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AUTOMORPHISM GROUPS OF P_2-LATTICES

The concept of a P_0-lattice was first introduced by T. Traczyk [4]. Epstein G. and Horn A. [1] used this concept to define a P_1- and P_2-lattices.

The aim of this paper is to describe the automorphism groups of finite P_2-lattices.

In this paper we use terminology and notation of [1] and [2]. In particular, V denotes the lattice join.

1. Preliminaries

A P_0-lattice is a bounded distributive lattice P which is generated by its center B and a finite subchain $o = e_0 < e_1 < \ldots < e_{n-1}$ containing o and 1. It is denoted by $P = \langle B, e_0, \ldots, e_{n-1} \rangle$.

(i) $e_0, \ldots, e_{n-1}$ is called a chain base of P.

(ii) A P_0-lattice P is of order n ($n > 1$), if n is the smallest integer such that P has a chain base with n terms.

(iii) Every element $x \in P$ can be written in the form

$$x = d_1e_1 V\, d_2e_2 V \ldots V\, d_{n-1}e_{n-1} = V_{i=1}^{n-1} d_ie_i, \quad d_i \in B,$$

$$i = 1, \ldots, n-1 \text{ and } d_1 \geqslant d_2 \geqslant \ldots \geqslant d_{n-1}.$$

Such a representation is called a monotonic representation (mon. rep.) of x.

Let P be a bounded distributive lattice with center B. Let $x \to y$ denotes the largest $z \in P$ (if it exists) such that

$xz \leqslant y$. Let $\neg x = x \to o$. P is a Heyting algebra if $x \to y$ exists for all $x, y \in P$.

An element $x \in P$ is called dense if $\neg x = o$. A P_1-lattice is a P_0-lattice which is a Heyting algebra together with a chain base,

$$e_0, \ldots, e_{n-1} \quad \text{such that} \quad e_{i+1} \to e_i = e_i.$$

Hence $e_i \to e_j = e_j$ if $i > j$ and $e_i \to e_j = 1$ if $i < j$. It was proved ([1], Theorem 3.3) that, if $P = \langle B, e_0, \ldots, e_{n-1} \rangle$ is a P_0-lattice which is a Heyting algebra, then there exists a chain base $o = f_0 < f_1 < \ldots < f_{n-1} = 1$ such that $P = \langle B, f_0, \ldots, f_{n-1} \rangle$ is a P_1-lattice. Such a chain is unique.

Let P be a bounded distributive lattice with center B. Let $x \Rightarrow y$ denotes the largest $b \in B$ (if it exists) such that $xb \leqslant y$. P is called a B-algebra, if $x \Rightarrow y$ exists for all $x, y \in P$. $!x = 1 \Rightarrow y$ is called the pseudo-supplement of x. If P is a B-algebra then $x \to y = y \; V(x \Rightarrow y)$ for all $x, y \in P$.

A P-algebra is a B-algebra satisfying $(x \Rightarrow y) \; V(y \Rightarrow x) = 1$ for all $x, y \in P$. P_0-lattice which are P-algebras are called P_0P-lattices [2].

Let the least Boolean element greater than or equal to x (if it exists) is noted by $x!$.

It was proved in [2] that, in any P_0P-lattice $P = \langle B, e_0, \ldots, e_{n-1} \rangle$:

(i) $(x \Rightarrow y)\;(y \Rightarrow z) \leqslant x \Rightarrow z$,

(ii) $x! = \overline{x \Rightarrow o}$,

(iii) $(x \; V \; y)! = x! \; V y!$ and $(xy)! = x! \; y!$

(iv) Every element $x \in P$ can be written in the form

$x = V_{i=1}^{n-1} D_i(x)\; e_i$ where $D_i(x) = x!\;(e_i \Rightarrow x)$,

$i = 1, \ldots, n-1$ and the following properties hold:

(a) $D_1(x) \geqslant D_2(x) \geqslant \ldots \geqslant D_{n-1}(x)$,

(b) $D_i(xVy) = D_i(x) V \; D_i(y)$,

(c) $D_i(xy) = D_i(x)\; D_i(y)$,

(d) $D_i(b) = b$ for $b \in B$,

(e) $D_i(e_j) = e_j!$ for $i \leqslant j$ and $D_i(e_j) = e_j!\,(e_i \Rightarrow e_j)$ for $i > j$ and in particular $D_{n-1}(e_j) = !e_j$.

A P_2-lattice is a P_1-lattice $P = \langle B, e_0, \ldots, e_{n-1} \rangle$ such that $!e_i$ exists for all i.

It was proved ([1], Theorem 4.4) that, if $P = \langle B, e_0, \ldots, e_{n-1} \rangle$ is a P_0-lattice of order n and P is a B-algebra, then there exists a unique chain $f_0, \ldots, f_{n-1}$ such that $P = \langle B, f_0, \ldots, f_{n-1} \rangle$ is a P_2-lattice.

2. The P_2-lattice automorphisms

Definition 2.1. Let $P = \langle B, e_0, \ldots, e_{n-1} \rangle$ and $P' = \langle B', e'_0, \ldots, e'_{m-1} \rangle$ be two P_2-lattices of orders n and m respectively, then a lattice homomorphism $h : P \to P'$ is a P_2-lattice homomorphism provided:

(i) $h \mid B$ is a Boolean homomorphism of B into B',

(ii) $h(e_i) \in \{e'_1, \ldots, e'_{m-1}\}$ for every $i = 1, \ldots, n-1$,

(iii) $h(x \Rightarrow y) = h(x) \Rightarrow h(y)$.

A one-to-one P_2-lattice homomorphism of a P_2-lattice P onto itself is a P_2-lattice automorphism. Hence, if h is an automorphism of a P_2-lattice $P = \langle B, e_0, \ldots, e_{n-1} \rangle$, then $h(B) = B$, $h(e_i) = e_i$, $i = 1, \ldots, n-1$ and $h(x \Rightarrow y) = h(x) \Rightarrow h(y)$, for all $x, y \in P$.

Since in a P_2-lattice $P = \langle B, e_0, \ldots, e_{n-1} \rangle$ $e_i \to e_j = e_j$ for $i > j$, $\neg e_i = e_i \to o = o$ for all $i = 1, \ldots, n-1$. Hence, if a is an atom in B ($a \in B$, $a \neq o$ and if $o \neq b \leqslant a$, then $b = a$), then either $a e_i = a$ or $o < a e_i < a$.

Lemma 2.1. In a P_2-lattice $P = \langle B, e_0, \ldots, e_{n-1} \rangle$ of order n, if $a e_i = a e_{i+1}$, $a \in B$, then $a e_i = a$, $i = 1, \ldots, n-2$.

Proof. It is obvious for $a = o$. Let $a \neq o$ and $a e_i = a e_{i+1}$, then $a \neq 1$. Hence $a < a e_{i+1} = a e_i < e_i$ (because $a e_i = e_i$ implies e_i is not dense) which implies $a \leqslant e_{i+1} \Rightarrow e_i = !(e_{i+1} \to e_i) = !e_i < e_i$ i.e. $a < e_i$.

D e f i n i t i o n 2.2. Let $P = \langle B, e_0, \ldots, e_{n-1} \rangle$ be a P_2-lattice of order n with an atomic center B. Let:

(i) $D_{e_i} = \{a : a \text{ is an atom in } B \text{ and } ae_i = a\}$, $i = 1,\ldots,n-1$.

(ii) $P_{e_i} = \{a : a \text{ is an atom in } B \text{ and } ae_i < a\}$, $i = 1,\ldots,n-1$.

It is clear that,

(a) $D_{e_1} \subseteq D_{e_2} \subseteq \ldots \subseteq D_{e_{n-1}}$,

(b) $P_{e_1} \supseteq P_{e_2} \supseteq \ldots \supseteq P_{e_{n-1}}$,

and if A is the set of all atoms in B, then

(c) $A = P_{e_i} \dot{\cup} D_{e_i}$ for all $i = 1,\ldots,n-1$, where $\dot{\cup}$ denotes the disjoint union of sets.

L e m m a 2.2. $A = \dot{\cup}_{i=1}^{n-2} (P_{e_i} - P_{e_{i+1}}) \dot{\cup} D_{e_1}$ provides a partition of A (some of the terms $P_{e_i} - P_{e_{i+1}}$) may be empty).

P r o o f . Since $P_{e_{n-1}} = \Phi$, $P_{e_1} = (P_{e_1} - P_{e_2}) \cup (P_{e_2} - P_{e_3}) \cup \ldots$ $\ldots \cup (P_{e_{n-2}} - P_{e_{n-1}}) = \cup_{i=1}^{n-2} (P_{e_i} - P_{e_{i+1}})$. (Some of the terms $P_{e_1} - P_{e_{i+1}}$) may be empty). Hence, $A = P_{e_1} \dot{\cup} D_{e_1} =$ $= \cup_{i=1}^{n-2} (P_{e_i} - P_{e_{i+1}}) \dot{\cup} D_{e_1}$ constitute a partition of A.

L e m m a 2.3. Let $P = \langle B, e_0, \ldots, e_{n-1} \rangle$ be a finite P_2-lattice of order n. Let h be an automorphism of P, then:

(i) $h(P_{e_i}) = P_{h(e_i)} = P_{e_i}$, $i = 1,\ldots,n-2$,

(ii) $h(D_{e_i}) = D_{h(e_i)} = D_{e_i}$, $i = 1,\ldots,n-2$.

P r o o f . (i): Let $a \in P_{e_i}$, $i = 1,\ldots,n-2$, then $o < ae_i < a$. Hence, $o < h(a)\, h(e_i) < h(a)$. Since the chain $e_0, \ldots, e_{n-1}$ is unique ([1], Theorem 4.4), $h(e_i) = e_i$, $i = 1,\ldots,n-1$ and $o < h(a)\, e_i < h(a)$ i.e. $h(a) \in P_{e_i}$. The converse implication is simple.

(ii): Let $a \in D_{e_i}$, then $ae_i = a$ and $h(a)\, e_i = h(a)$ which implies $h(a) \in D_{e_i}$.

The converse implication is simple:

L e m m a 2.4. Let $P = \langle B, e_0, \dots, e_{n-1} \rangle$ be a finite P_2-lattice of order n. Let a be an atom in B, then for any $x \in P$, if $a \leqslant D_j(x)$ and $a \in De_j$, then $a \leqslant D_i(x)$ for all $i = 1, \dots, n-1$.

P r o o f . The proof is clear for $i \leqslant j$. Suppose $i > j$ and $a \in D_{e_j}$. Now, $a \leqslant D_j(x) = x! \, (e_j \Rightarrow x) \leqslant e_j \Rightarrow x$ which implies $ae_j \leqslant x$. Hence, $ae_i \leqslant x (a = ae_j \leqslant ae_i \leqslant a)$ which implies $a \leqslant e_i \Rightarrow x$. Since $a \leqslant x!$, $a \leqslant D_i(x)$.

T h e o r e m 2.1. If $h_0 : B \to B$ is an automorphism of a finite Boolean algebra B and $P = \langle B, e_0, \dots, e_{n-1} \rangle$ is a P_2-lattice, then there exists an automorphism h of P such that $h \mid B = h_0$ if and only if $h_0(P_{e_i}) = P_{e_i}$, $i = 1, \dots, n-2$ and $h_0(D_{e_i}) = D_{e_i}$, $i = 1, \dots, n-2$.

P r o o f . Since P is finite, P is a P-algebra ([1], Theorem 4.11) and hence is a P_0P-lattice. Suppose h_0 satisfies $h_0(P_{e_i}) = P_{e_i}$ and $h_0(D_{e_i}) = D_{e_i}$, $i = 1, \dots, n-2$. Let $x, y \in P$, then x and y have the monotonic representations.

$$x = \mathrm{V}_{i=1}^{n-1} D_i(x) e_i \quad \text{and} \quad y = \mathrm{V}_{i=1}^{n-1} D_i(y) e_i$$

where

$$D_i(x) = x! (e_i \Rightarrow x) \quad \text{and} \quad D_i(y) = y! (e_i \Rightarrow y).$$

Let $h : P \to P$ be defined by

$$h(x) = \mathrm{V}_{i=1}^{n-1} h_0(D_i(x)) e_i .$$

Then,

$$(1) \quad h(D_i(x)) = \mathrm{V}_{j=1}^{n-1} h_0(D_j(D_i(x))) e_j = \mathrm{V}_{j=1}^{n-1} h_0(D_i(x)) e_j = h_0(D_i(x)).$$

Since $D(x)$ is Boolean, $h \mid B = h_0$.

(2) $D_i(h(x)) = D_i(V_{j=1}^{n-1} h_o(D_j(x))e_j) =$

$= V_{j=1}^{n-1} D_i(h_o(D_j(x)))D_i(e_j)$ (D_i are homomorphisms)

$= V_{j=1}^{n-1} h_o(D_j(x))D_i(e_j) =$

$= V_{j=1}^{i-1} h_o(D_j(x))(!e_j) V\ V_{j=i}^{n-1} h_o(D_j(x))e_j!.$

Now, $e_j! = \overline{e_i \Rightarrow o} = !\overline{(e \rightarrow o)} = \overline{!o}$ (e_i is dense) $= \bar{o} = 1.$

Hence $D_i(h(x)) = V_{j=1}^{i-1} h_o(D_j(x))(!e_j) V\ h_o(D_i(x)).$

It is clear that $!e_j = V\ D_{e_j}$, for $j = 1,\dots,n-1$. In order to prove that $D_i(h(x)) = h_o(D_i(x))$, it remains to show that:

(i) $h_o(D_j(x))(!e_j) \leqslant h_o(D_i(x))$, $j = 1,\dots,i-1$.

We shall do it in two steps:

a) If $a \in P_{e_j}$ and $a \leqslant D_j(x)$ for certain $j < i$, then $h_o(a) \in P_{e_j}$. Since $P_{e_j} \cap D_{e_j} = \Phi$, $h_o(a)(!e_j) = o$ and hence (i) holds.

b) If $a \in D_{e_j}$ and $a \leqslant D_j(x)$, then $a \in D_i(x)$ (Lemma 2.4). Hence, $h_o(a)\ (!e_j) = h_o(a) \leqslant h_o(D_i(x))$ and again (i) holds. Since there is no third possibility, (i) has been proved.

Hence $D_i(h(x)) = V_{j=1}^{i-1} h_o(D_j(x))(!e_j) V\ h_o(D_i(x)) =$

$= h_o(D_i(x)).$

(3) $h(xVy) = h(x)\ Vh(y)$ and $h(xy) = h(x)h(y)$. The easy proof is omitted.

(4) We shall prove here that $h(e_i) = e_i$, $i = 1,\dots,n-1$. Since $e_i! = 1$ for $i = 1,\dots,n-1$ and

$$e_j \Rightarrow e_i = \begin{cases} 1 & \text{if } j \leqslant i \\ !e_i & \text{if } j > i \end{cases}.$$

We have $h(e_i) = V_{j=1}^{n-1} h_o(D_j(e_i))e_j = V_{j=1}^{n-1} h_o(e_i!)(e_j \Rightarrow e_i))e_j =$

$= V_{j=1}^{n-1} h_o(e_j \Longrightarrow e_i)e_j = V_{j=1}^{i} e_j V\ V_{j=i+1}^{n-1} h_o(!e_i)e_j =$

$= e_i V\ h_o(!e_i) = e_i\ (!e_i \leqslant e_i).$

(5) To prove h is one-to-one, let $x,y \in P$, $x = V_{i=1}^{n-1} D_i(x)e_i$, $y = V_{i=1}^{n-1} D_i(y)e_i$ and $h(x) = h(y)$. Since the representation is unique, $h(x) = h(y)$ implies that $D_i(h(x) = D_i(h(y))$ for $i = 1,\dots,n-1$. Hence, (2) implies $h_o(D_i(x)) = h_o(D_i(y)$ for $i = 1,\dots,n-1$. Hence $x = y$.

(6) To prove h is onto, let $y = V_{i=1}^{n-1} D_i(y)e_i$ be a mon. rep. of $y \in P$. Let $x = V_{i=1}^{n-1} h_o^{-1} - 1\ (D_i(y))e_i$. It is easy to show that $x = V_{i=1}^{n-1} h_o^{-1}(D_i(y))e_i$, is a mon. rep. of x; and $h(x) = y$.

(7) We shall prove that, $h(x \Rightarrow y) = h(x) \Longrightarrow h(y)$ for all $x,y \in P$. Since h is a lattice automorphism, h is order embedding i.e. $x \leqslant y$ if and only if $h(x) \leqslant h(y)$ ([3] Theorem III. 1.4), hence,

(i) $h(x \Rightarrow y)\ h(x) = h(x(x \Rightarrow y) \leqslant h(y)$ which implies $h(x \Rightarrow y) \leqslant h(x) \Rightarrow h(y)$.

(ii) Let $b = h(x) \Rightarrow h(y)$, then $b\ h(x) \leqslant h(y)$ which implies $h_o^{-1}(b)\ x \leqslant y$ (h is order embedding). Hence, $h_o^{-1}(b) \leqslant x \Rightarrow y$ and $b \leqslant h_o(x \Rightarrow y) = h(x \Rightarrow y)$ i.e. $h(x) \Rightarrow h(y) \leqslant h(x \Rightarrow y)$ and (7) is proved.

Now, (i),...,(7) imply that h is a P_2-lattice automorphism of P with $h(e_i) = e_i$ for $i = 1,\dots,n-1$ and $h \mid B = h_o$. The converse implication follows by Lemma 2.3.

It is clear that the automorphism h defined by Theorem 2.1 is unique.

R e m a r k 2.1. The following theorem is well known in abstract algebra "If A is a set of m elements, then the number of permutations on A that leaves a set $S \subset A$ or r elements $(r < m)$ fixed is $(m-r)!\ r!$".

T h e o r e m 2.2. Let $P = \langle B, e_0, \dots, e_{n-1} \rangle$ be a finite P_2-lattice with the center $B = 2^m$, then the number of automorphisms of P is

$$\prod_{i=1}^{n-2}(p_i)!\,(d_1)! \quad \text{where} \quad d_1 = |D_{e_1}|,\ p_i = |P_{e_i} - P_{e_{i+1}}|$$

$i = 1, \dots, n-2$, D_{e_1} and P_{e_i}, $i = 1, \dots, n-2$ defined earlier.

P r o o f . Let A be the set of all atoms in B. Hence $|A| = m$ and by Lemma 2.2 A has the partition

$$A = \dot{\bigcup}_{i=1}^{n-2}(P_{e_i} - P_{e_{i+1}})\,\dot{\cup}\, D_{e_1}. \quad \text{Hence, if} \quad m = |A|,$$

$$p_i = |P_{e_i} - P_{e_{i+1}}|, \quad i = 1, \dots, n-2 \text{ and } d_1 = |D_{e_1}|,$$

then,

$$m = \sum_{i=1}^{n-2} p_i + d_1.$$

By Theorem 2.1, the only automorphisms h_0 of B that can be extended to automorphisms of P are the automorphisms that leave each of the sets $P_{e_i} - P_{e_{i+1}}$, $i = 1, \dots, n-2$ and D_{e_1} fixed. Hence, by Remark 2.1, the number of automorphisms of P is greater than or equal to $\prod_{i=1}^{n-2}(p_i)!\,(d_1)!$. Since the extension of h_0 is unique, the number is exactly $\prod_{i=1}^{n-2}(p_i)!(d_1)!$.

REFERENCES

[1] G. E p s t e i n , A. H o r n : Chain based lattices, Pacific J. Math. 55 (1974), 65-84.

[2] J. K l u k o w s k i , N . Z w o r s k i : On the representation of P_0-lattices being P-algebras, Demonstratio Math. 18 (1985) 103-114.

[3] H. R a s i o w a : An algebraic approach to o non--classical logics. North Holland Publishing Company, Amsterdam, 1974.

[4] T. T r a c z y k : Axioms and some properties of Post--algebras, ibidem 10 (1963) 193-209.

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