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Bogusław Merdas

REMARKS ON SCHATTEN – VON NEUMANN CLASSES C_p

Operator ideals C_p $(0 < p \leqslant \infty)$ have been introduced by J. von Neumann and R. Schatten [7] in 1946. They are natural generalizations of the nuclear (trace-class) operators and the Hilbert-Schmidt operators. In §1 we define the p-projective tensor product $X \hat{\otimes}_p Y$ of Banach spaces X and Y and describe its dual space. In §2 we show that the analogue of the Grothendieck theorem is not true for C_p. In §3 we show that C_p $(0 < p < 1)$ does not have the minorant property. In §4 we consider Schur multipliers on C_p, $0 < p \leqslant 1$. In this paper we use the following notation: $\mathbb{N}$ is the set of positive integers, $\mathbb{R}$ - real numbers, $\mathbb{C}$ - complex numbers, H - infinite dimensional separable Hilbert space, l_2^n - n-dimensional Hilbert space, $\langle \cdot, \cdot \rangle$ is the scalar product in a Hilbert space. We write "operator" ("functional") instead of "linear operator" (respectively "linear functional"). If X and Y are Banach spaces then L(X,Y) is the space of all continuous operators from X into Y with the usual operator norm. L(X) is the space of all continuous operators on X. All considered linear spaces are complex.

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1. p-norms $(0 < p \leqslant 1)$. p-projective tensor product of Banach spaces

Let us recall that a real-valued function $\|\cdot\|_p$ on a linear space X is said to be a p-norm if for all $x, y \in X$ and $\lambda \in \mathbb{C}$

(i) $\|x\|_p \geq 0$; $\|x\|_p = 0$ if and only if $x = 0$,

(ii) $\|\lambda x\|_p = |\lambda| \|x\|_p$,

(iii) $\|x + y\|_p^p \leq \|x\|_p^p + \|y\|_p^p$ (p-triangle inequality).

It follows from (iii) that

$$\|x + y\|_p \leq 2^{\frac{1}{p}-1} (\|x\|_p + \|y\|_p).$$

Putting $d(x,y) = \|x - y\|_p^p$ we define a (translation invariant) metric on X. $(X, \|\cdot\|_p)$ is said to be a p-Banach space, if the above metric is complete.

Let X and Y be Banach spaces and $X \otimes Y$ their algebraic tensor product. We define the function $\|\cdot\|_p$ on $X \otimes Y$ as follows

$$\|u\|_p = \inf\Big(\sum_{i=1}^{n} \|x_i\|^p \|y_i\|^p\Big)^{\frac{1}{p}}$$

where the infimum is taken over all representations

$$u = \sum_{i=1}^{n} x_i \otimes y_i, \quad x_i \in X,\ y_i \in Y.$$

Lemma 1.1. The function $\|\cdot\|_p$ is a p-norm on $X \otimes Y$.

Proof. (ii) is obvious. Let $u,v \in X \otimes Y$. Take $\varepsilon > 0$ and such representations

$$u = \sum_{i=1}^{n} x_i \otimes y_i \quad \text{and} \quad v = \sum_{j=1}^{m} t_j \otimes s_j$$

that

$$\sum_{i=1}^{n} \|x_i\|^p \|y_i\|^p \leq \|u\|_p^p + \frac{\varepsilon}{2} \quad \text{and} \quad \sum_{j=1}^{m} \|t_j\|^p \|s_j\|^p \leq \|v\|_p^p + \frac{\varepsilon}{2}.$$

Then we have

$$\|u+v\|_p^p \leq \sum_{i=1}^{n} \|x_i\|^p \|y_i\|^p + \sum_{j=1}^{m} \|t_j\|^p \|s_j\|^p \leq \|u\|_p^p + \|v\|_p^p + \varepsilon.$$

Letting $\varepsilon \to 0$ we get (iii). Now we prove (i). Let $x^* \in X^*$, $y^* \in Y^*$, u as above. Then $(x^* \otimes y^*)(u) = \sum_{i=1}^{n} x^*(x_i)y^*(y_i)$ and

$$\left|\sum_{i=1}^{n} x^*(x_i)y^*(y_i)\right|^p \leqslant \sum_{i=1}^{n} \left|x^*(x_i)\right|^p \left|y^*(y_i)\right|^p \leqslant$$

$$\leqslant \|x^*\|^p \|y^*\|^p (\|u\|_p^p + \varepsilon),$$

hence $|x^* \otimes y^*(u)| \leqslant \|x^*\| \cdot \|y^*\| \cdot \|u\|_p$. If $u \neq 0$ then we take such a representation $u = \sum_{i=1}^{n} x_i \otimes y_i$ that $x_1, \ldots, x_n$ are linearly independent. We define the functional $\bar{x}^*$ on the subspace $\mathrm{lin}\{x_1, \ldots, x_n\}$ by $\bar{x}^*(x_i) = \delta_{i1}$ (δ_{ij} - Kronecker's symbol) and extend it to a functional $x^* \in X$. Taking $y^* \in Y$ so that $y_1 \notin \ker y^*$ we have $x^* \otimes y^*(u) = x^*(x_1)y^*(y_1) \neq 0$, hence $\|u\|_p \neq 0$. ■

R e m a r k [6]. If $p > 1$ then we have for every $u \in X \otimes Y$

$$\|u\|_p = \inf\left(\sum_{i=1}^{n} \|x_i\|^p \|y_i\|^p\right)^{\frac{1}{p}} = 0.$$

The space $(X \otimes Y, \|\cdot\|_p)$, $0 < p \leqslant 1$, need not to be complete. Let $X \hat{\otimes}_p Y$ denote its completion and we call it the p-projective tensor product of X and Y. Using the same idea as in [3] with obvious modifications one can show

T h e o r e m 1.2. Let $u \in X \hat{\otimes}_p Y$ and $\varepsilon > 0$. Then there exist sequences (x_n) in X and (y_n) in Y such that $u = \sum_{n=1}^{\infty} x_n \otimes y_n$ and $\sum_{n=1}^{\infty} \|x_n\|^p \|y_n\|^p \leqslant \|u\|_p^p + \varepsilon$.

C o r o l l a r y 1.3. If X and Y are Banach spaces then $X \hat{\otimes}_p Y$ is p-Banach space.

P r o o f . Let $\varepsilon > 0$ and $u, v \in X \hat{\otimes}_p Y$. Take representations

$$u = \sum_{i=1}^{\infty} x_i \otimes y_i, \quad v = \sum_{j=1}^{\infty} t_j \otimes s_j \quad \text{and} \quad n, m \in \mathbf{N}$$

such that

$$\sum_{i=1}^{n} \|x_i\|^p \|y_i\|^p \leq \|u\|_p^p + \frac{\varepsilon}{2}, \quad \sum_{j=1}^{\infty} \|t_j\|^p \|s_j\|^p \leq \|v\|_p^p + \frac{\varepsilon}{2}.$$

Reasoning analogically as in the proof of Lemma 1 we show that $\|\cdot\|_p$ is a p-norm on $X \hat{\otimes}_p Y$. ■

Let X be a p-Banach space. By the dual space X^* of X we mean the set of all continuous functionals on X.

L e m m a 1.4. Let X be a p-Banach space. For $f \in X^*$

$$\|f\| = \sup\left\{|f(u)| : u \in X \text{ and } \|u\|_p \leq 1\right\}$$

is a norm and X^* with the above norm becomes a Banach space.

P r o o f . The triangle inequality follows from the properties of supremum. It is clear that other properties of a norm also hold. The proof of the completness of X^* is identical to that in case of a normed space X. ■

T h e o r e m 1.5. Let X and Y be Banach spaces. Then $(X \hat{\otimes}_p Y)^*$ is isometrically isomorphic to $L(X, Y^*)$, the space of all bounded operators from X to Y^*. The isomorphism is given by the correspondence $F: L(X, Y^*) \to (X \hat{\otimes}_p Y)^*$, $F(A) = f_A$, where

$$f_A\Big(\sum_{i=1}^{\infty} x_i \otimes y_i\Big) = \sum_{i=1}^{\infty} (Ax_i)(y_i) \quad \text{for } A \in L(X, Y^*).$$

Moreover, $\|f_A\| = \sup\left\{|f_A(u)| : \|u\|_p \leq 1, u \in D\right\}$, where

$$D = \{x \otimes y : x \in X, y \in Y\}.$$

P r o o f . Let $A \in L(X, Y^*)$. We show that f_A is a continuous functional on $X \hat{\otimes}_p Y$ and $\|f_A\| \leq \|A\|$. Let $\varepsilon > 0$. Take such a representation $u = \sum_{i=1}^{\infty} x_i \otimes y_i$ that $\sum_{i=1}^{\infty} \|x_i\|^p \|y_i\|^p \leq \|u\|_p^p + \varepsilon$. Then

$$|f_A(u)|^p \leqslant \Big|\sum_{i=1}^{\infty} (Ax_i)(y_i)\Big|^p \leqslant \sum_{i=1}^{\infty} \|Ax_i\|^p\|y_i\|^p \leqslant \|A\|^p(\|u\|_p^p + \varepsilon).$$

Letting $\varepsilon \to 0$ we obtain $|f_A(u)| \leqslant \|A\|\cdot\|u\|_p$, so $\|f_A\| \leqslant \|A\|$. Conversely, consider sequences (x_n) in X and (y_n) in Y such that $\|x_n\| \leqslant 1$, $\|y_n\| \leqslant 1$ and

$$\|Ax_n\| \geqslant \|A\| - \frac{1}{2n}\ ;\ |(Ax_n)(y_n)| \geqslant \|Ax_n\| - \frac{1}{2n}\ .$$

Then we have

$$\sup\Big\{|f_A(u)| : \|u\|_p \leqslant 1,\ u \in D\Big\} \geqslant |f_A(x_n \otimes y_n)| = |(Ax_n)(y_n)| \geqslant$$

$$\geqslant \|Ax_n\| - \frac{1}{2n} \geqslant \|A\| - \frac{1}{n}\ .$$

Consequently, $\|f_A\| \geqslant \sup\Big\{|f_A(u)| : u \in D,\ \|u\|_p \leqslant 1\Big\} \geqslant \|A\|$ and F is "1-1". On the other hand, if $f \in (X \hat{\otimes}_p Y)^*$ then we define $A \in L(X,Y^*)$ as follows: for $x \in X$ and $y \in Y$ we put $(Ax)(y) = f(x \otimes y)$. Then $f_A = f$. ■

2. $C_p(H)$ -spaces

Let $A: H \to H$ be a compact operator. Then it has the Schmidt representation $A = \sum_{n=1}^{\infty} \mu_n f_n \otimes g_n$, where (f_n), (g_n) are orthonormal sets in H and (μ_n) is the sequence of eigenvalues of the operator $|A| = (A^*A)^{\frac{1}{2}}$ rearranged in decreasing order. Let us recall that $f \otimes g: H \to H$ is the operator of rank one defined by $(f \otimes g)(x) = \langle x,g\rangle f$, for $x \in H$.

Let $0 < p < \infty$. We say that A belongs to the Schatten-von Neumann class $C_p(H) = C_p$, if $\|A\|_p = \Big(\sum_{n=1}^{\infty} \mu_n^p\Big)^{\frac{1}{p}} < \infty$. We define $C_\infty = L(H)$ and $\|A\|_\infty = \|A\|$. In the case $p = 1$ we have the trace class operators; p=2 - the Hilbert-Schmidt operators. It is well known that if $1 \leq p \leq \infty$, C_p is a Banach space with the norm $\|\cdot\|_p$, and, if $0<p<1$, a p-Banach space with the p-norm $\|\cdot\|_p$. If $0<p\leq q \leq \infty$ then $C_p \subset C_q$ and for $A \in C_p$ $\|A\|_q \leq \|A\|_p$, see [4]. C_p are also operator ideals i.e. if $A \in C_p$, $T,S \in L(H)$ then $TAS \in C_p$ and $\|TAS\|_p \leq \|T\|\cdot\|A\|_p\|S\|$. Since $\|A\| = \|A\|_\infty \leq \|A\|_p$, C_p are algebras under the operator composition. If $A \in C_1$ we define the trace of A by $Tr(A) = \sum_{n=1}^{\infty} \langle Ae_n, e_n\rangle$, where (e_n) is (any) orthonormal basis of H. Tr(A) does not depend on the choice of the basis (e_n). If $\frac{1}{p}+\frac{1}{q}=1$, $1 \leq p < \infty$, $A \in C_p$, $B \in C_q$ then AB, $BA \in C_1$, $Tr(AB) = Tr(BA)$ and $|Tr(AB)| \leq \|A\|_p\|B\|_q$, $\|AB\|_1 \leq \|A\|_p\|B\|_q$. Moreover,

$$\|A\|_p = (Tr|A|^p)^{\frac{1}{p}} = \sup\{|Tr(AB)| : \|B\|_q \leq 1\}.$$

The dual C_p^* is isometrically isomorphic to C_q and $(K(H))^* = C_1$, where K(H) is the space of all compact operators on H. The isomorphism is given by $B \in C_q \to \{C_p \ni A \to Tr(AB) \in \mathbf{C}\}$. We shall use the following result [10]:

$$H \hat{\otimes}_p H = C_p \quad \text{for} \quad 0<p\leq 1.$$

C o r o l l a r y 2.1. If $0<p\leq 1$ then C_p^* is isometrically isomorphic to $C_\infty = L(H)$. The isomorphism is given by the mapping

$$(*) \qquad C_p \ni U \to Tr(AU) \in \mathbf{C}, \quad \text{where} \quad A \in C_\infty.$$

P r o o f. This is an immediate consequence of Theorem 1.5 and the result mentioned above. We only check that (*) and the formula given in Theorem 1.5 are the same. In fact, if $U = \sum_{n=1}^{\infty} \mu_n f_n \otimes g_n$ then $f_A(U) = \sum_{n=1}^{\infty} (A(\mu_n f_n))(g_n) = \sum_{n=1}^{\infty} \mu_n\langle Af_n, g_n\rangle$. In turn, if (e_i) is an orthonormal basis in H, we have

$$\mathrm{Tr}(AU) = \mathrm{Tr}\Big(\sum_{n=1}^{\infty} \mu_n (Af_n)\otimes g_n\Big) = \sum_{i=1}^{\infty} \Big\langle\Big(\sum_{n=1}^{\infty} \mu_n (Af_n)\otimes g_n\Big)(e_i), e_i\Big\rangle =$$

$$= \sum_{i=1}^{\infty}\sum_{n=1}^{\infty} \langle \mu_n\langle e_i, g_n\rangle Af_n, e_i\rangle = \sum_{n=1}^{\infty} \mu_n \sum_{i=1}^{\infty}\langle Af_n, e_i\rangle\langle e_i, g_n\rangle =$$

$$= \sum_{n=1}^{\infty} \mu_n \langle Af_n, g_n\rangle. \blacksquare$$

Some properties of C_p (we may also define $C_o = K(H)$), i.e. inclusions, duality, are similar to those of sequence spaces c_o, l_p. We recall an interesting result of Grothendieck and we show that its analogue for C_p is not true. Let X and Y be Banach spaces. A bounded operator $T: X \to Y$ is said to be p-absolutely summing ($0 < p < \infty$) if there exists a positive constant K such that for every $n \in \mathbf{N}$ and $x_1, \dots, x_n \in X$

$$\sum_{i=1}^{n} \|Tx_i\|^p \leqslant K^p \cdot \sup\Big\{\sum_{i=1}^{n} |x^*(x_i)|^p : x^* \in X^*, \|x^*\| \leqslant 1\Big\}.$$

Grothendieck has proved [5] that every bounded operator $T: l_1 \to l_2$ is 1-absolutely summing.

T h e o r e m 2.2. The injection $C_p \hookrightarrow C_2$ is not a p-absolutely summing operator for $1 \leqslant p < 2$.

P r o o f . Let (e_i) be an orthonormal basis in H, $n \in \mathbf{N}$. Let us define $P_{ik}: H \to H$, $P_{ik}(e_l) = \delta_{il} e_k$. Then $\|P_{ik}\|_2 = 1$ and $\sum_{k=1}^{n} \|P_{ik}\|_2^p = n$. Now we estimate $\sup\Big\{\sum_{k=1}^{n} |\mathrm{Tr}(P_{1k}B)|^p : B \in C_\infty, \|B\| \leqslant 1\Big\}$. Let $(b_{ij}) = B(i,j)$ be the matrix of B in the basis (e_i). Then

$$P_{1k}B(i,j) = \begin{cases} b_{1j} & \text{if } i = k \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \mathrm{Tr}(P_{1k}B) = b_{1k}.$$

Since $\|B\| = \|B^*\| \geqslant \|B^* e_1\| = \left\| \sum_{k=1}^{\infty} \bar{b}_{1k} e_k \right\| \geqslant \left(\sum_{k=1}^{n} |b_{1k}|^2 \right)^{\frac{1}{2}}$, it is sufficient to compute the maximum of the function $h_p(b_{11},\dots,b_{1n}) = \sum_{k=1}^{n} |b_{1k}|^p$, $1 \leqslant p < 2$ if $\sum_{k=1}^{n} |b_{1k}|^2 = 1$.
It is well known, that h_p has its maximum if $|b_{11}| = \dots = |b_{1n}| = n^{-\frac{1}{2}}$. Consequently, we have

$$\max h_p(b_{11},\dots,b_{1n}) = n\cdot\left(n^{-\frac{1}{2}}\right)^p = n^{1-\frac{p}{2}}.$$

If the injection $C_p \hookrightarrow C_2$ is a p-absolutely summing operator, then we have for some positive constant K

$$n = \sum_{k=1}^{n} \|P_{1k}\|_2^p \leqslant K^p \sup\left\{\sum_{k=1}^{n} |Tr(P_{1k}B)|^p : B \in C_q,\ \|B\|_q \leqslant 1,\ \frac{1}{p}+\frac{1}{q}=1\right\}$$

$$\leqslant K^p \sup\left\{\sum_{k=1}^{n} |Tr(P_{1k}B)|^p : B \in C_\infty,\ \|B\| \leqslant 1\right\} = K^p \cdot n^{1-\frac{p}{2}}.$$

Since n may be arbitrarily large, we get the contradiction.■

3. The minorant property in C_p

Let (e_i) be an orthonormal basis in H. For $A \in L(H)$ we set $a_{ij} = \hat{A}(i,j) = \langle Ae_j, e_i\rangle$, $i,j \in \mathbf{N}$. Let $A,B \in L(H)$. We say that A is a minorant of B if $|a_{ij}| \leqslant b_{ij}$ for every $i,j \in \mathbf{N}$. Then we write $|\hat{A}| \leqslant \hat{B}$. $C_p(H)$ is said to have the minorant property (positive minorant property) if for every $A,B \in C_p(H)$ from $|\hat{A}| \leqslant \hat{B}$ ($0 \leqslant \hat{A} \leqslant \hat{B}$ respectively) it follows that $\|A\|_p \leqslant \|B\|_p$. It has been proved by Peller [9] that $C_p(H)$ has the minorant property if and only if $p = 2k$, for some $k \in \mathbf{N}$. Moreover, for every $p \neq 2k$ there exists $n \in \mathbf{N}$ such that $C_p(l_2^n)$ does not have the minorant property. Let $N(p) = \min\{n \in \mathbf{N} : C_p(l_2^n)$ does not have the minorant property$\}$. Dechamps-Gondim, Lust-Piquard and Queffélec proved [2] that

1) $N(p) \leqslant \left[\frac{p}{2}\right] + 2$, $1 \leqslant p < \infty$, $p \neq 2k$;

2) $C_p(H)$ has the minorant property if and only if it has the positive minorant property;

3) $C_p(l_2^2)$ does not have the positive minorant property for $1 \leqslant p < 2$.

We show that result for the case $0 < p < 1$.

Proposition 3.1. If $0 < p < 1$ then $C_p(l_2^2)$ does not have the minorant property nor the positive minorant property.

Proof. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Then $0 \leqslant \hat{A} \leqslant \hat{B}$ and $\|A\|_p = (\mathrm{Tr}\,|A|^p)^{\frac{1}{p}} = 2^{\frac{1}{p}}$. On the other hand, $B = |B|$ and the matrix of B in the basis of its eigenvectors is $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$, so $\|B\|_p = (\mathrm{Tr}\,|B|^p)^{\frac{1}{p}} = (2^p)^{\frac{1}{p}} = 2.$ ■

4. Schur multipliers on C_p, $0 < p \leqslant \infty$

Throughout this section we fix an orthonormal basis (e_i) in H. We consider bounded operators on H as matrices i.e. functions on $\mathbb{N} \times \mathbb{N}$. For $A \in L(H)$ let $a_{ij} = A(i,j) = \langle Ae_j, e_i \rangle$. If A, B are infinite matrices, we define their Schur product to be the matrix $A \bullet B$ such that $A \bullet B(i,j) = A(i,j) \cdot B(i,j)$. An infinite matrix Φ is said to be a Schur multiplier on C_p, if $\Phi \bullet A \in C_p$ whenever $A \in C_p$. Let $M(C_p)$ denote the set of all Schur multipliers on C_p. It follows from closed graph theorem (which is also true for p-Banach spaces) that

$$\|\Phi\|_{M(C_p)} = \sup\left\{\|\Phi \bullet A\|_p :\ A \in C_p,\ \|A\|_p \leqslant 1\right\} < \infty$$

for $\Phi \in M(C_p)$. It is easy to see that $M(C_p)$ is a Banach space with the norm $\|\cdot\|_{M(C_p)}$, if $1 \leqslant p \leqslant \infty$, and a p-Banach space with the p-norm $\|\cdot\|_{M(C_p)}$, if $0 < p < 1$. It is known that $M(C_1) = M(C_\infty)$ consists of all matrices of the form $(\langle x_i, y_j \rangle)$, where x_i, y_j are vectors in a Hilbert space, $\|x_i\| \leqslant c$, $\|y_j\| \leqslant c$ for some constant c (Grothendieck [5], Bennett [1]). $M(C_2)$ is the set of all bounded matrices, because $\|A\|_2 = \left(\sum_{i,k}^{\infty} |a_{ik}|^2\right)^{\frac{1}{2}}$. We are interested in the case $0 < p < 1$.

P r o p o s i t i o n 4.1. Let $A, B \in C_p$, $0 < p \leq 1$. Then $A \cdot B \in C_p$ and $\|A \cdot B\|_p \leq \|A\|_p \|B\|_p$.

P r o o f . We use the characterisation of C_p: $C_p = H \hat{\otimes}_p H$. Let $\varepsilon > 0$. Take such representations $A = \sum_{n=1}^{\infty} x_n \otimes y_n$ and $B = \sum_{m=1}^{\infty} u_m \otimes v_m$ that

$$\sum_{n=1}^{\infty} \|x_n\|^p \|y_n\|^p \leq \|A\|_p^p + \varepsilon \quad \text{and} \quad \sum_{m=1}^{\infty} \|u_m\|^p \|v_m\|^p \leq \|B\|_p^p + \varepsilon .$$

For $x, y \in H$ we define $x \cdot y \in H$ as follows: $\langle x \cdot y, e_i \rangle = \langle x, e_i \rangle \langle y, e_i \rangle$. Then

$$\|x \cdot y\| = \left(\sum_{i=1}^{\infty} |\langle x \cdot y, e_i \rangle|^2\right)^{\frac{1}{2}} \leq \sum_{i=1}^{\infty} |\langle x \cdot y, e_i \rangle| =$$

$$= \sum_{i=1}^{\infty} |\langle x, e_i \rangle| \, |\langle y, e_i \rangle| \leq \left(\sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{\infty} |\langle y, e_i \rangle|^2\right)^{\frac{1}{2}} =$$

$$= \|x\| \cdot \|y\|.$$

We have

$$A \cdot B = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (x_n \cdot u_m) \otimes (y_n \cdot v_m)$$

and

$$\|A \cdot B\|_p^p \leq \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \|x_n \cdot u_m\|^p \|y_n \cdot v_m\|^p \leq$$

$$\leq \sum_{n=1}^{\infty} \|x_n\|^p \|y_n\|^p \sum_{m=1}^{\infty} \|u_m\|^p \|v_m\|^p \leq (\|A\|_p^p + \varepsilon)(\|B\|_p^p + \varepsilon). \blacksquare$$

Consequently, we have $C_p \subset M(C_p)$ and $\|A\|_{M(C_p)} \leq \|A\|_p$.

P r o p o s i t i o n 4.2. $M(C_p) \subsetneq M(C_1)$ for $0 < p < 1$.

P r o o f . We use Corollary 2.1. Let $\Phi \in M(C_p)$, $B \in C_\infty$, $A \in C_p$. Then

$$|Tr((\Phi \cdot B)A)| = |Tr(B(\Phi^T \cdot A))| = |Tr(B^T(\Phi \cdot A^T))| \leq$$

$$\leq \|B\|_\infty \|\Phi \cdot A\|_1 \leq \|B\|_\infty \|\Phi \cdot A\|_p \leq \|B\|_\infty \|\Phi\|_{M(C_p)} \|A\|_p,$$

so $\Phi \in M(C_\infty) = M(C_1)$. A^T is the matrix transposed to A.

Now we give an example of a multiplier on C_1 which is not a multiplier on C_p for $0 < p < 1$. Let I be the matrix of the identity operator. Since $I = (\langle e_i, e_j \rangle)$, $I \in M(C_1)$. Let us define the matrices A_n for $n \in \mathbb{N}$:

$$A_n(i,j) = \begin{cases} \frac{1}{n} & i,j \leq n \\ 0 & \text{otherwise.} \end{cases}$$

Then we have $\|I \cdot A_n\|_p = n^{\frac{1}{p}-1}$, hence, $\|I \cdot A_n\|_p \to \infty$ if $n \to \infty$. On the other hand, $\|A_n\|_p = 1$, so I can not be a bounded (equivalently [11], a continuous) operator on C_p. ■

P r o p o s i t i o n 4.3. If $0 < p < q < 1$ then $M(C_p) \subset M(C_q)$.

P r o o f . We use the following result of Oloff [8]. Using the so called K-method of real interpolation he showed that, if $0 < p < r < \infty$ and $0 < \theta < 1$ then

$$(C_p, C_r)_{\theta,q;K} = C_q$$

where $\frac{1}{q} = \frac{1-\theta}{p} + \frac{\theta}{r}$ and the quasi-norms are equivalent. If $p < q < 1$ then we have for $\theta = \frac{q-p}{q-pq}$

$$(C_p, C_1)_{\theta,q;K} = C_q.$$

Let $\Phi \in M(C_p)$. Then $\Phi \in M(C_1)$ (Proposition 4.2), hence, $\Phi \in M(C_q)$ and

$$\|\Phi\|_{M(C_q)} \leqslant k\cdot\|\Phi\|^{1-\theta}_{M(C_p)}\|\Phi\|^{\theta}_{M(C_1)} \leqslant k\cdot\|\Phi\|_{M(C_p)}$$

for some constant k.■

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INSTITUTE OF MATHEMATICS, WROCŁAW UNIVERSITY,
50-384 WROCŁAW, POLAND

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