DEMONSTRATIO MATHEMATICA
Vol. XXII No 4 1989

Marek Balcerzak

ON σ-IDEALS HAVING PERFECT MEMBERS IN ALL PERFECT SETS

0. Preliminaries

Let X be a perfect Polish space. We consider σ-ideals $\mathcal{J}$ of subsets of X such that each perfect subset of X contains a perfect set belonging to $\mathcal{J}$. We find new examples of these σ-ideals and apply to them the Sierpiński-Erdös duality principle by assuming CH.

We use the standard set-theoretical notation (cf. e.g. [Kn]). We identify an ordinal with the set of its predecessors. Thus $\omega = \{0,1,2,...\}$ and $n+1 = \{0,1,...,n\}$ for $n \in \omega$. We denote by ${}^{B}A$ the set of all functions from B into A.

In general, we consider proper σ-ideals in the power set $\mathcal{P}(X)$ where X is an uncountable set. For any sets X and Y, a set $E \subset X \times Y$ and an element x of X, we denote

$$E_x = \{y \in Y : \langle x,y \rangle \in E\};$$

this set is called the X-section of E generated by x. If X,Y are uncountable and $\mathcal{J} \subset \mathcal{P}(X)$, $\mathcal{j} \subset \mathcal{P}(Y)$ are σ-ideals, then the family

$$\mathcal{J} \times \mathcal{j} = \{E \subset X \times Y : \{x \in X : E_x \notin \mathcal{j}\} \in \mathcal{J}\}$$

forms a σ-ideal in $\mathcal{P}(X \times Y)$ called the product of the σ ideals $\mathcal{J}$ and $\mathcal{j}$ (cf. [CP]).

We mostly assume that the sets X and Y are perfect Polish spaces. The terminology and notation concerning descriptive set theory are derived from [Mo]. We use the Borel pointclasses

Σ^0_α, Π^0_α for $0<\alpha<\omega_1$ and the pointclass $\mathcal{B}$ of all Borel sets; if they are taken in a space X, we shall write $\Sigma^0_\alpha(X)$, $\Pi^0_\alpha(X)$ and $\mathcal{B}(X)$. Analogous remarks should be repeated for projective pointclasses. If X is a perfect Polish space, $\mathcal{J}\subset\mathcal{P}(X)$ is a σ-ideal and $\mathcal{S}$ is a pointclass, we denote

$$\mathcal{J}\,|\,\mathcal{S}=\{E\subset X : E\subset D \text{ for some } D\in\mathcal{S},\ D\subset X\}.$$

The final results of the paper concern the three σ-ideals: $\mathcal{K}$ of all meager sets, $\mathcal{L}$ of all Lebesgue null sets and the σ-ideal $\mathcal{M}$ of Mycielski, described in [My 2]. For simplicity, all of them will be considered in the Cantor set C where $\mathcal{M}$ was originally defined. Here C is treated as ${}^{\omega}2$ with the Tychonov topology generated by the discrete topology in $2=\{0,1\}$. Note that C can be metrized by $d(x,y)=\sum_{i=0}^{\infty}|x_i-y_i|/2^i$ for $x=\langle x_0,x_1,\dots\rangle$, $y=\langle y_0,y_1,\dots\rangle$. In C one may introduce a group structure where the group operation is coordinatewise addition modulo 2. Let μ denote the product measure in C generated by the measure ν on $\{0,1\}$ such that $\nu(\{0\})=\nu(\{1\})=1/2$. Let $h : C\to[0,1]$ be given by $h(x)=\sum_{i=0}^{\infty}x_i/2^{i+1}$ for $x=\langle x_0,x_1,\dots\rangle\in C$. Note that the Lebesgue measure of $A\subset[0,1]$ is zero if and only if $\mu(h^{-1}(A))=0$. Similarly, A is meager in $[0,1]$ if and only if $h^{-1}(A)$ is meager in C. Thus we may (without loss of generality) consider meager sets in C instead of meager sets in $[0,1]$ and μ-null sets instead of Lebesgue null sets. The families of all meager sets in C and of all μ-null sets will be denoted by $\mathcal{K}$ and $\mathcal{L}$, respectively. Let $\mathcal{L}_1=\mathcal{L}|\Sigma^0_2$, i.e. $\mathcal{L}_1$ consists of all subsets of F_σ sets belonging to $\mathcal{L}$. Then $\mathcal{L}_1$ forms a σ-ideal properly included in $\mathcal{K}\cap\mathcal{L}$ (cf. [B 1]).

Recall some definitions from [My 2]. For given sets $E\subset C$ and $K\subset\omega$, we define a positional game (E,K) with perfect information between two players: I and II. They choose consecutive terms of a sequence $\langle x_0,x_1,\dots\rangle\in C$. The value of x_i is chosen by Player I if $i\notin K$, and by Player II otherwise. Each player choosing x_i knows E,K and $x_0,x_1,\dots,x_{i-1}$. Player I wins if $\langle x_0,x_1,\dots,\rangle \in E$ and Player II wins in the opposite case. Let $V_{II}(K)$ denote the family of sets $E \subset C$ for which Player II has a winning strategy in the game $\Gamma(E,K)$ (for the notions concerning game theory, we refer the reader to [My 1] or [Mo]). Let Sq denote the set $\bigcup_{n\in\omega} {}^n2$ of all finite sequences of zeros and ones (where ^{02} consists of the empty sequence written as $\langle\rangle$). For s and t from Sq, the condition $s \subset t$ means that the sequence t extends s. We say that $\mathcal{G} = \{K_s : s \in Sq\}$ is a generating family if the following conditions hold:

(a) the sets K_s are infinite subsets of ω,

(b) $K_s \subset K_t$ whenever $t \subset s$,

(c) K_s and K_t are disjoint whenever s and t are different of the same length.

Assume that $\mathcal{G}$ is a fixed generating family and put

$$\mathcal{M} = \bigcap \{V_{II}(K_s) : K_s \in \mathcal{G}\}.$$

Then $\mathcal{M}$ is a σ-ideal in $\mathcal{P}(C)$ invariant with respect to the group operation. We call it the -ideal of Mycielski, generated by $\mathcal{G}$.

1. The property (P)

For a σ-ideal $\mathcal{I} \subset \mathcal{P}(X)$ where X is a perfect Polish space, consider the following property:

(P) each perfect subset of X contains a perfect set belonging to $\mathcal{I}$.

(We mean a perfect set as a closed set without isolated points). Observe that if $\mathcal{I}$ and $\mathcal{J}$ are σ-ideals such that $\mathcal{I} \subset \mathcal{J} \subset \mathcal{P}(X)$ and $\mathcal{I}$ fulfils (P), then $\mathcal{J}$ fulfils (P). It is well known that the σ-ideal of meager sets and the σ-ideal of Lebesgue null sets on the real line (and, similarly, $\mathcal{K}$ and $\mathcal{L}$ in C) have the property (P) (cf. [O], Lemma 5.1). On the other hand, there exists a class of σ-ideals on the real line which do not contain any perfect set (cf. [Mi]), thus do not fulfil (P). In this section we show that the σ-ideal of Mycielski fulfils (P).

Thus, consider a fixed generating family

$$\mathcal{G} = \{K_s : s \in Sq\}$$

and the σ-ideal $\mathcal{M}$ of Mycielski, generated by $\mathcal{G}$.

(1.1) Lemma. For each $K \subset \omega$ and each perfect set $P \subset C$, there is a perfect set $Q \subset P$ such that $Q \in V_{II}(K)$.

Proof. If $y|K = z|K$ for all $y,z \in P$, let $Q = P$. The strategy in which Player II chooses $x_i = 1 - y_i$ for $i \in K$, where y is an arbitrary element of P, is winning for him in the game $\Gamma(Q,K)$. If there are $y,z \in P$ such that $y|K \neq z|K$, then observe that the open set $U = \{x \in C : x|K \neq y|K\}$ intersects P and we can therefore find a perfect subset Q of $P \cap U$. The strategy in which Player II chooses $x_i = y_i$ for $i \in K$ is winning for him in the game $\Gamma(Q,K)$.

Further, for any perfect set $P \subset C$ and for $K \subset \omega$, let $\Phi(P,K)$ denote a fixed perfect set Q satisfying the assertion of Lemma 1.1.

From the definition of V_{II} we easily deduce the next two lemmas.

(1.2) Lemma. If $K^{(i)}$ for $i = 1,2,\dots,r$ are disjoint infinite subsets of ω, and $A_i \in V_{II}(K^{(i)})$ for $i=1,2,\dots,r$, then $\bigcup_{i=1}^{r} A_i \in V_{II}\left(\bigcup_{i=1}^{r} K^{(i)}\right)$.

(1.3) Lemma. If $K \subset \omega$ is infinite and $A,B \subset C$ are such that $A \subset B$, then $B \in V_{II}(K)$ implies that $A \in V_{II}(K)$.

(1.4) Lemma. Let $K_1,K_2,\dots,K_n$ be infinite disjoint subsets of ω and let $P_1,P_2,\dots,P_r$ be disjoint perfect subsets of C. Then there are perfect sets $Q_1,Q_2,\dots,Q_r$ contained in $P_1,P_2,\dots,P_r$, respectively, and such that $\bigcup_{i=1}^{r} Q_i \in \bigcup_{j=1}^{n} V_{II}(K_j)$.

Proof. For each $j \in \{1,2,\dots,n\}$, consider any partition $\{K_j^{(i)} : i = 1,2,\dots,r\}$ of K_j into disjoint infinite sets. Fix any $i \in \{1,2,\dots,r\}$ and define inductively

$$Q_i^{(1)} = \Phi(P_i, K_1^{(i)}),$$
$$Q_i^{(2)} = \Phi(Q_i^{(1)}, K_2^{(i)}),$$
$$\dots\dots\dots\dots\dots\dots\dots$$
$$Q_i^{(n)} = \Phi(Q_i^{(n-1)}, K_n^{(i)}).$$

Put $Q_i = Q_i^{(n)}$. By Lemma 1.2, we have

$$\bigcup_{i=1}^{r} Q_i^{(j)} \in V_{II}(K_j) \quad \text{for} \quad j = 1,2,\dots,n.$$

Since $\bigcup_{i=1}^{r} Q_i \subset \bigcup_{i=1}^{r} Q_i^{(j)}$ for $j = 1,2,\dots,n$, therefore, by Lemma 1.3, we have $\bigcup_{i=1}^{r} Q_i \in \bigcap_{j=1}^{n} V_{II}(K_j)$.

R e m a r k . In the same way one can show that the property described in Lemma 1.4 holds for each countable family $P_0, P_1, \dots,$ of disjoint perfect sets, as well.

The next lemma, called the "fusion lemma", is quite well known (cf. e.g. [J], Lemma 93).

(1.5) L e m m a . Let $\{P_s : s \in Sq\}$ be a family of perfect subsets of C, such that

(a) $P_s \subset P_t$ whenever s extends t,

(b) P_s and P_t are disjoint whenever s and t are fifferent of the same length,

(c) the diamenter of P_s tends to 0 if the length of s tends to infinity.

Then the set $P^* = \bigcap_{n\in\omega} \bigcup_{s\in {}^n2} P_s$ is perfect.

(1.6) T h e o r e m . The σ-ideal $\mathcal{M}$ has the property (P).

P r o o f . Let P be a perfect subset of C. We shall define by induction perfect subsets P_s, $s \in Sq$, of P fulfilling the assumptions of Lemma 1.5 and the condition

$$\bigcup_{s\in {}^n2} P_s \in \bigcap_{s\in {}^n2} V_{II}(K_s) \quad \text{for all} \quad n \in \omega .$$

At first, by Lemma 1.1, we choose a perfect set $Q \subset P$ such that $Q \in V_{II}(K_{\langle\rangle})$. Put $P_{\langle\rangle} = Q$. Assume that $n \in \omega$, $n > 0$, and that we have defined the sets P_s for $s \in {}^n2$. By Lemma 1.4, there are perfect sets $Q_s \subset P_s$ for $s \in {}^n2$, such that

$$\bigcup_{s \in {}^n2} Q_s \in \bigcap_{s \in {}^{n+1}2} V_{II}(K_s).$$

In each set Q_s where $s \in {}^n2$, $s = \langle s_0, \ldots, s_{n-1}\rangle$, we find disjoint perfect subsets $P_{\langle s_0,\ldots,s_{n-1},0\rangle}$ and $P_{\langle s_0,\ldots,s_{n-1},1\rangle}$ such that the diameter of each of them does not exceed $(1/2)^{n+1}$. From Lemma 1.3 we easily conclude that

$$\bigcup_{s \in {}^{n+1}2} P_s \in \bigcap_{s \in {}^{n+1}2} V_{II}(K_s).$$

In that way, all sets P_s are defined and they have the required properties. Consider P^* constructed as in Lemma 1.5. It is a perfect subset of P. Let $t \in Sq$ be arbitrary and assume that $t \in {}^n2$. Then $\bigcup_{s \in {}^n2} P_s \in V_{II}(K_t)$ and, by Lemma 1.3, we have $P \in V_{II}(K_t)$. Thus it follows that $P^* \in \mathcal{M}$, which ends the proof.

Finally, we give one more result on the property (P), concerning products of σ-ideals.

(1.7) Theorem. If $\mathcal{I} \subset \mathcal{P}(X)$ and $\mathcal{J} \subset \mathcal{P}(Y)$ are σ-ideals containing all singletons, where X and Y are perfect Polish spaces, then $(\mathcal{I} \times \mathcal{J}) | \mathcal{B}$ has the property (P).

Proof. Let P be a perfect subset of X×Y. If there exists an uncountable section P_x, thus it (being, at the same time, closed) contains a perfect set Q. Then $\{x\} \times Q$ is a perfect subset of P and belongs to $(\mathcal{I} \times \mathcal{B})|\mathcal{B}$. Next, assume that all X-sections of P are countable. Then the projection $pr_X P$ of P onto X must be uncountable. The set P has a Borel uniformization $\hat{P}$ ([Mo], 4F. 17) which is uncountable. Choose a perfect subset of $\hat{P}$ (cf. [Mo], 2C. 3). It obviously belongs to $(\mathcal{I} \times \mathcal{J})|\mathcal{B}$.

2. The duality principle

Let $\mathfrak{J} \subset \mathcal{P}(X)$ be a -ideal where X is a uncountable set. We say that a family $\mathcal{H} \subset \mathfrak{J}$ is cofinal in $\mathfrak{J}$ if each set from $\mathfrak{J}$ can be covered by a set from $\mathcal{H}$. We denote (see [CP])

$$cf(\mathfrak{J}) = \min\{|\mathcal{H}| : \mathcal{H} \subset \mathfrak{J} \text{ and } \mathcal{H} \text{ is cofinal in } \mathfrak{J}\}.$$

We say that σ-ideals $\mathfrak{J}$ and $\mathfrak{j}$ in $\mathcal{P}(X)$ have the decomposition property (abbr. d.p.) if there are two disjoint sets belonging to $\mathfrak{J}$ and $\mathfrak{j}$, respectively, and their union is equal to X (cf. [B 1]).

The following (purely set theoretic) theorem is essential in the proof of the Sierpiński-Erdos theorem.

(2.1) Theorem (see [O], Th. 19.5). Let X be a set of power ω_1 and let $\mathfrak{J} \subset \mathcal{P}(X)$ be a σ-ideal containing all singletons, such that $cf(\mathfrak{J}) \leq \omega_1$. Assume that the complement of each member of $\mathfrak{J}$ contains a set of power ω_1 belonging to $\mathfrak{J}$. Then X can be decomposed into ω_1 disjoint sets, each of power ω_1, such that a subset E of X belongs to $\mathfrak{J}$ if and only if E is contained in a countable union of sets making the above decomposition.

(2.2) Theorem (see [O], Th. 19.6). Let X be a set of power ω_1. Let $\mathfrak{J}$ and $\mathfrak{j}$ be a σ-ideals in $\mathcal{P}(X)$ and let each of them fulfil the assumptions of (2.1). Assume further that $\mathfrak{J}$ and $\mathfrak{j}$ have the d.p. Then there is a one-one mapping f of X onto itself such that $f = f^{-1}$ and such that $f(E) \in \mathfrak{j}$ if and only if $E \in \mathfrak{J}$.

If X is an uncountable set and, for σ-ideals $\mathfrak{J}$ and $\mathfrak{j}$ in $\mathcal{P}(X)$, the assertion of Theorem 2.2 holds, we say that $\mathfrak{J}$ and $\mathfrak{j}$ fulfil the Sierpiński-Erdös duality principle (abbr. SEDP). The duality between the families of meager sets and of Lebesgue null sets on the real line is discussed in detail in [O]. Mendez in [Me] showed that if CH is assumed, the σ-ideals $\mathcal{K} \times \mathcal{L}$ and $\mathcal{L} \times \mathcal{K}$ (on the plane) fulfil SEDP. We shall establish SEDP for some other pairs of σ-ideals by using Theorem 2.2, if CH is assumed.

(2.3) L e m m a . Let X be a perfect Polish space and let $\mathcal{J} \subset \mathcal{P}(X)$ be a σ-ideal fulfilling (P) such that $\mathcal{J} = \mathcal{J}|\Pi_1^1$. Then complement of each member of $\mathcal{J}$ contains a set of power 2^ω belonging to $\mathcal{J}$.

P r o o f . Let $E \in \mathcal{J}$ and let $D \in \Pi_1^1(X)$ be such that $E \subset D \in \mathcal{J}$. Then $X \setminus D$ is an uncountable analytic set, thus contains a perfect set (cf. [Mo], 2C.3) in which, by (P), we can choose a perfect subset belonging to $\mathcal{J}$. This last set fulfils the assertion.

Consider the σ-ideals $\mathcal{M}$ and $\mathcal{L}_1$ in (C). It is known that $\mathcal{M} \cap \Pi_2^0(C)$ is cofinal in $\mathcal{M}$ (cf. [My 2]), and $\mathcal{L}_1 \cap \Sigma_2^0(C)$ is obviously cofinal in $\mathcal{L}_1$. Moreover, $\mathcal{M}$ and $\mathcal{L}_1$ have the d.p. (cf. [Ba 1]) and they fulfil (P) (see our Theorem 1.6 and Lemma 5.1 from [O]). If we additionaly apply Lemma 2.3, it will be clear that all assumptions of Theorem 2.2 are satisfied by $X = C$, $\mathcal{J} = \mathcal{M}$, $\mathcal{Y} = \mathcal{L}_1$ when CH is assumed. Thus we have

(2.4) C o r o l l a r y . Assume CH. Then the σ-ideals $\mathcal{L}_1$ and $\mathcal{M}$ in $\mathcal{P}(C)$ fulfil SEDP.

(2.5) T h e o r e m . Assume CH. Let X and Y be perfect Polish spaces. Let $\mathcal{J} \subset \mathcal{P}(X)$, $\mathcal{Y} \subset \mathcal{P}(X)$ and $\mathcal{U} \subset \mathcal{P}(Y)$ be σ-ideals containing all singletons. Assume that $\mathcal{J}$ and $\mathcal{Y}$ have the d.p., and $\mathcal{J} = \mathcal{J}|\Pi_1^1$, $\mathcal{Y} = \mathcal{Y}|\Pi_1^1$. Then each of the pairs

(a) $(\mathcal{J} \times \mathcal{U})|\Pi_1^1$ and $(\mathcal{Y} \times \mathcal{U})|\Pi_1^1$,

(b) $(\mathcal{U} \times \mathcal{J})|\Pi_1^1$ and $(\mathcal{U} \times \mathcal{Y})|\Pi_1^1$

has the d.p. and fulfils SEDP. The analogous theorem with Π_1^1 replaced by $\mathcal{B}$ is also true.

P r o o f . Let A and Be be disjoint sets belonging to $\mathcal{J}$ and $\mathcal{Y}$, respectively, such that their union is equal to X. Then the pairs of sets

(a′) $A \times Y$ and $B \times Y$,

(b′) $Y \times A$ and $Y \times B$,

guarantee the d.p. for pairs (a) and (b), respectively. This verifies one assumption of Theorem 2.2. Another assumption can be verified by Theorem 1.7 and Lemma 2.3. Since CH is supposed and the cardinality of the family of all coanalytic sets in a perfect Polish space is 2^{ω}, the remaining assumption holds. So, it suffices to apply Theorem 2.2. The proof with Π^1_1 replaced by $\mathcal{B}$ is analogous.

(2.6) Corollary. Assume CH. Then each of the pairs

(a) $(\mathcal{K}\times\mathcal{L})\,|\,\mathcal{B}$ and $(\mathcal{L}\times\mathcal{K})\,|\,\mathcal{B}$,

(b) $(\mathcal{K}\times\mathcal{L})\,|\,\Pi^1_1$ and $(\mathcal{L}\times\mathcal{K})\,|\,\Pi^1_1$

fulfils SEDP.

Note that $(\mathcal{K}\times\mathcal{L})\,|\,\Pi^1_1 \neq \mathcal{K}\times\mathcal{L}$ and $(\mathcal{L}\times\mathcal{K})\,|\,\Pi^1_1 \neq \mathcal{L}\times\mathcal{K}$ (see [Me], 1.3). However, we do not know whether $(\mathcal{K}\times\mathcal{L})\,|\,\mathcal{B} \neq$ $\neq (\mathcal{K}\times\mathcal{L})\,|\,\Pi^1_1$ and $(\mathcal{L}\times\mathcal{K})\,|\,\mathcal{B} \neq (\mathcal{L}\times\mathcal{K})\,|\,\Pi^1_1$ (cf. [B 2] where the problem is posed). Finally, let us remark that CH is essential in (2.6) (a) since, without CH, some purely set-theoretic properties of $(\mathcal{K}\times\mathcal{L})\,|\,\mathcal{B}$ and $(\mathcal{L}\times\mathcal{K})\,|\,\mathcal{B}$ can be different (cf. [CP] where $\mathrm{add}((\mathcal{K}\times\mathcal{L})|\mathcal{B})$ and $\mathrm{add}((\mathcal{L}\times\mathcal{K})|\mathcal{B})$ are studied).

REFERENCES

[B1] M. Balcerzak: The decomposition property of σ-ideals, Radovi Matematički 2 (1986), 305-315.

[B2] M. Balcerzak: On C-sets and products of ideals, Colloq. Math. (to appear).

[CP] J. Cichoń, J. Pawlikowski: On ideals of subsets of the plane and on Cohen reals, J. Symb. Logic 51 (1986), 560-569.

[J] T. Jech: Lectures in set theory with particular emphasis on the method of forcing, Lecture Notes in Math. 217, Springer Verlag, Berlin 1971.

[Kn] K. Kunen: Set theory. An introduction to independence proofs, North-Holland, Amsterdam 1980.

[Me] C.G. M e n d e z : On sigma-ideals of sets, Proc. Amer. Meth. Soc. 60 (1976), 124-128.

[Mi] A.W. M i l l e r : Special subsets of the real line, in Handbook of Set Theoretic Topology (K. Kunen, J.E. Vaughan eds.), North-Holland, Amsterdam 1984.

[Mo] Y.N. M o s c h o v a k i s : Desciptive set theory, North-Holland, Amsterdam 1980.

[My 1] J. M y c i e l s k i : On the axiom of determinateness, Fund. Math. 53 (1964), 205-224.

[My 2] J. M y c i e l s k i : Some new ideals of sets on the real line, Colloq. Math. 20 (1969), 71-76.

[O] J.C. O x t o b y : Measure and category, Springer--Verlag, New York 1971.

INSTITUTE OF MATHEMATICS, ŁÓDŹ UNIVERSITY, 90-238 ŁÓDŹ, POLAND

Received April 25, 1988.