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A STATE CRITERION OF THE COMPLETENESS FOR INNER PRODUCT SPACES

We show that a separable real or complex inner product space is complete if and only if its orthomodular orthocomplemented orthoposet of all splitting subspaces possesses at least one state. This gives a new measure-theoretic characterization of Hilbert spaces among inner product spaces.

Preliminaries

Let V be a real or complex inner product space with an inner product (.,.). By a subspace of V we shall understand a linear closed submanifold of V. By $\bar{M}$ we denote the completion of M.

A subspace M of V is said to be strongly closed if $M = M^{\perp\perp}$, where $M^{\perp} = \{x \in V: (x,y) = 0 \text{ for all } y \in M\}$. Let us denote by L(V) the set of all strongly closed subspaces of V. It is well-known [1], that L(V) is an orthocomplemented complete latice with the operations $\wedge_L$, $\vee_L$ satisfying equalities

$$(1) \qquad \bigwedge_{t\in T}^{L} M_t = \bigcap_{t\in T} M_t, \qquad \bigvee_{t\in T}^{L} M_t = sp\Big(\bigcap_{t\in T} M_t\Big)^{\perp\perp},$$

where the symbol sp means the linear span.

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The main object of our study will be the set $E(V)$ of all splitting subspaces of V, i.e., the set of all subspaces M of V for which the condition $M \oplus M^{\perp} = V$ holds. The simple verification shows that any complete subspace, and therefore any finite-dimensional subspace, of V belongs to $E(V)$, and $E(V)$ is an orthocomplemented orthomodular orthoposet [6] for which

(2) $$E(V) \subseteq L(V).$$

Denote by $\wedge_E$ and $\wedge_E$ the meet and join in $E(V)$. Then $\bigwedge_{t\in T^E} M_t = \bigcap_{t\in T} M_t$ whenever the left-hand side exists in $E(V)$. Moreover, the de Morgan laws hold in $E(V)$.

T h e o r e m 1. Let $\{M_t : t \in T\}$ be a system of mutually orthogonal splitting subspaces of V. If $\bigvee_{t\in T^E} M_t$ exists in $E(V)$, then

(3) $$\bigvee_{t\in T^E} M_t = \Big(\sum_{t\in T} M_t\Big)^{\perp\perp} = \bigvee_{t\in T^L} M_t,$$

where $\sum_{t\in T} M_t = \{x \in V : x = \sum_{t\in T} x_t,\ x_t \in M_t\}$. If $T = \{1,2\}$, then $M_1 \oplus M_2 \in E(S)$ and

(4) $$M_1 \vee_E M_2 = M_1 \oplus M_2.$$

If $\{x_t : t \in T\}$ is an orthonormal basis of a splitting subspace M, then

(5) $$M = \bigvee_{t\in T^E} sp(x_t) = \sum_{t\in T} sp(x_t).$$

P r o o f . It is evident that $M_o = \sum_t M_t$ is a submanifold of V. Let $x_n \in M_o$ and $x_n \to x \in V$. Then $x_n = \sum_t x_t^n$, and $x = x_t + x_t^{\perp}$, where $x_t^n, x_t \in M_t$, $x_t^{\perp} \in M_t$ for all $n \geq 1$ and $t \in T$. Calculate

$$\|x_n - x\|^2 = \|\sum_t x_t^n - x_s - x_s^{\perp}\|^2 = \|(x_s^n - x_s) + \sum_{t\neq s} x_t^n - x_s^{\perp}\|^2 =$$

$$= \|x_s^n - x_s\|^2 + \|\sum_{t\neq s} x_t^n - x_s^{\perp}\|^2 \geq \|x_s^n - x_s\|^2 \to 0.$$

Hence, $x_t^n \to x_t$ for any $t \in T$.

If $x_t \neq 0$, then $(x, x_t/\|x_t\|) = \|x_t\|$ and using the Bessel inequality, $x_0 = \sum_t' x_t \in \bar{V}$, where $\bar{V}$ denotes the completion of V. We claim to show $x_0 = x$. Let $y \in M_t$, then $(x-x_0, y) = 0$. Hence, $x-x_0 \perp M_t$, and, consequently, $x-x_0 \perp \bar{M}_0$. On the other hand, $x-x_0 \in \bar{M}_0$, which gives $x = x_0$, and $x \in \sum_t M_t$.

Now we show that (3) holds. Let $x \in M_0$, then $x = \sum_t x_t$, and there exists a countable subset $T_0 = \{t_1, t_2, \ldots\} \subseteq T$ such that $\|x_t\|^2 = 0$ for any $t \in T - T_0$. Putting $x_n = \sum_{i=1}^{n} x_{t_i} \in M :=$ $:= \bigvee_{t\in T^E} M_t$, we have $x_n \to x$, so that $x \in M$. Therefore, $M_0 \subseteq M$ and $M_0^{\perp\perp} \subseteq M^{\perp\perp} = M$.

On the other hand, $M_t \subseteq M_0$. Hence, $M_0^{\perp} \subseteq M_t^{\perp}$ which yields $M_0^{\perp} \subseteq \bigcap_t M_t^{\perp} = M^{\perp}$. Therefore, $M \subseteq M_0^{\perp\perp}$ and $M = M_0^{\perp\perp}$.

Due to (1), we see that $\bigvee_{t^L} M_t = sp\left(\bigcup_t M_t\right)^{\perp\perp} \subset M_0^{\perp\perp}$, since $sp\left(\bigcup_t M_t\right) \subseteq M_0$. Inasmuch as $M_0^{\perp} = sp\left(\bigcup_t M_t\right)^{\perp}$, we conclude that (3) holds. Q.E.D.

The following two assertions follows either easily from the above or directly from [2]. The equality (5) is a consequence of (3). Q.E.D.

Let m be a mapping from $E(S) \to [0,1]$ such that

(i) $m(S) = 1$;

(ii) $m\left(\bigvee_{t\in T^E} M_t\right) = \sum_{t\in T} m(M_t)$ whenever $\{M_t : t \in T\}$ is a system of mutually orthogonal splitting subspaces for which the join exists in E(S).

If (ii) holds for any finite index set, any countable or any T, m is said to be a finitely additive state, state or totally additive state.

Analogically we define these notions for L(V).

T h e o r e m 2. Let x be a unit vector of V. The mapping $m_x: E(V) \to [0,1]$ defined via

$$m_x(M) = \|x_M\|^2, \quad M \in E(V),$$

where x_M is a unique vector from M such that $x = x_M + x_{M^\perp}$, $x_{M^\perp} \in M$, is a finitely additive state on E(V). Moreover, the system $\{m_x : x \in V, \|x\| = 1\}$ is a quite full system of finitely additive states, that is, the statement "if $m_x(M) = 1$, then $m_x(N) = 1$" implies $M \subseteq N$.

P r o o f . See [3].

It is well-known that Gleason [5] described all states for the case of a separable Hilbert space V, dim V $\neq$ 2.

The systems L(V) and E(V) have been used to determine the algebraic conditions in order to V would be a Hilbert space (i.e., $V = \bar{V}$).

T h e o r e m 3. The following statements are equivalent:

(i) V is complete.

(ii) L(V) is orthomodular ([1]).

(iii) E(V) is complete ([6]).

(iv) L(V) = E(V).

(v) E(V) is a σ-lattice ([2]).

(vi) E(V) is a quantum logic ([3]).

(vii) E(V) contains the join of any sequence of mutually orthogonal atoms of E(V) ([3]).

<u>State criterion of the completeness</u>

A very elegant characterization of Hilbert spaces among inner product spaces is given by Hamhalter and Pták [7]: A separable real inner product space is complete iff L(V) possesses at least one state. We note the assertion is valid also for any complex separable inner product space. This result has been generalized in [4] for inner product spaces whose orthogonal dimension is a non-measurable cardinal and also for any arbitrary space (in the latter case we demand the existence of a totally additive state on L(V)).

In the present section, we show that the assertion of Hamhalter and Pták is valid if we assume the existence of a state on the system of all splitting subspaces.

T h e o r e m 4. A separable inner product space is complete whenever E(V) possesses at least one state.

P r o o f . We try to extend m to $L(V) \supseteq E(V)$ using Theorem 1.

First of all we describe the state m on E(V). Let $S(V) = \{x \in V: \|x\| = 1\}$ and define a mapping $f: S(V) \to [0,1]$ via

$$f(x) = m(sp(x)), \quad x \in S(V).$$

Applying the Gleason theorem [5] on any finite-dimensional subspace M of V, we may show that there is a unique symmetric bilinear form t_M such that

$$f(x) = t_M(x,x), \quad x \in S(V) \cap M.$$

We shall now define a bilinear form t on V as follows: let x,y be two vectors of V and let M be a two-dimensional subspace containing f and g; then we put $t(x,y) = t_M(x,y)$. It is straightforward to verify that t is a well-defined positive symmetric bilinear form, moreover,

$$t(x,x) = m(sp(x)) \leqslant 1, \quad x \in S(V).$$

The form t may be extended uniquely to a bounded positive symmetric bilinear form $\bar{t}$ defined on whole $\bar{V}$. Hence, there is a unique positive Hermitian operator $U: \bar{V} \to \bar{V}$ such that

$$t(x,x) = (Ux,x), \quad x \in V.$$

The separability of V entails that V contains (and also any subspace M of V) at least one orthonormal basis $\{x_n\}$ of it. In view of (5),

$$1 = m(V) = m\Big(\bigvee_{n}{}^{E}\ sp(x_n)\Big) = \sum_n (Ux_n, x_n),$$

which gives U is of finite trace. Let M be an arbitrary splitting subspace. (5) entails that if $\{e_i\}$ is an orthonormal basis of M, then

$$(6)\qquad m(M) = m\Big(\bigvee_{i}{}^{E}\ sp(e_i)\Big) = \sum_i (Ue_i, e_i) = tr(UP_{\bar{M}}),$$

where $P_{\bar{M}}$ is the orthoprojector of $\bar{V}$ into $\bar{P}$. In the last equality we have used the fact that if $\{e_i\}$ is an orthonormal basis of M, so it is for $\bar{M}$.

The equality (6) enables us to extend m onto $\bar{m}$: L(V) → [0,1] via

$$(7)\qquad \bar{m}(M) = tr(U\bar{M}),\quad M \in L(V).$$

Hence, if $\{x_i\}$ is an orthonormal basis for M, then

$$\bar{m}(M) = \sum_i m(sp(x_i)).$$

Suppose now that $\{e_n\}_{n=1}^{\infty}$ is a maximal orthonormal system of vectors from V. It may be shown that

$$(8)\qquad \bigvee_{n=1}^{\infty}{}^{E}\ sp(x_n) = V.$$

Let $\{x_i\}$ and $\{y_j\}$ be orthonormal basis in M and $M^{\perp}$, respectively. Then $\{x_i\} \cup \{y_j\}$ is a maximal orthonormal system in V. Indeed, if there is $x \perp x_i$, $x \perp y_j$ for all i,j, then $x \perp M$, $x \perp M^{\perp}$ which gives $x \in M \cap M^{\perp}$, so that x = 0. According to (8),

(9) $$\bar{m}(M) + \bar{m}(M^{\perp}) = 1 \quad \text{for any} \quad M \in L(V).$$

Let now $\{M_n\}_{n=1}^{\infty}$ be a sequence of mutually orthogonal subspaces from L(V) with the join M in L(V). Let $\{x_j^n : j \in N_n\}$ and $\{x_j : j \in N_o\}$ be orthonormal bases in M_n and M. Then $\bigcup_{n=1}^{\infty} \{x_j^n : j \in N_n\} \cup \{x_j : j \in N_o\}$ is a maximal orthonormal system in V. Suppose the converse, then $x \perp x_j^n$, x_j, so that $x \perp M_n$, $x \perp M^{\perp}$. Therefore $M_n \subseteq sp(x)^{\perp}$, $M \subset \bigvee_n L\, M_n \subseteq sp(x)^{\perp}$, which gives $x \perp M$ and simultaneously $x \perp M^{\perp}$, so that $x = 0$.

In view of (8) and (9),

$$1 = \sum_{n=1}^{\infty} \sum_{j \in N_n} m(sp(x_j^n)) + \sum_{j \in N_o} m(sp(x_j)) = \sum_{n=1}^{\infty} \bar{m}(M_n) + \bar{m}(M^{\perp}) =$$

$$= \bar{m}(M) + \bar{m}(M^{\perp}).$$

Consequently, $\bar{m}(\bar{M}) = \sum_{n=1}^{\infty} \bar{m}(M_n)$, in other words, $\bar{m}$ is a state on L(V).

Applying the theorem of Hamhalter and Pták [7], we conclude finally V is complete. Q.E.D.

We remark in conclusion that E(V), for a separable incomplete inner product space V, gives an example of a stateless orthocomplemented orthomodular orthoposet with a quite full system of finitely additive states which is not a quantum logic.

In addition, we note that the presented methods are not applicable to nonseparable inner product spaces because of the possible nonexistence of any orthonormal basis in V, in general.

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