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SOME PROPERTIES OF TOPOLOGICAL σ-IDEALS

This note is concerned with σ-ideals in a measurable space $(X, \mathcal{S})$, such that convergence with respect to a σ-ideal $\mathcal{J}$ induces the Fréchet topology on the space of all $\mathcal{S} \vee \mathcal{J}$-measurable real functions, where $\mathcal{S} \vee \mathcal{J}$ is the smallest σ-field containing both $\mathcal{S}$ and $\mathcal{J}$. These σ-ideals are called $\mathcal{S}$-topological.

Suppose that we are given a nonempty set X. In all that follows below, we consider σ-fields and σ-ideals of subsets of X.

To begin with, let us recall two definitions.

D e f i n i t i o n 1. (cf. [4]). Let $\mathcal{J}$ be a σ-ideal. We say that a property holds $\mathcal{J}$-almost everywhere on X (abbr. $\mathcal{J}$ - a.e.) if the set of all points which do not have this property belongs to $\mathcal{J}$.

D e f i n i t i o n 2. (cf. [4]). We say that a sequence $\{f_n\}_{n\in N}$ of real functions defined on X converges with respect to a σ-ideal $\mathcal{J}$ to a function f (abbr. $\{f_n\} \xrightarrow[n\to\infty]{} f$) if each subsequence of $\{f_n\}_{n\in N}$ contains a subsequence converging $\mathcal{J}$-a.e. on X to f.

For a σ-field $\mathcal{S}$ and a σ-ideal $\mathcal{J}$, denote by $\mathcal{S} \vee \mathcal{J}$ the smallest σ-field containing both $\mathcal{S}$ and $\mathcal{J}$. One can easy check that is the collection of all sets of the form $(A-B) \cup C$ where $A \in \mathcal{S}$ and $B, C \in \mathcal{J}$; moreover, $\mathcal{S} \vee \mathcal{J} = \mathcal{S}$ if and only if $\mathcal{J} \subset \mathcal{S}$.

Without difficulty we can check that, for every σ-field $\mathcal{S}$ and any σ-ideal $\mathcal{J}$, the set $\mathcal{L}^*(\mathcal{S},\mathcal{J})$ of all $\mathcal{S} \vee \mathcal{J}$-measurable real functions defined on X, equipped with convergence with respect to $\mathcal{J}$, is an $\mathcal{L}^*$-space (cf. [1; Problem Q, p.89]). Hence it is possible to define the closure operator on $\mathcal{L}^*(\mathcal{S},\mathcal{J})$ by letting $f \in \bar{A}$ if and only if A contains a sequence converging with respect to $\mathcal{J}$ to the function f (cf. [4]). This closure operator has the properties: $\bar{\emptyset} = \emptyset$, $A \subset \bar{A}$ and $\overline{A \cup B} = \bar{A} \cup \bar{B}$ for any sets $A,B \subset \mathcal{L}^*(\mathcal{S},\mathcal{J})$; however, the equality $\bar{\bar{A}} = \bar{A}$ holds for each $A \subset \mathcal{L}^*(\mathcal{S},\mathcal{J})$ if and only if the following condition, usually labelled by (L4), is satisfied:

(L4) If $f_{j,n} \xrightarrow[n\to\infty]{\mathcal{J}} f_j$ and $f_j \xrightarrow[j\to\infty]{\mathcal{J}} f$ for each $j \in N$, then there exist sequences $\{j_p\}_{p\in N}$ and $\{n_p\}_{p\in N}$ of positive integers such that $f_{j_p,n_p} \xrightarrow[p\to\infty]{\mathcal{J}} f$.

D e f i n i t i o n 3. (cf. [4] and [5]). Let $\mathcal{S}$ be a σ-field. We say that a σ-ideal $\mathcal{J}$ is $\mathcal{S}$-topological if the space $\mathcal{L}^*(\mathcal{S},\mathcal{J})$ fulfils (L4).

The above definition characterizes $\mathcal{S}$-topological σ-ideals by their relations to space of $\mathcal{S}$-measurable functions. In order to give some characterization of $\mathcal{S}$-topological σ-ideals $\mathcal{J}$ in terms of members of $\mathcal{S}$ and $\mathcal{J}$, we shall now introduce condition (E″). We concentrate on this condition, but some similar conditions (E), (E′) also exist (cf. [4] and [5]).

D e f i n i t i o n 4. Let $\mathcal{S}$ be a σ-field and $\mathcal{J}$ a σ-ideal. We say that the pair $(\mathcal{S},\mathcal{J})$ fulfils condition (E″) if, for each set $D \in \mathcal{S} - \mathcal{J}$ and for each double sequence $\{B_{j,n}\}_{j,n\in N}$ of members of $\mathcal{S}$ such that

(1) $$B_{j,n} \subset B_{j,n+1} \quad \text{for any } j,n \in N,$$

(2) $$\bigcup_{n=1}^{\infty} B_{j,n} = D \quad \text{for each } j \in N,$$

there exists a positive integer-valued function $n(i,j)$ of positive integers i and j such that $\left(D - \bigcup_{i=1}^{\infty} \bigcap_{j=1}^{\infty} B_{j,n(i,j)}\right) \in \mathcal{J}$.

Studying condition (I) of [5; p. 99] formulated in the language of Boolean algebras, one easily observes that, in the case where $\mathcal{J} \subset \mathcal{S}$, our condition (E″) is equivalent to (I) of [4].

T h e o r e m 1. For any σ-ideal $\mathcal{J}$ and any σ-field $\mathcal{S}$, the following conditions are equivalent:

(i) the σ-ideal $\mathcal{J}$ is $\mathcal{S}$-topological;

(ii) the pair $(\mathcal{S}\triangle\mathcal{J};\ \mathcal{J})$ fulfils (E″);

(iii) the pair $(\mathcal{S};\ \mathcal{J})$ fulfils (E″).

P r o o f . Since $\mathcal{J} \subset \mathcal{S}\triangle\mathcal{J}$, the equivalence (i) $\Longleftrightarrow$ (ii) is an immediate consequence of Theorem 1 of [5]. The implication (ii) $\Longrightarrow$ (iii) follows from the inclusion $\mathcal{S} \subset \mathcal{S}\triangle\mathcal{J}$. Hence, the proof will be completed if we show that (iii) $\Longrightarrow$ (ii).

Let us consider any sets $D \in (\mathcal{S}\triangle\mathcal{J}) - \mathcal{J}$ and $B_{j,n} \in \mathcal{S}\triangle\mathcal{J}$ $(j,n \in N)$ which satisfy conditions (1) and (2) of definition 4. There exist sets $A_{j,n} \in \mathcal{S}$, $C_{j,n} \in \mathcal{J}$ and $E_{j,n} \in \mathcal{J}$ such that $B_{j,n} = (A_{j,n} - C_{j,n}) \cup E_{j,n}$ where $j,n \in N$. Denote $D^* = \bigcap_{j=1}^{\infty} \bigcup_{n=1}^{\infty} A_{j,n}$ and $B^*_{j,n} = \bigcup_{k=1}^{n} A_{j,k} \cap D^*$ for $j,n \in N$. Of course, $D^* \in \mathcal{S}$ and $B^*_{j,n} \in \mathcal{S}$ for $j,n \in N$. Moreover, $B^*_{j,n} \subset B^*_{j,n+1}$ and $D^* = \bigcup_{n=1}^{\infty} B^*_{j,n}$ whenever $j,n \in N$. As $D = \bigcap_{j=1}^{\infty} \bigcup_{n=1}^{\infty} B_{j,n}$, then $D - D^* =$

$$= \bigcap_{j=1}^{\infty} \left[\bigcup_{n=1}^{\infty} (A_{j,n} - C_{j,n}) \cup \bigcup_{n=1}^{\infty} E_{j,n} \right] - \bigcap_{j=1}^{\infty} \bigcup_{n=1}^{\infty} A_{j,n} \subset$$

$$\subset \left[\left(\bigcap_{j=1}^{\infty} \bigcup_{n=1}^{\infty} A_{j,n} \right) \cup \left(\bigcup_{j=1}^{\infty} \bigcup_{n=1}^{\infty} E_{j,n} \right) \right] - \bigcap_{j=1}^{\infty} \bigcup_{n=1}^{\infty} A_{j,n} \subset \bigcup_{j=1}^{\infty} \bigcup_{n=1}^{\infty} E_{j,n}.$$

Since $\left(\bigcup_{j=1}^{\infty} \bigcup_{n=1}^{\infty} E_{j,n} \right) \in \mathcal{J}$, we have that $D - D^* \in \mathcal{J}$. This implies that $D^* \notin \mathcal{J}$, because $D \notin \mathcal{J}$ and $D = (D \cap D^*) \cup (D \setminus D^*)$. If the pair $(\mathcal{S}, \mathcal{J})$ fulfils (E″), then there exists a positive integer-valued function $n(i,j)$ such that $\left(D^* - \bigcup_{i=1}^{\infty} \bigcap_{j=1}^{\infty} B^*_{j,n(i,j)} \right) \in \mathcal{J}$.

Denote $B = \bigcup_{i=1}^{\infty} \bigcap_{j=1}^{\infty} B_{j,n(i,j)}$, $B^* = \bigcup_{i=1}^{\infty} \bigcap_{j=1}^{\infty} B^*_{j,n(i,j)}$ and

$C = \bigcup_{i=1}^{\infty} \bigcap_{j=1}^{\infty} \bigcup_{k=1}^{n(i,j)} C_{j,k}$. Since $D-B \subset (D-D^*) \cup (D^*-B)$, $C \in \mathcal{J}$ and $D^*-B^* \in \mathcal{J}$, in order to prove that $D-B \in \mathcal{J}$, it suffices to show that $D^*-B \subset (D^*-B^*) \cup C$.

Let us observe that $B = \bigcup_{i=1}^{\infty} \bigcap_{j=1}^{\infty} \bigcup_{k=1}^{n(i,j)} \left[(A_{j,k} - C_{j,k}) \cup \cup E_{j,k}\right] \supset \bigcup_{i=1}^{\infty} \bigcap_{j=1}^{\infty} \bigcup_{k=1}^{n(i,j)} (A_{j,k} - C_{j,k}) \supset \left(\bigcup_{i=1}^{\infty} \bigcap_{j=1}^{\infty} \bigcup_{k=1}^{n(i,j)} A_{j,k}\right) - C$; this yields that $D^*-B \subset D^* - \left[\bigcup_{i=1}^{\infty} \bigcap_{j=1}^{\infty} \bigcup_{k=1}^{n(i,j)} A_{j,k} - C\right] \subset \subset (D^*-B^*) \cup C$. Hence, $D-B \in \mathcal{J}$ and (iii) $\Rightarrow$ (ii).

It is worth mentioning that, in the case where a σ-ideal $\mathcal{J}$ is contained in a σ-field $\mathcal{S}$, there are weaker conditions (E) or (E′) which, together with the countable chain condition for the pair $(\mathcal{S},\mathcal{J})$, are equivalent for $\mathcal{J}$ to be $\mathcal{S}$-topological (cf. [3]).

T h e o r e m 2. Let $\mathcal{S}$ be a σ-field. If a σ-ideal $\mathcal{J}_1$ is $\mathcal{S}$-topological, then any σ-ideal $\mathcal{J}_2 \supset \mathcal{J}_1$ is $\mathcal{S}$-topological.

P r o o f . If $\mathcal{J}_1$ is an $\mathcal{S}$-topological σ-ideal, then, by theorem 1, the pair $(\mathcal{S}, \mathcal{J}_1)$ fulfils (E″). This implies that every pair $(\mathcal{S}, \mathcal{J}_2)$, where $\mathcal{J}_2$ is a σ-ideal containing $\mathcal{J}_1$, fulfils (E″). Hence, the proposition follows from Theorem 1.

One can easily prove the following

L e m m a . Let $\mathcal{S}$ be a σ-field and $\mathcal{J}$ - a σ-ideal. The pair $(\mathcal{S},\mathcal{J})$ fulfils condition (E″) if and only if, for an arbitrary double sequence $\{B_{j,n}\}_{j,n\in N}$ of $\mathcal{S}$-measurable sets such that

(1) $$B_{j,n} \subset B_{j,n+1} \quad \text{for any} \quad j,n \in N,$$

(2) $$\bigcup_{n=1}^{\infty} B_{j,n} = X \quad \text{for each } j \in N,$$

there exists a positive integer-valued function $n(i,j)$ such that $\left(X - \bigcup_{i=1}^{\infty} \bigcap_{j=1}^{\infty} B_{j,n(i,j)}\right) \in \mathcal{J}$.

T h e o r e m 3. Suppose that $\mathcal{S}$ is a σ-field and $\mathcal{J}_1$, $\mathcal{J}_2$ are σ-ideals. Then the σ-ideal $\mathcal{J}_1 \cap \mathcal{J}_2$ is $\mathcal{S}$-topological if and only if both $\mathcal{J}_1$ and $\mathcal{J}_2$ are $\mathcal{S}$-topological.

P r o o f . By virtue of Theorem 2, it suffices to assume that both $\mathcal{J}_1$ and $\mathcal{J}_2$ are $\mathcal{S}$-topological σ-ideals, and to show that the σ-ideal $\mathcal{J}_1 \cap \mathcal{J}_2$ is $\mathcal{S}$-topological.

Let $\{B_{j,n}\}_{j,n\in N}$ be an arbitrary double sequence of $\mathcal{S}$-measurable sets which satisfies conditions (1) and (2) of our lemma. According to Theorem 1 and the lemma, there exist positive integer-valued functions $n_1(i,j)$ and $n_2(i,j)$ such that $\left(X - \bigcup_{i=1}^{\infty} \bigcap_{j=1}^{\infty} B_{j,n_k(i,j)}\right) \in \mathcal{J}_k$ for $k = 1,2$. Let $n(i,j) = \max(n_1(i,j); n_2(i,j))$ for any $i,j \in N$. As $B_{j,n} \subset B_{j,n+1}$ for any $j,n \in N$, then $X - \bigcup_{i=1}^{\infty} \bigcap_{j=1}^{\infty} B_{j,n(i,j)} \in \mathcal{J}_1 \cap \mathcal{J}_2$. Thus, it follows from our lemma that the pair $(\mathcal{J}_1 \cap \mathcal{J}_2; \mathcal{S})$ fulfils condition (E''). Using Theorem 1, we conclude that the σ-ideal $\mathcal{J}_1 \cap \mathcal{J}_2$ is $\mathcal{S}$-topological.

C o r o l l a r y . Let $\mathcal{S}$ be a σ-field. If, for a σ-ideal $\mathcal{J}_1$, there exists an $\mathcal{S}$-topological σ-ideal $\mathcal{J}_2$ such that the σ-ideal $\mathcal{J}_1 \cap \mathcal{J}_2$ is not $\mathcal{S}$-topological, then the σ-ideal $\mathcal{J}_1$ is not $\mathcal{S}$-topological.

In view of Theorems 2 and 3, a σ-ideal is $\mathcal{S}$-topological if and only if it contains the common part of two $\mathcal{S}$-topological σ-ideals.

Let us note still another consequence of the lemma and Theorem 1.

T h e o r e m 4. Let $\mathcal{J}$ be a σ-field. If a σ-ideal $\mathcal{J}_1$ is $\mathcal{S}$-topological, then every σ-ideal $\mathcal{J}_2$, for which $\mathcal{J}_1 \cap \mathcal{S} = \mathcal{J}_2 \cap \mathcal{S}$, is $\mathcal{S}$-topological, too.

For any σ-ideal $\mathcal{J}$ and any σ-field $\mathcal{S}$, let us denote

$$\mathcal{J}[\mathcal{S}] = \{A \subset X: \text{ there is } B \in \mathcal{S} \subset \mathcal{J} \text{ such that } A \subset B\}.$$

It is readily seen that the family $\mathcal{J}[\mathcal{S}]$ is a σ-ideal for which $\mathcal{J} \cap \mathcal{S} = \mathcal{J}[\mathcal{S}] \cap \mathcal{S}$; moreover, $\mathcal{J}[\mathcal{S}]$ is contained in every σ-ideal $\mathcal{J}^*$ such that $\mathcal{J}^* \cap \mathcal{S} = \mathcal{J} \cap \mathcal{S}$.

Theorems 2 and 4 imply

T h e o r e m 5. Let $\mathcal{S}$ be a σ-field. Then a σ-ideal $\mathcal{J}$ is $\mathcal{S}$-topological if and only if the σ-ideal $\mathcal{J}[\mathcal{S}]$ is $\mathcal{S}$-topological.

Denote by $\mathcal{B}$ the σ-field of all Borel subsets of the real line R.

The last theorem leads to the notion of Borel σ-ideals.

D e f i n i t i o n 5. (cf. [1]). We say that a σ-ideal $\mathcal{J}$ of subsets of R is Borel if $\mathcal{J}[\mathcal{B}] = \mathcal{J}$.

E x a m p l e 1. It is well known and easy to observe that the Borel σ-ideal $\mathcal{L}$ of all subsets of R of Lebesgue measure zero is $\mathcal{B}$-topological. Let $\mathcal{J}_0$ denote the σ-ideal of all subsets of R having the property (s_0) (cf. [2]). Then the smallest σ-ideal $\mathcal{J}_1$, containing both $\mathcal{J}_0$ and $\mathcal{L}$, is $\mathcal{B}$-topological. At the same time, there exists a nonmeasurable set having the property (s_0) (cf. [2]), hence $\mathcal{J}_1 \supsetneq \mathcal{L}$. But the σ-ideal $\mathcal{J}_1$ is not Borel because $\mathcal{J}_1[\mathcal{B}] = \mathcal{L}$. In connection with this example, one can ask the following questions:

<u>Question 1.</u> Does there exist a proper $\mathcal{B}$-topological Borel σ-ideal $\mathcal{J}$ of subsets of R such that $\mathcal{J} \supsetneq \mathcal{L}$?

<u>Question 2.</u> Does there exist a $\mathcal{B}$-topological Borel σ-ideal $\mathcal{J}$ of subsets of R such that $\mathcal{J} \subsetneq \mathcal{L}$?

In the case of the second question, it would be sufficient to find some $\mathcal{B}$-topological σ-ideal $\mathcal{J}$ such that $\mathcal{J} \subsetneq \mathcal{L}$, because the σ-ideal $\mathcal{J}[\mathcal{B}]$ would be Borel.

E x a m p l e 2. The σ-ideal $\mathcal{K}$ of all sets of the first category in R is not $\mathcal{B}$-topological (cf. [4]). Let $\mathcal{J}_p$ be the σ-ideal of porous sets in R (cf. [6]). The σ-ideal $\mathcal{J}_p$ is Borel and, obviously, $\mathcal{J}_p \subset \mathcal{K}$. Hence, by Theorem 2, $\mathcal{J}_p$ is not $\mathcal{B}$-topological. We can formulate the following problem:

<u>Question 3.</u> Does there exist a non-$\mathcal{B}$-topological Borel σ-ideal $\mathcal{J}$ of subsets of R such that $\mathcal{J} \supsetneq \mathcal{K}$?

The answers to questions 1-3 are not known to the author.

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