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TRANSITIVE n-QUASIGROUPS

In the present note we generalize the notion of right transitivity to n-quasigroups and define it by the constraints (T_n) and (U_n). It is shown that for every right transitive n-quasigroup (Q,A) there is a group $(Q,+)$ such that

$$A(x_1,x_2,\dots,x_n) = x_1 - x_2 - \dots - x_n.$$

The law of right transitivity for binary quasigroups $(Q,\circ)$ is given by the identity

$$xz \circ yz = xy$$

and it is well-known that the operation $\circ$ in a right transitive quasigroup is in fact subtraction in a certain group, [2], [3]. Namely, if $(Q,\circ)$ is a right transitive quasigroup, then there is a group $(Q,+)$ such that

$$xy = x-y$$

for all $x,y \in Q$, where $-y$ is the inverse of y in the group $(Q,+)$ and $x-y = x+(-y)$.

Now, for a group $(Q,+)$ let us introduce an n-ary operation A on Q, $A\colon Q^n \to Q$, $n \in N \setminus \{1\}$, by the definition

$$A(x_1^n) = A(x_1,x_2,\dots,x_n) = x_1 - x_2 - \dots - x_n.$$

It is easy to verify that (Q,A) is an n-quasigroup (see [1] for the definitions and notation) which satisfies the following two conditions:

$$(T_n)\quad A(A(x_1^j,y_1^{n-j}),y_{n-j+1}^{n-1},\ A(x_{j+1},y_1^{n-1}),\ x_{j+2}^n) = A(x_1^n)$$

for all $x_1,\dots,x_n,\ y_1,\dots,y_n \in Q$ and every $j \in \{1,2,\dots,n-1\}$

(U_n) There is an element $0 \in Q$ such that

$$A(x,\overset{j}{0},x,\overset{n-j-2}{0}) = 0$$

for all $x \in Q$ and every $j \in \{0,1,\dots,n-2\}$.

Indeed, if one takes the zero of the group $(Q,+)$ as the element 0, then (U_n) is fulfilled, and the condition (T_n) is valid too since we have

$$A(A(x_1^j,y_1^{n-j}),y_{n-j+1}^{n-1},A(x_{j+1},y_1^{n-1})x_{j+2}^n) =$$

$$= (x_1-x_2-\dots-x_j-y_1-\dots-y_{n-j})-y_{n-j+1}-\dots-y_{n-1}-$$
$$-(x_{j+1}-y_1-\dots-y_{n-1})-x_{j+2}-\dots-x_n =$$
$$= x_1-x_2-\dots-x_j-y_1-\dots-y_{n-1}+y_{n-1}+\dots+y_1-x_{j+1}-x_{j+2}-\dots-x_n =$$
$$= x_1-x_2-\dots-x_j-x_{j+1}-\dots-x_n = A(x_1^n).$$

In particular, the identity (T_2) has the following form

$$A(A(x_1,y_1),A(x_2,y_1)) = A(x_1,x_2),$$

i.e. it is the right transitivity, and in that case (U_2) is satisfied too.

Therefore, we shall say that an n-quasigroup (Q,A) is a right transitive n-quasigroup iff it satisfies the constraints (T_n) and (U_n). Such an n-quasigroup will be shortly called transitive n-quasigroup.

Namely, let be reminded that the identity

$$zx\cdot zy = xy$$

is the law of left transitivity and it is obvious that a binary quasigroup $(Q,\cdot)$ is left transitive iff the quasigroup

$(Q,\circ)$, defined by $x\circ y = yx$, is right-transitive. Hence one possibility for a generalization of left transitivity to n-quasigroups is the following: an n-quasigroup (Q,A) is left transitive iff n-quasigroup $(Q,\hat{A})$ defined by $\hat{A}(x_1^n) = A(x_n,\dots,x_2,x_1)$ is right transitive. Therefore, it is enough to investigate only right transitive n-quasigroups and, as we said before, they are shortly called transitive n-quasigroups.

Particularly, for a transitive 3-quasigroup, resp. 4-quasigroup (Q,A) we get:

(T_3)
$$\begin{aligned} A(A(x_1,y_1,y_2),A(x_2,y_1,y_2),x_3) &= \\ = A(A(x_1,x_2,y_1),y_2,A(x_3,y_1,y_2)) &= \\ = A(x_1,x_2,x_3); \end{aligned}$$

(U_3) There is $0\in Q$ such that

$$A(x,x,0) = A(x,0,x) = 0;$$

(T_4)
$$\begin{aligned} A(A(x_1,y_1,y_2,x_3),A(x_2,y_1,y_2,y_3),x_3,x_4) &= \\ = A(A(x_1,x_2,y_1,y_2),y_3,A(x_3,y_1,y_2,y_3),x_4) &= \\ = A(A(x_1,x_2,x_3,y_1),y_2,y_3,A(x_4,y_1,y_2,y_3)) &= \\ = A(x_1,x_2,x_3,x_4); \end{aligned}$$

(U_4) There is $0\in Q$ such that

$$A(x,x,0,0) = A(x,0,x,0) = A(x,0,0,x) = 0.$$

P r o p o s i t i o n 1. If (Q,A) is a transitive n-quasigroup then the following assertions hold:

(i) $$A(x,\overset{n-1}{0}) = x, \quad \text{for all } x\in Q,$$

(ii) $$A(x,y,\overset{n-2}{0}) = A(x,\overset{j-2}{0},y,\overset{n-j}{0}), \quad \text{for all } x,y\in Q$$
and every $j\in\{2,\dots,n-2\}$,

(ii) $$A(A(x_1^j,\overset{n-j}{0}),x_{j+1},\overset{n-2}{0}) = A(x_1^{j+1},\overset{n-j-1}{0})$$
for all $x_1,\dots,x_{j+1}\in Q$ and every $j\in\{1,\dots,n-1\}$.

P r o o f . (i) Because of (U_n) it follows $A(\overset{n}{0}) = 0$ and hence $A(x,\overset{n-1}{0}) = A(A(x,\overset{n-1}{0}),A(0,\overset{n-1}{0}),\overset{n-2}{0}) = A(A(x,\overset{n-1}{0}),\overset{n-1}{0})$ which implies (i),

(ii) $$A(x,\overset{j-2}{0},y,\overset{n-j}{0}) =$$
$$= A(A(x,\overset{j-2}{0},\overset{n-j+1}{0}),y,\overset{j-3}{0},A(y,\overset{n-j+1}{0},y,\overset{j-3}{0}),\overset{n-j}{0}) =$$
$$= A(x,y,\overset{n-2}{0}),$$

(iii) $$A(A(x_1^j,\overset{n-j}{0}),x_{j+1},\overset{n-2}{0}) = A(A(x_1^j,\overset{n-j}{0}),\overset{j-1}{0},x_{j+1},\overset{n-j-1}{0}) =$$
$$= A(A(x_1^j,\overset{n-j}{0}),\overset{j-1}{0},A(x_{j+1},\overset{n-1}{0}),\overset{n-j-1}{0}) =$$
$$= A(x_1^{j+1},\overset{n-j-1}{0}).$$

P r o p o s i t i o n 2. Let (Q,A) be a transitive n-quasigroup and $+$ binary operation on Q defined by

$$x+y = A(x,A(0,y,\overset{n-2}{0}),\overset{n-2}{0}).$$

Then $(Q,+)$ is a group.

P r o o f . Obviously, $(Q,+)$ is a quasigroup. Moreover, it is a loop with the unit 0 since

$$x+0 = A(x,A(\overset{n}{0}),\overset{n-2}{0}) = A(x,\overset{n-1}{0}) = x$$

$$0+x = A(0,A(0,x,\overset{n-2}{0}),\overset{n-2}{0}) =$$
$$= A(A(x,x,\overset{n-2}{0}),A(0,x,\overset{n-2}{0}),\overset{n-2}{0}) =$$
$$= A(x,0,\overset{n-2}{0}) = x.$$

Further, for all $x,y,z \in Q$ we have

$$A(x,y,\overset{n-2}{0}) = A(A(x,A(0,z,\overset{n-2}{0}),\overset{n-2}{0}),A(y,A(0,z,\overset{n-2}{0}),\overset{n-2}{0}),\overset{n-2}{0}) =$$
$$= A(x+z,y+z,\overset{n-2}{0})$$

and hence

$$A(x+y,y,\overset{n-2}{0}) = A(x+y,0+y,\overset{n-2}{0}) = A(x,\overset{n-1}{0}) = x$$

and the associativity follows:

$$(x+y)+z = A((x+y)+z,\overset{n-1}{0}) =$$

$$= A(A((x+y)+z,y+z,\overset{n-2}{0}),A(0,y+z,\overset{n-2}{0}),\overset{n-2}{0}) =$$

$$= A(A(x+y,y,\overset{n-2}{0}),A(0,y+z,\overset{n-2}{0}),\overset{n-2}{0}) =$$

$$= A(x,A(0,y+z,\overset{n-2}{0}),\overset{n-2}{0}) = x+(y+z),$$

i.e. $(Q,+)$ is a group with the unit 0. The inverse $-x$ of an element $x \in Q$ is given by $-x = A(0,x,\overset{n-1}{0})$, because of $-x = A(-x+x,x,\overset{n-2}{0})$.

P r o p o s i t i o n 3. For a transitive n-quasigroup (Q,A) there is a group $(Q,+)$ such that

$$A(x_1^n) = x_1-x_2-\dots-x_n.$$

P r o o f . Let $(Q,+)$ be the group defined in the previous proposition. Then we get

$$A(x,y,\overset{n-2}{0}) = A(x-y,y-y,\overset{n-2}{0}) = A(x-y,\overset{n-1}{0}) = x-y$$

and Proposition 1 implies

$$A(x_1^n) = A(A(x_1^{n-1},0),x_n,\overset{n-2}{0}) =$$

$$= A(x_1^{n-1},0)-x_n =$$

$$= \dots =$$

$$= A(x_1^{j+1},\overset{n-j-1}{0})-x_{j+2}-\dots-x_n =$$

$$= A(A(x_1^j, \overset{n-j}{0}), x_{j+1}, \overset{n-2}{0}) - x_{j+2} - \dots - x_n =$$

$$= A(x_1^j, \overset{n-j}{0}) - x_{j+1} - x_{j+2} - \dots - x_n =$$

$$= \dots =$$

$$= A(x_1, x_2, \overset{n-2}{0}) - x_3 - \dots - x_n =$$

$$= x_1 - x_2 - \dots - x_n.$$

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