B_{\varrho|X\times X}(x', \delta_0)$ is compact, there is a positive number α_0, such that $\alpha_0 < \alpha_1$ and

(3) $$|\Delta(y, y'')| < \tfrac{\varepsilon}{2} \quad \text{for} \quad y \in U_{\alpha_0} \cap C(c_0, B_{\varrho|X\times X}(x', \varrho_0)).$$

Let $U = U_{\alpha_0} \cap C(c_0, B_{\varrho|X\times X}(x', \delta_0))$. Since $x \in \operatorname{Int} U$ there is $\delta > 0$ such that $B_{\varrho|A\times A}(x, \delta) \subset U$. We shall prove that $B_{\varrho|A\times A}(x, \delta) \subset B_{(\varrho|A\times A)^*}(x, \varepsilon)$. Indeed, take $y \in B_{\varrho|A\times A}(x, \delta)$. Then $y' \in B_{\delta|A\times A}(x', \delta_0)$ and, by (2), there is an arc L' in X, with $L' < \frac{\varepsilon}{2}$, joining x' and y'. Let $L'' = C(c_0, L') \cap H_x$, then L'' is an arc joining x and y''. Moreover

(4) $$|L''| < |L'| < \tfrac{\varepsilon}{2}.$$

Consider $L = \Delta(y, y'') \cup L''$. Clearly L is an arc in $C(c_0, X)$ joining y and x. Moreover (3) and (4) imp.y $|L| < \varepsilon$. Hence (1) is proved. ■

The following proposition will be used in the sequel (comp. [7]).

1.5. If $f_i : X_i \to Y_i$ is an intrinsic isometry for $i = 1, 2$ then $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is an intrinsic isometry. Observe that the implication converse to 1.5 is also true:

1.6. If $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is an intrinsic isometry, then $f_i : X_i \to Y_i$ is an intrinsic isometry for $i = 1,2$.

2. Rigid and weakly rigid subsets of metric spaces

All spaces considered in this section will be GA. From the definition of rigidity (weak rigidity) follows immediately.

2.1. Let $A \subset X$, and let $f : A \to Y$ be an intrinsic embedding. If A is rigid (weakly rigid) in X then f(A) is rigid (weakly rigid) in X.

Now we shall prove:

2.2. Theorem. If A is rigid in $X \times Y$, and $p_X(A), p_Y(A) \in GA$ then $p_X(A)$ and $p_Y(A)$ are rigid in X and Y respectively.

Proof. We shall prove that $p_X(A)$ is rigid in X. Let $h : p_X(A) \to X$ be an intrinsic embedding. Take $x_1, x_2 \in p_X(A)$ and $y_1, y_2 \in Y$ such that $(x_1,y_1),(x_2,y_2) \in A$. Let us define $\bar{h} = h \times id_Y$. By 1.5 we infer that $\bar{h}$ is an intrinsic embedding as well as $\bar{h}|A$. Then $\bar{h}|A$ is an isometric embedding. Hence

$$\sqrt{\rho_X^2(x_1,x_2) + \rho_Y^2(y_1,y_2)} = \rho_{X\times Y}((x_1,y_1),(x_2,y_2)) =$$

$$= \rho_{X\times Y}(\bar{h}(x_1,y_1),\bar{h}(x_2,y_2)) = \rho_{X\times Y}((h(x_1),y_1),(h(x_2),y_2)) =$$

$$= \sqrt{\rho_X^2(h(x_1),h(x_2)) + \rho_Y^2(y_1,y_2)}.$$

Therefore $\rho_X(x_1,x_2) = \rho_X(h(x_1),h(x_2))$; hence h is an isometric embedding.

Now we shall give an example of an affine automorphism in R^2 which preserve neither rigidity nor weak rigidity.

2.3. Example. Let $T_1 = \Delta((0,0),(4,0)(0,2))$, $T_2 = \Delta(a,b,c)$ for $a = \left(\frac{1}{2}, \frac{3}{2}\right)$, $b = \left(\frac{5}{2}, \frac{1}{2}\right)$, $c = \left(\frac{1}{2}, \frac{1}{2}\right)$ and let $T_3 = \mathrm{Conv}\left\{\left(\frac{3}{4}, 0\right),\left(\frac{3}{4}, \frac{1}{2}\right),\left(\frac{5}{2}, \frac{1}{2}\right),\left(\frac{7}{2}, 0\right)\right\}$. Let $A_1 = \mathrm{Int}\, T_2 \cup \{a\}$, $A_2 = T_1 - (T_2 \cup T_3) \cup \{a\}$ and let $A = A_1 \cup A_2$ (fig.2).

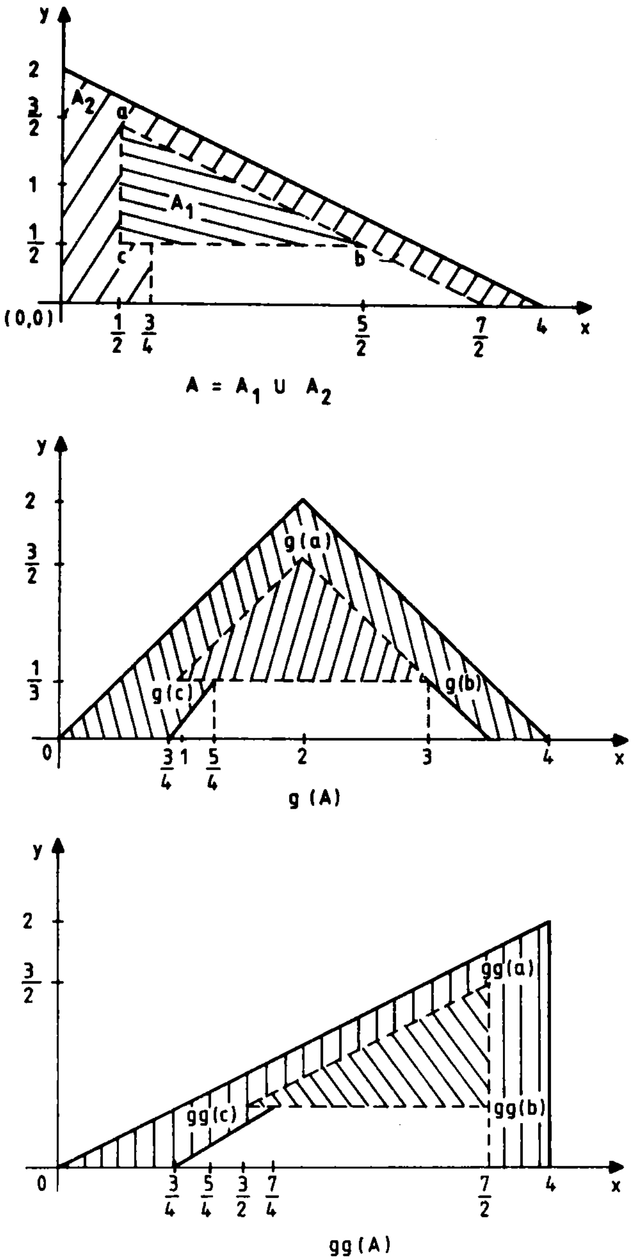


Fig.2

Evidently $A \in GA$. To prove that A is rigid in R^2 it suffices to show that

(1) the only one isometric embedding $f : A_1 \to R^2$ such that $f(A_1) \cap A_2 = \{a\}$ and $f(a) = a$ is $f = id_{A_1}$. Indeed, assume (1) and let $h : A \to R^2$ be an intrinsic embedding. By 0.5, $h|A_i$ is an isometric embedding for $i = 1,2$. Let $h_2 : R^2 \to R^2$ be an isometric extension of $h|A_2$. Then $h_2^{-1}\, h|A_1$ is an isometric embedding and $h_2^{-1}\, h|A_2 = id_{A_2}$. Hence $h_2^{-1}\, h(a) = $ $= h^{-1}h(a) = a$ and $h_2^{-1}|h(A_1) \cap A_2 = h_2^{-1}h(A_1 \cap A_2) = \{a\}$. Then, by (1) $h_2^{-1}\, h|A_1 = id_{A_1}$ whence $h_2^{-1}h = id_{A_1 \cup A_2}$. Therefore h is an isometric embedding.

Let us now prove (1). Let $f : A_1 \to R^2$ be an isometric embedding, $f(A_1) \cap A_2 = \{a\}$ and $f(a) = a$. Let $f_1 : R^2 \to R^2$ be an isometric extension of f. It is easy to verify that $f_1(b) = b$ and $f_1(c) = c$ thus $f_1 = id_{R_2}$. Hence $f = id_{A_1}$ and A is rigid in R^2. Let us now define an affine automorphism $g : R^2 \to R^2$ by the formula $g(x,y) = (x+y,y)$. By Corollary 1.3, $g(A) \in GA$. It can be proved that $g(A)$ is weakly rigid in R^2 and it is evident that $gg(A)$ is neither rigid nor weakly rigid in R^2 (fig.2). ■

We shall now prove some statements concerning the closure operation of rigid and weakly rigid sets. It is evident that

2.3. If A is rigid in X and $Cl_X A \in GA$ then $Cl_X A$ is rigid in X.

The analogue of 2.3 for weakly rigid sets is not true (see example 2.6). However we can easily prove

2.4. If A is weakly rigid in a complete space X and $Cl_X A \in GA$ then $Cl_X A$ is weakly rigid in X.

P r o o f . Let $h : Cl_X A \to X$ be an intrinsic embedding. Since A is weakly rigid, there is an isometry $f : A \to h(A)$. By virtue of [4] th. 4.3.10 there is an extension $\bar{f} : Cl_X A \to Cl_X h(A)$ of f. Hence $\bar{f}(Cl_X A) = h(Cl_X A)$. Thus $Cl_X A$ is weakly rigid. ■

Similarly it can be proved

2.5. If A is weakly rigid in perfectly homogeneous space X and $Cl_X A \in GA$ then $Cl_X A$ is weakly rigid in X.

2.6. Example. Let $X = \{(x,y) \in R^2 : (x+1)^2+y^2 < 1 \vee (x-1)^2+y^2 < 1\} \cup \{(-1,1),(0,0)\}$. Let ρ be the Cartesian metric restricted to X. Let $A = \{(x,y) \in X : (x \leq 0 \wedge y < 1) \vee (x > 0 \wedge y \leq \frac{1}{2})\}$ (fig.3). It can be easily proved that A is weakly rigid in (X,ρ). However $Cl_X A = A \cup \{(-1,1)\}$ is not weakly rigid in X.

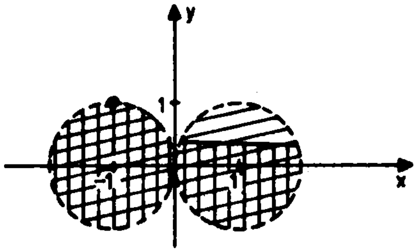


Fig.3

Finally let us notice that sometimes the Cartesian product of rigid sets is neither rigid nor weakly rigid.

2.7. Example. Let $X = \{(x,y) \in R^2 : x^2+y^2 = 1\}$ and let ρ_X be the Cartesian metric restricted to X. Let $Y = R$ and let ρ_Y be the Cartesian metric in R. We define

$$A = \{(\cos\varphi, \sin\varphi) \in X : 0 \leq \varphi \leq \pi\}, \quad B = \{\psi \in Y : 0 \leq \psi \leq 1\}.$$

Then $A \times B = \{(\cos\varphi, \sin\varphi, \psi) : 0 \leq \varphi \leq \pi, 0 < \psi \leq 1\}$. By 0.3 and 0.4 the sets A and B are rigid in X and Y respectively.

Let us define an intrinsic embedding $h : A \times B \to X \times Y$ by the formula $h(x,y,z) = (\cos z, \sin z, \arccos x)$, (fig.4). It is evident that h is not an isometric embedding. Moreover the sets $A \times B$ and $h(A \times B)$ are not isometric in $X \times Y$. ■

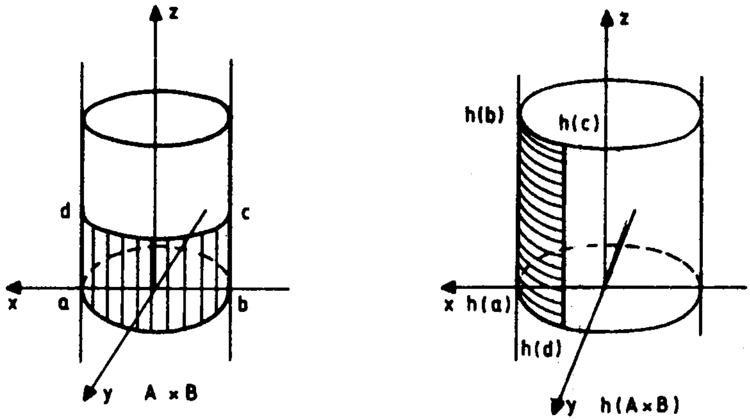


Fig. 4

3. Rigid and weakly rigid subsets of Euclidean spaces

In this section we shall restrict our consideration to subsets of Euclidean spaces.

3.1. The union of rigid sets

We start with some conditions under which the union of rigid sets is rigid.

3.1.1. Theorem. Let $A_1, A_2 \in E \in \varepsilon^n$, $A_1 \cup A_2 \in GA$, and $\dim Af(A_1 \cap A_2) = n$. If A_1 and A_2 are rigid in E then $A_1 \cup A_2$ is rigid in E.

Proof. Let $h : A_1 \cup A_2 \to E$ be an intrinsic embedding. Then $h|A_i$ is an isometric embedding for $i = 1,2$. Let $h_i : E \to E$ be an isometric extension of f_i for $i = 1,2$. Let $(a_0,\dots,a_n)$ be an affine base in $A_1 \cap A_2$. Since $h_1(a_j) = h(a_j) = h_2(a_j)$ for $j = 0,\dots,n$, we get $h_1 = h_2$. Hence h is an isometric embedding. ∎

3.1.2. Lemma. Let $E \in \varepsilon^n$. If A is weakly rigid in E and $\bar{\bar{A}} > 1$ then $\dim Af(A) = n$.

Proof. Assume $\dim Af\ A = k < n$. By [7] th. 1.5 there is an intrinsic embedding $h : Af\ A \to E$ such that $\operatorname{diam} h(Af\ A) < \operatorname{diam} A$. Thus, $\operatorname{diam} h(A) < \operatorname{diam} A$; hence A and $h(A)$ are not isometric, a contradiction. ∎

By Theorem 3.1.1 and Lemma 3.1.2 we obtain immediately

3.1.3. C o r o l l a r y . Let $A_1, A_2 \subset E \in \mathcal{E}^n$, $A_1 \cup A_2 \in GA$ and $\overline{\overline{A_1 \cap A_2}} > 1$. If A_1, A_2 and $A_1 \cap A_2$ are rigid in E then $A_1 \cup A_2$ is rigid in E.

Finally let us prove

3.4.1. T h e o r e m . Let $A_1, A_2 \subset E \in \mathcal{E}^n$, $A_1 \cup A_2 \in GA$, and let there is a set U open and convex in E, $U \subset A_1 \cup A_2$, $U \cap \operatorname{Int} A_1 \neq \emptyset \neq U \cap \operatorname{Int} A_2$. If A_1 and A_2 are rigid in E then $A_1 \cup A_2$ is rigid in E.

P r o o f . Let $h : A_1 \cup A_2 \to E$ be an intrinsic embedding. By 0.3 the set U is rigid in E. The rigidity of U, A_1, and A_2 implies that $h|U$, $h|A_1$, and $h|A_2$ are isometric embeddings. Let $h_i : E \to E$ and $g : E \to E$ be isometric extensions of $h|A_i$ and $h|U$ respectively, for i = 1,2. Consider an affine base $(a_0^i, \dots, a_n^i)$ in $U \cap \operatorname{Int} A_i$, i = 1,2. Since $h_i(a_j^i) = h(a_j^i) = g(a_j^i)$, for j = 0,...,n and i = 1,2, we have $h_i = g$. Hence $h = g|A_1 \cup A_2$ is an isometric embedding. ■

3.2. <u>Cartesian product of rigid (weakly rigid) sets</u>

The example 2.7 shows that the Cartesian product of rigid sets need not be rigid. For subsets of Euclidean spaces the problem is open. We shall give a partial solution, for some classes of subsets of Euclidean spaces (Cor. 3.2.4, 3.2.6, Th. 3.2.7, Cor. 3.2.8).

For any $S_0 \in \Omega(A)$ we define the following sequence $(\Omega_j(S_0))_{j \in N}$:

$$\Omega_1(S_0) = \{S_0\}, \quad \Omega_{j+1}(S_0) = \{S \in \Omega(A) - \Omega_j(S_0) : S \cap (\cup\, \Omega_j(S_0)) \neq \emptyset\}$$

for $j \geq 1$.

3.2.1. L e m m a . Let $A \subset E$. If $A = \bigcup_{j=1}^{\infty} \bigcup \Omega_j(S_0)$ for some $S_0 \in \Omega(A)$, then $A = \bigcup_{j=1}^{\infty} \bigcup \Omega_j(S)$ for every $S \in \Omega(A)$.

We omit the easy proof of this lemma.

Let us now define a class $\sum$ of subsets of E by the formula

$$\Sigma(E) := \{A \subset E : \exists S_0 \in \Omega(A),\ A = \bigcup_{j=1}^{\infty} \bigcup \Omega_j(S_0)\}.$$

We shall prove that for Euclidean spaces E_1 and E_2 the Cartesian product of rigid sets $A \in \Sigma(E_1)$ and $B \in \Sigma(E_2)$ is rigid in $E_1 \times E_2$ (Cor. 3.2.4).

For this purpose we shall first prove that the Cartesian product of rigid subsets is rigid with respect to some class of intrinsic embeddings (Th. 3.2.3).

Let $(E_1, \rho_1) \in \mathcal{E}^k$, $(E_2, \rho_2) \in \mathcal{E}^n$, and let $A \subset E_1$, $B \subset E_2$.

We define the class $\mathcal{F}(A,B)$ as follows: $h \in \mathcal{F}(A,B)$ if and only if h is an intrinsic embedding and for every $(a,b) \in A \times B$ there is a pair of orthogonal affine subspaces (H_1, H_2) in $E_1 \times E_2$ such that

(*) $$h(A \times \{b\}) \subset H_1 \quad \text{and} \quad h(\{a\} \times B) \subset H_2.$$

3.2.2. Theorem. If A and B are rigid in E_1 and E_2 respectively, $\overset{=}{A} > 1$, $\overset{=}{B} > 1$, then $A \times B$ is rigid with respect to $\mathcal{F}(A,B)$.

Proof. Let $(a,b) \in A \times B$ and let $h \in \mathcal{F}(A,B)$. Let H_1 and H_2 be the orthogonal subspaces in $E_1 \times E_2$ satisfying (*). From Lemma 3.1.2 we infer that $\dim H_1 = k$ and $\dim H_2 = n$. Let $f : E_1 \times E_2 \to E_1 \times E_2$ be an isometry such that

$$f(H_1) = E_1 \times \{b\} \quad \text{and} \quad f(H_2) = \{a\} \times E_2.$$

Then $fh(A \times \{b\}) \subset E_1 \times \{b\}$ and $fh(\{a\} \times B) \subset \{a\} \times E_2$. Since A and B are rigid, $fh|A \times \{b\}$ and $fh|\{a\} \times B$ are isometric embeddings. We shall prove that

(1) $fh(A \times \{b\}) \cup (\{a\} \times B)$ is an isometric embedding.

Let $(x,b) \in A \times \{b\}$ and $(a,y) \in \{a\} \times B$. If $fh(x,b) = (x',b)$ and $fh(a,y) = (a,y')$, then

$$(\varrho_1\times\varrho_2)(fh(a,y),fh(x,b)) = (\varrho_1\times\varrho_2)((a,y'),(x',b)) =$$
$$= (\varrho_1^2(a,x') + \varrho_2^2(y',b))^{\frac{1}{2}} = \left[(\varrho_1\times\varrho_2)^2((a,b),(x',b)) +\right.$$
$$\left. + (\varrho_1\times\varrho_2)^2((a,b),(a,y'))\right]^{\frac{1}{2}} = \left[(\varrho_1\times\varrho_2)^2((a,b),(x,b)) +\right.$$
$$\left. + (\varrho_1\times\varrho_2)^2((a,b),(a,y))\right]^{\frac{1}{2}} = (\varrho_1^2(a,x) + \varrho_2^2(y,b))^{\frac{1}{2}} =$$
$$= (\varrho_1\times\varrho_2)((a,y),(x,b)),$$

whence (1) is proved. Since f is an isometry, by (1), we obtain

(2) $h\,(\{a\}\times B)\cup(A\times\{b\})$ is an isometric embedding.

Let $(a_i,b_i)\in A\times B$, $i = 1,2$. Since $(a_i,b_i)\in\{a_1\}\times B\cup A\times\{b_2\}$ by (2) we infer that $h|(\{a_1\}\times B)\cup(A\times\{b_2\})$ is an isometric embedding. Thus $(\varrho_1\times\varrho_2)(h(a_1,b_1),h(a_2,b_2)) = (\varrho_1\times\varrho_2)((a_1,b_1),(a_2,b_2))$, whence h is an isometric embedding. This completes the proof.

3.2.3. **Lemma.** If $A\in\sum(E_1)$, $B\in\sum(E_2)$, and $h : A\times B \to E_1\times E_2$ is an intrinsic embedding, then $h\in\mathcal{F}(A,B)$.

Proof. We may assume that $A\neq\emptyset\neq B$. Let $h : A\times B\to E_1\times E_2$ be an intrinsic embedding. Take $(a,b)\in A\times B$. If $S\in\Omega(A)$ and $T\in\Omega(B)$, then $S\times T\in\Omega(A\times B)$ and by 1.3

(1) $h|S\times T$ is an intrinsic embedding.

Choose $S_0\in\Omega(A)$ and $T_0\in\Omega(B)$ such that $(a,b)\in S_0\times T_0$. By Lemma 1.1(i) there is exactly one pair (H_1,H_2) of orthogonal affine subspaces such that $\dim H_1 = k$, $\dim H_2 = n$, and

(2) $h(S_0\times\{b\})\subset H_1$, $h(\{a\}\times T_0)\subset H_2$.

We shall prove by induction that for every $j\in N$

(3) $h((\cup\,\Omega_j(S_0))\times\{b\})\subset H_1$.

By (2) the condition (3) holds for $j = 1$. Take an $i \geq 1$ and suppose that (3) holds for $j \leq i$. Choose $S_1 \in \Omega_{i+1}(S_0)$ and let $x \in S_1 \cap \cup\, \Omega_i(S_0)$. Thus, there is $S_2 \in \Omega_i(S_0)$ such that $x \in S_2$. By the inductive assumption

$$(4) \qquad h(S_2 \times \{b\}) \subset H_1.$$

According to (1), $h|S_2 \times T_0$ is an isometric embedding. Hence, by (4) and Lemma 1.1(i) we obtain

$$(5) \qquad h(\{x\} \times T_0) \subset H_1^{\perp}(x,b).$$

By (1), $h|S_1 \times T_0$ is an isometric embedding. Let H_0 be an affine k-dimensional subspace such that $h(S_1 \times \{b\}) \subset H_0$. By (5), applying once again Lemma 1.1(i) we conclude that $H_0 = H_1$. Thus (3) holds for $j = i+1$. Then by Lemma 3.2.1

$$h(A \times \{b\}) = h\Big(\bigcup_{j=1}^{\infty} \bigcup \Omega_j(S_0) \times \{b\}\Big) \subset H_1.$$

Similarly we prove that $h(\{a\} \times B) \subset H_2$. The proof is complete.∎

By Theorem 3.2.2 and Lemma 3.2.3 we obtain immediately

3.2.4. Corollary. Let $A \in \Sigma(E_1)$, $B \in \Sigma(E_2)$. If A and B are rigid in E_1 and E_2 respectively then $A \times B$ is rigid in $E_1 \times E_2$.

Now, let us define a subclass $\mathcal{L}$ of the class Σ. Let $E \in \mathcal{E}^n$. Then we define

$$\mathcal{L}(E) := \{A \subset E : A \text{ is connected, } A = \mathrm{Cl}_A \mathrm{Int}\, A,\ \Gamma(A) \text{ is locally finite in } A\}.$$

3.2.5. Lemma. $\mathcal{L}(E) \subset \Sigma(E)$.

The straightforward proof of this lemma will be omitted.

Combining Corollary 3.2.4 and Lemma 3.2.5 we obtain

3.2.6. Corollary. Let $A \in \mathcal{L}(E_1)$, $B \in \mathcal{L}(E_2)$.

If A and B are rigid in E_1 and E_2 respectively, then $A \times B$ is rigid in $E_1 \times E_2$.

Finally let us prove

3.2.7. Theorem. Let $A \in \Sigma(E_1)$, $B \in \Sigma(E_2)$. IF A and B are weakly rigid respectively in E_1 and E_2 then $A\times B$ is weakly rigid in $E_1\times E_2$.

Proof. The implication is evident for $A = \emptyset$ or $B = \emptyset$. Suppose now that $A \neq \emptyset \neq B$ and let $(a,b) \in A\times B$. Let $h : A\times B \to E_1\times E_2$ be an intrinsic embedding. By virtue of Lemma 3.2.3, $h \in \mathcal{F}(A\times B)$; hence there are orthogonal subspaces H_1 and H_2 in $E_1\times E_2$ such that

$$h(A\times\{b\}) \subset H_1 \quad \text{and} \quad h(\{a\}\times B) \subset H_2.$$

By Lemma 3.1.2 we obtain $\dim H_1 = k$ and $\dim H_2 = n$.

Let $f : E_1\times E_2 \to E_1\times E_2$ be an isometry such that $f(H_1) = E_1\times\{b\}$ and $f(H_2) = \{a\}\times E_2$. Then $fh(A\times\{b\}) \subset E_1\times\{b\}$ and $fh(\{a\}\times B) \subset \{a\}\times E_2$. Let us define $f_1 : A \to E_1$ and $f_2 : B \to E_2$ by the formulae $f_1(x) = p_{E_1} fh(x,b)$, $f_2(x) = p_{E_2} fh(a,y)$.

We shall prove that

$$fh = f_1\times f_2. \tag{1}$$

It is evident that

$$fh|(A\times\{b\}) \cup \{a\}\times B) = (f_1\times f_2)|(A\times\{b\} \cup \{a\}\times B). \tag{2}$$

Choose $S_0 \in \Omega(A)$ and $T_0 \in \Omega(B)$ such that $(a,b) \in S_0\times T_0$. We shall use the following notation

$$A_j = \cup\Omega_j(S_0),\ B_j = \cup\Omega_j(T_0), \quad \text{for} \quad j \in N;\ A_0 = \{a\},\ B_0 = \{b\}.$$

We shall prove by induction on j+k that

$$fh|A_j\times B_k = (f_1\times f_2)|A_j\times B_k \quad \text{for} \quad j,k \in N \cup \{0\}. \tag{3}$$

By (2) we infer that (3) holds for $j = 0$ or $k = 0$ and

$$fh(S_0\times\{b\} \cup \{a\}\times T_0) = (f_1\times f_2)|(S_0\times\{b\} \cup \{a\}\times T_0). \tag{4}$$

By 0.5, $fh|S_0\times T_0$ is an isometric embedding. Therefore, by (4) and Lemma 1.1(ii) we obtain $hf|S_0\times T_0 = (f_1\times f_2)|S_0\times T_0$. Hence (3) holds for $j = k = 1$. Take an $i \geq 2$ and suppose that (3) holds for $j+k \leq i$. To prove (3) it suffices to show that (3) holds for indexes $j+1$ and k. Let $S_1 \in \Omega_{j+1}(S_0)$, $T_1 \in \Omega_k(T_0)$, and $x \in S_1 \cap A_j$, and $y \in T_j \cap B_{k-1}$. According to the inductive assumption,

$$(5)\qquad fh(S_1\times\{y\} \cup \{x\}\times T_1) = (f_1\times f_2)(S_1\times\{y\} \cup \{x\}\times T_1).$$

Since by 0.5, $fh|S_1\times T_1$ is an isometric embedding by (5), applying Lemma 1.1(ii) we obtain $fh|S_1\times T_1 = (f_1\times f_2)|S_1\times T_1$. This completes the proof of (3). Since $A \in \Sigma(E_1)$ and $B \in \Sigma(E_2)$, by (3) and Lemma 3.2.1 we obtain (1). By (1) and 1.6 we infer that f_1 and f_2 are intrinsic embeddings. Hence there are isometries $g_1 : A \twoheadrightarrow f_1(A)$ and $g_2 : B \twoheadrightarrow f_2(B)$. By (1),

$$(g_1\times g_2)(A\times B) = f_1(A)\times f_2(B) = (f_1\times f_2)(A\times B) = fh(A\times B)$$

thus $f^{-1}(g_1\times g_2)(A\times B) = h(A\times B)$. Hence $A\times B$ is weakly rigid. ■

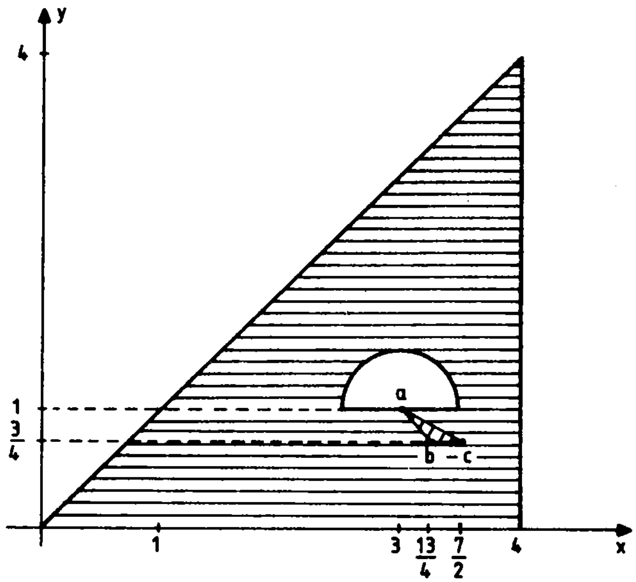
By Theorem 3.2.7 and Lemma 3.2.5 we obtain

3.2.8. C o r o l l a r y . Let $A \in \mathcal{L}(E_1)$ and $B \in \mathcal{L}(E_2)$. If A and B are weakly rigid in E_1 and E_2 respectively then $A\times B$ is weakly rigid in $E_1\times E_2$.

3.3. Cones over rigid sets

Let now $E_1 \in \mathcal{E}^n$ and $E_2 \in \mathcal{E}^1$. The Corollary 3.2.4 says, in particular, that a cylinder in $E_1\times E_2$ over a rigid set in E_1 is rigid in $E_1\times E_2$. We may ask if the similar implication holds for cones. The following example shows that without additional assumptions the answer to this question is negative.

3.3.1. E x a m p l e . Let $T = \Delta((0,0),(4,0),(4,4))$, $T_0 = \Delta(a,b,c)$ where $a = (3,1)$, $b = \left(\frac{13}{4}, \frac{3}{4}\right)$, $c = \left(\frac{7}{8}, \frac{3}{4}\right)$, and let $P = \left\{(x,y) \in R^2 : (x-3)^2 + (y-1)^2 < \frac{1}{4} \wedge y > 1\right\}$. Let us define $A_1 = T - P - T_0 \cup \{a\}$, $A_2 = \operatorname{Int} T_0 \cup \{a\}$, and $A = A_1 \cup A_2$ (fig.5).



$A = A_1 \cup A_2$

Fig.5

It can be easily shown that A is rigid in R^2. Let $d_1 = (0,4,3\sqrt{2})$. We shall prove that $C(d_1, A\times\{0\})$ is neither rigid nor weakly rigid in R^3 (fig.6). Let $g:\{(a,0),(b,0),(c,0),d\} \to \{a_1,b_1,c_1,d_1\}$ be defined as follows:

$$g(a,0) = g(3,1,0) = (3,1,0) = a_1,$$

$$g(b,0) = g\left(\frac{13}{4}, \frac{3}{4}, 0\right) = (3, 1, -\frac{1}{4}\sqrt{2}) = b_1,$$

$$g(c,0) = g(\frac{7}{2}, \frac{3}{4}, 0) = (\frac{25}{8}, \frac{9}{8}, -\frac{3\sqrt{2}}{8}) = c_1,$$

$$g(d_1) = d_1.$$

Simple calculations show that g is an isometry. Let $\bar{g} : R^3 \to R^3$ be an isometric extension of g. Let us define $h : C(d_1, A\times\{0\}) \to R^3$ by the formula

$$h(x) = \begin{cases} x & \text{for } x \in C(d_1, A_1\times\{0\}), \\ \bar{g}(x) & \text{for } x \in C(d_1, A_2\times\{0\}). \end{cases}$$

We shall show that

(1) h is an intrinsic embedding.

Let $C_1 = C(d_1, A_1 \times \{0\})$ and $C_2 = C(d_1, A_2 \times \{0\})$. Since $h|C_1$ and $h|C_2$ are isometries, and $h|C_1 \cap C_2 = id_{C_1 \cap C_2}$, to prove (1) it suffices to show that $h(C_1) \cap h(C_2) = C_1 \cap C_2$. Let $a_2 = a_1$ and let $b_2 \in (b_1 \vee d_1) \cap (R^2 \times \{0\})$, and $c_2 \in (c_1 \vee d_1) \cap (R^2 \times \{0\})$. It can be checked that

(2) $$b_2, c_2 \in P \times \{0\}.$$

Let $T_i = \Delta(a_i, b_i, c_i)$, for $i = 1,2$. By (2), $C_1 \cap C(d_1, T_2) = \Delta(a_1, d_1)$. Since $C_1 = h(C_1)$, we obtain

(3) $$h(C_1) \cap C(d_1, T_2) = \Delta(a_1, d_1).$$

Since

$$h(C_2) \subset C(d_1, T_1) = C(d_1, T_2) \cup \{(x,y,z) \in C(d_1, T_1) : z < 0\}$$

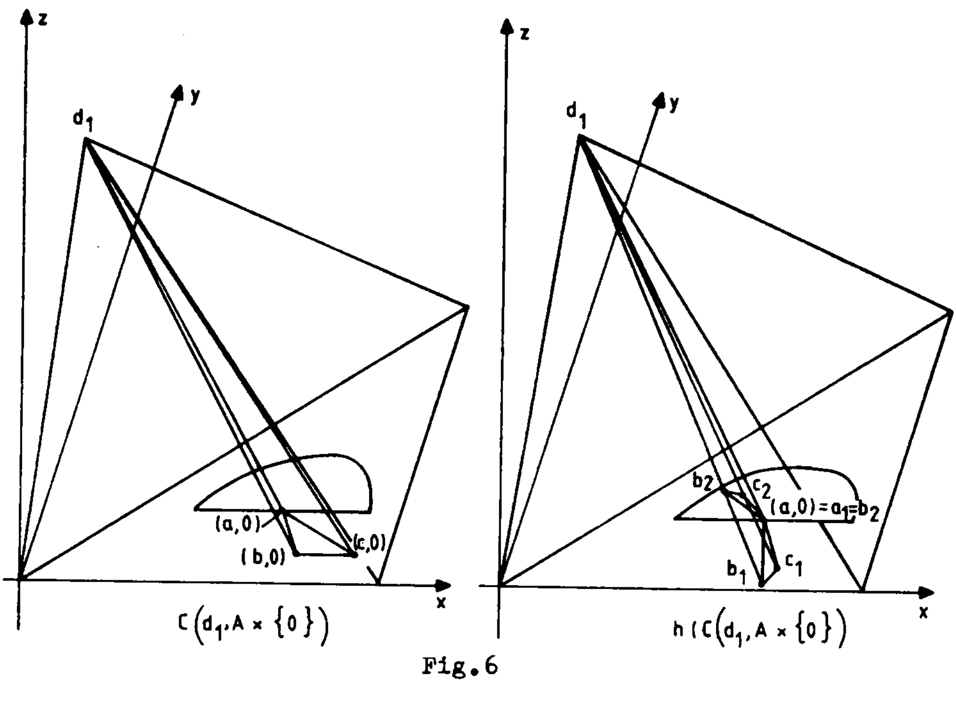


Fig.6

and $h(C_1) \subset \{(x,y,z) \in R^3 : z \geq 0\}$ by (3), $h(C_1) \cap h(C_2) \subset \Delta(a_1,d_1)$. It is evident that $\Delta(a_1,d_1) \subset h(C_1) \cap h(C_2)$, whence $h(C_1) \cap h(C_2) = \Delta(a_1,d_1) = C_1 \cap C_2$, which proves (1). Hence h is an intrinsic embedding. However h is not an isometric embedding. Moreover there is no isometry of $C(d_1, A\times\{0\})$ onto $h(C(d_1, A\times\{0\}))$.

We shall show that under additional assumptions a cone over a rigid set is rigid.

3.3.2. Let $E \in \mathcal{E}^2$ and let $L \subset E$ be an Euclidean line. Let $c_0 \in E-L$, $A \in GA$, $A \subset L$ and $\bar{\bar{A}} > 1$. Then $C(c_0,A)$ is rigid in E.

P r o o f . Since A is a connected subset of L and $\bar{\bar{A}} > 1$, the cone $C(c_0,A)$ satisfies the assumptions of 0.5, whence it is rigid in E. ∎

Let us define some class of intrinsic embeddings of a cone. Let $E \in \mathcal{E}^{n+1}$ and let H be a hyperplane in E; let $c_0 \in E-H$, and $A \subset H$. Let

$$\mathcal{H}(c_0,A) = \{h : C(c_0,A) \to E : h \text{ is an intrinsic embedding and } \dim \operatorname{Af} h(A) \leq n\}.$$

3.3.3. T h e o r e m . Let $A \in \Sigma(H)$. If A is rigid in H then $C(c_0,A)$ is rigid in E with respect to $\mathcal{H}(c_0,A)$.

P r o o f . Let A be rigid in H and $h \in \mathcal{H}(c_0,A)$. Then there is a hyperplane H_0 in E such that $h(A) \subset H_0$. Take an isometry $f : E \to E$ with $f(H_0) = H$. Since A is rigid in H, $fh|A$ is an isometric embedding. Hence

(1) $h|A$ is an isometric embedding.

Since $A \in \Sigma(H)$, we have

$$C(c_0,A) = C(c_0, \cup\Omega(A)) = \cup \{C(c_0,S) : S \in \Omega(A)\}.$$

Take $x_1,x_2 \in C(c_0,A) - \{c_0\}$. Let $S_i \in \Omega(A)$ satisfy the condition $x_i \in C(c_0,S_i)$ for $i = 1,2$. Let $x_i' \in (x_i \vee c_0) \cap H_0$ for $i = 1,2$. By (1) we obtain

(2) $\varrho(h(x_1'),h(x_2')) = \varrho(x_1',x_2')$.

According to 0.5 the cone $C(c_o,S_i)$ is rigid in E, whence $h|C(c_o,S_i)$ is an isometric embedding for i = 1,2. Therefore

(3) $$\rho(h(c_o),h(x_i')) = \rho(c_o,x_i') \quad \text{for } i = 1,2,$$

(4) $$\rho(h(c_o),h(x_i)) = \rho(c_o,x_i) \quad \text{for } i = 1,2.$$

By (2)-(4) we obtain $\rho(h(x_1),h(x_2)) = \rho(x_1,x_2)$. This completes the proof. ∎

Let us now define a class $\mathcal{J}(E)$ of cones as follows

$$C(c_o,A) \in \mathcal{J}(E) :\Longleftrightarrow \forall S_i,S_j \in$$

$$\in \Omega_H(A)(S_i \cap S_j \neq \emptyset \Rightarrow \pi_H(c_o) \in Af(S_i \cap S_j)).$$

3.3.4. Theorem. If $A \in \Sigma(H)$ and $C(c_o,A) \in \mathcal{J}(E)$ then every intrinsic embedding $h : C(c_o,A) \to E$ belongs to $\mathcal{H}(c_o,A)$.

Proof. We may assume that $A \neq \emptyset$. Let $h:C(c_o,A) \to E$ be an intrinsic embedding. Take $S_o \in \Omega_H(A)$. According to 0.5 the cone $C(c_o,S_o)$ is rigid in E. Thus, $h|C(c_o,S)$ is an intrinsic embedding. Let H_o be a hyperplane in E such that

(1) $$h(S_o) \subset H_o.$$

We shall prove by induction that for $j \in N$

(2) $$h(\cup \Omega_j(S_o)) \subset H_o.$$

By (1), the condition (2) holds for j = 1. Suppose that it holds for some $k \in N$ and let $j = k+1$. Given $S_1 \in \Omega_{k+1}(S_o)$ and $S_2 \in \Omega_k(S_o)$ such that $S_1 \cap S_2 \neq \emptyset$, consider $C(c_o,S_i)$ for i = 1,2. By 0.5 the cone $C(c_o,S_i)$ is rigid in E, thus $h|C(c_o,S_i)$ is an isometric embedding for i = 1,2. Let $g_i : E \to E$ be an isometric extension of $h|C(c_o,S_i)$, for i = 1,2. It is evident that

(3) $$g_1|Af(S_1 \cap S_2) = g_2|Af(S_1 \cap S_2)$$

and

(4) $$g_1(c_0) = g_2(c_0) = h(c_0).$$

By the inductive assumption, $g_2(S_2) \subset H_0$. Let H_1 be a hyperplane in E such that $g_1(S_1) \subset H_1$. Let $c_0' := \pi_H(c_0)$. The vector $\overrightarrow{c_0, c_0'}$ is orthogonal to H, thus $\overrightarrow{g_1(c_0), g_1(c_0')}$ and $\overrightarrow{g_2(c_0), g_2(c_0')}$ are orthogonal to H_1 and H_2 respectively. By the assumption $c_0' \in Af(S_1 \cap S_2)$, thus by (3) and (4) we conclude that $\overrightarrow{g_1(c_0), g_1(c_0')} = \overrightarrow{g_2(c_0), g_2(c_0')}$. Hence $H_0 = H_1$. Therefore (2) holds for $j = k+1$. Since $A \in \Sigma(H)$, by Lemma 3.2.1, we obtain $h(A) \subset H_0$. ■

As a consequence of Theorems 3.3.3 and 3.3.4 we obtain

3.3.5. Corollary. Let $A \in \Sigma(H)$ and $C(c_0, A) \in \mathcal{J}(E)$. If A is rigid in H then $C(c_0, A)$ is rigid in E. Finally, by Corollary 3.3.5 and Lemma 3.2.5 we obtain

3.3.6. Corollary. Let $A \in \mathcal{L}(H)$ and $C(c_0, A) \in \mathcal{J}(E)$. If A is rigid in H then $C(c_0, A)$ is rigid in E.

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