DEMONSTRATIO MATHEMATICA
Vol XXII No 4 1989

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ON RICCI-PSEUDO-SYMMETRIC WARPED PRODUCTS

1. Introduction

Let (M,g) be a connected n-dimensional ($n \geq 3$) Riemannian manifold of class C^∞ with not necessarily definite metric g. We denote by ∇, $\tilde{R}$, R, S and K the Levi-Civita connection, the curvature tensor, the Riemann-Christoffel curvature tensor, the Ricci tensor and the scalar curvature of (M,g), respectively.

A manifold (M,g) is said to be pseudo-symmetric [6] if the following condition is satisfied:

(*) at every point of M the tensors $\tilde{R}\cdot R$ and $Q(g,R)$ are linearly dependent.

This condition is trivially satisfied at points at which $R = R(1)$ (we note that the tensor $Q(g,R)$ vanishes at a point $x \in M$ if and only if $R(x) = R(1)(x)$). Thus the condition (*) is equivalent to the following relation

$$\tilde{R}\cdot R = L\, Q(g,R) \tag{1}$$

on the set $W = \{x \in M: R \neq R(1) \text{ at } x\}$, where L is a function on W. Recently, pseudo-symmetric manifolds were studied by various authors. It is easy to see that if the condition (*) holds on a Riemannian manifold (M,g) then also

(**) at every point of M the tensors $\tilde{R}\cdot S$ and $Q(g,S)$ are linearly dependent.

The condition (**) is equivalent to the following relation

$$\tilde{R}\cdot S = LQ(g,S) \tag{2}$$

on the set $U = \{x \in M : S \neq \frac{K}{n} g \text{ at } x\}$. Manifolds satisfying the condition (**) are called Ricci-pseudo-symmetric [8]. Obviously, any Ricci-semi-symmetric manifold ($\tilde{R}\cdot S = 0$, cf. [11]) is Ricci pseudo-symmetric. The conditions (*) and (**) are equivalent at all points of a manifold (M,g) at which the Weyl conformal curvature tensor C vanishes [3, Lemma 1.2] (cf. also [7, Lemma 3]).

Let I be an open interval of $\mathbb{R}$ considered with its standard metric $\underset{(1)}{g}$ and F a positive smooth function on I. If $(M_2, \underset{(2)}{g})$, $\dim M_2 \geq 2$, is an Einstein manifold, then the warped product $(I\times M_2, \underset{(1)}{g} \oplus F \underset{(2)}{g})$ is Ricci-pseudo-symmetric [8]. Moreover, all such Ricci-pseudo-symmetric warped products for which the manifold $(M_2, \underset{(2)}{g})$ is not necessarily Einstein manifold are determined in [8]. The Ricci-pseudo-symmetric warped products $(I\times M_2, \underset{(1)}{g} \oplus F \underset{(2)}{g})$, $\dim M_2 \geq 3$, are non pseudo-symmetric and non Ricci-semi-symmetric in general.

This paper contains some results on Ricci-pseudo-symmetric warped products $(M_1\times M_2, \underset{(1)}{g} \oplus F \underset{(2)}{g})$ for which $\dim M_1 \geq 1$. We give necessary and sufficient conditions for a warped product to be Ricci-pseudo-symmetric. In particular, we obtain necessary and sufficient conditions for a warped product of two Einstein manifolds to be Ricci-pseudo-symmetric. With the help of the above results, we construct various examples of manifolds of this type.

2. Ricci-pseudo-symmetric warped products

Let (M,g) be a Riemannian manifold. For a tensor field A of type $(0,p)$, $p \geq 1$, on M we define the tensor fields $\tilde{R}\cdot A$ and $Q(g,A)$ by the formulas

$$(\tilde{R}\cdot A)(X_1,\ldots,X_p;X,Y) = (\tilde{R}(X,Y)\cdot A)(X_1,\ldots,X_p) =$$

$$= -A(\tilde{R}(X,Y)X_1,X_2,\ldots,X_p) - \ldots - A(X_1,\ldots,X_{p-1},\tilde{R}(X,Y)X_p)$$

and

$$Q(g,A)(X_1,\dots,X_p;X,Y) = -((X\wedge Y)\cdot A)(X_1,\dots,X_p) =$$

$$= A((X\wedge Y)X_1,X_2,\dots,X_p) + \dots + A(X_1,\dots,X_{p-1},(X\wedge Y)X_p)$$

respectively, where $\tilde{R}(X,Y)$ and $X\wedge Y$ are derivations of the algebra of the tensor fields on M and $X_1,\dots,X_p,X,Y\in\mathfrak{X}(M)$, $\mathfrak{X}(M)$ being the Lie algebra of vector fields on M. These derivations are the extensions of the endomorphisms $\tilde{R}(X,Y)$ and $X\wedge Y$ of $\mathfrak{X}(M)$ defined by

$$\tilde{R}(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

and

$$(X\wedge Y)Z = g(Z,Y)X - g(Z,X)Y$$

respectively, where $X,Y,Z\in\mathfrak{X}(M)$.

For the Riemann-Christoffel curvature tensor R we define the tensor R(1) by $R(1) = \frac{K}{n(n-1)}G$, where G is given by

$$G(X_1,X_2,X_3,X_4) = g((X_1\wedge X_2)X_3,X_4),$$

$X_k\in\mathfrak{X}(M)$, $k = 1,\dots,4$.

Let $(M_i, \underset{(i)}{g})$ ($i = 1,2$, $\dim M_1 = p$, $\dim M_2 = n-p$, $1\leqslant p<n$) are Riemannian manifolds covered by systems of charts $\{V';x^a\}$ and $\{V';y^\alpha\}$, respectively. Let F be a positive smooth function on M_1. The warped product of $(M_1, \underset{(1)}{g})$ and $(M_2, \underset{(2)}{g})$ ([9], [2]) is the Cartesian product $M_1\times M_2$ with the metric $\underset{(1)}{g}\oplus F\underset{(2)}{g}$ (more precisely, $g = \pi_1^*\underset{(1)}{g} + (f\circ\pi_1)\pi_2^*\underset{(2)}{g}$, $\pi_i : M_1\times M_2 \longrightarrow M_i$ being the natural projections). Let $\{V'\times V''; u^1 = x^1,\dots,u^p = x^p, u^{p+1} = y^1,\dots,u^n = y^{n-p}\}$ be a product chart for $M_1\times M_2$. The components of g with respect to this chart are following

$$(3)\qquad g_{rs} = \begin{cases} \underset{(1)}{g}_{ab} & \text{if } r = a,\ s = b \\ F\underset{(2)}{g}_{\alpha\beta} & \text{if } r = \alpha,\ s = \beta, \\ 0 & \text{otherwise} \end{cases}$$

where a,b,c,d $\{1,\dots,p\}$, $\alpha,\beta,\gamma,\delta \in \{p+1,\dots,n\}$ and r,s,t,u,w $\in \{1,\dots,n\}$. The local components Γ^{r}_{st} of the Levi-Civita connection ∇ of $g \oplus F \underset{(2)}{g}$ are the following

$$\text{(4)} \quad \begin{cases} \Gamma^{a}_{bc} = \underset{(1)}{\Gamma^{a}_{bc}}, \quad \Gamma^{\alpha}_{\gamma\beta} = \underset{(2)}{\Gamma^{\alpha}_{\gamma\beta}}, \quad \Gamma^{a}_{\alpha\beta} = \frac{1}{2}\, \underset{(1)}{g^{ab}} F_b \underset{(2)}{g_{\alpha\beta}}, \\ \Gamma^{\alpha}_{a\beta} = \frac{1}{2F} F_a \delta^{\alpha}_{\beta}, \quad \Gamma^{\alpha}_{ab} = \Gamma^{a}_{\alpha b} = 0, \quad F_a = \frac{\partial}{\partial x^a}(F). \end{cases}$$

We shall indicate each object formed from $\underset{(i)}{g}$ by (i). The local components

$$R_{rstu} = g_{rw}\tilde{R}^{w}{}_{stu} = g_{rw}\left(\partial_u \Gamma^{w}_{st} - \partial_t \Gamma^{w}_{su} + \Gamma^{v}_{st}\Gamma^{w}_{vu} - \Gamma^{v}_{su}\Gamma^{w}_{vt}\right),$$

$$\partial_u = \frac{\partial}{\partial x^u},$$

of the tensor R and the local components S_{ts} of the tensor S of $\underset{(1)}{g} \oplus F \underset{(2)}{g}$ which may not vanish identically are the following

$$\text{(5)} \quad R_{abcd} = \underset{(1)}{R_{abcd}},$$

$$\text{(6)} \quad R_{\alpha ab\beta} = -\frac{1}{2} T_{ab} \underset{(2)}{g_{\alpha\beta}},$$

$$\text{(7)} \quad R_{\alpha\beta\gamma\delta} = F \underset{(2)}{R_{\alpha\beta\gamma\delta}} - \frac{1}{4} \Delta_1 F \underset{(2)}{G_{\alpha\beta\gamma\delta}},$$

$$\text{(8)} \quad S_{ab} = \underset{(1)}{S_{ab}} - \frac{n-p}{2F} T_{ab},$$

$$\text{(9)} \quad S_{\alpha\beta} = \underset{(2)}{S_{\alpha\beta}} - \frac{1}{2}\left(\operatorname{tr}(T) + \frac{n-p-1}{2F}\Delta_1 F\right) \underset{(2)}{g_{\alpha\beta}},$$

where

$$\text{(10)} \quad T_{ab} = \nabla_b F_a - \frac{1}{2F} F_a F_b, \quad \operatorname{tr}(T) = \underset{(1)}{g^{ab}} T_{ab}, \quad \Delta_1 F = \underset{(1)}{g^{ab}} F_a F_b.$$

The scalar curvature K of $\underset{(1)}{g} \oplus F\underset{(2)}{g}$ satisfies the equation

$$(11) \qquad K = \underset{(1)}{K} + \frac{1}{F}\underset{(2)}{K} - \frac{n-p}{F}\left(tr(T) + \frac{n-p-1}{4F}\Delta_1 F\right).$$

The only components of $\tilde{R}\cdot S$ which may not vanish are those related to

$$(12) \qquad (\tilde{R}\cdot S)_{abcd} = (\underset{(1)}{\tilde{R}}\cdot\underset{(1)}{S})_{abcd} - \frac{n-p}{2F}(\underset{(1)}{\tilde{R}}\cdot T)_{abcd},$$

$$(13) \quad (\tilde{R}\cdot S)_{a\alpha\beta b} = \frac{1}{2F}\left(\underset{(1)}{S}_{ac} - \frac{n-p}{2F}T_{ac} + \frac{1}{2F}\left(tr(T) + \frac{n-p-1}{2F}\Delta_1 F\,\underset{(1)}{g}_{ac}\right)\right)T^c{}_b g_{\alpha\beta}$$
$$- \frac{1}{2F}T_{ab}\underset{(2)}{S}_{\alpha\beta}, \quad T^c{}_b = g^{ca}T_{ab},$$

$$(14) \qquad (\tilde{R}\cdot S)_{\alpha\beta\gamma\delta} = (\underset{(2)}{\tilde{R}}\cdot\underset{(2)}{S})_{\alpha\beta\gamma\delta} - \frac{1}{4F}\Delta_1 F\, Q(\underset{(2)}{g},\underset{(2)}{S})_{\alpha\beta\gamma\delta}.$$

Further, in virtue of (3), (8) and (9), we can easily show that the only components of $Q(g,S)$ not identically zero are those related to

$$(15) \qquad Q(g,S)_{abcd} = Q(\underset{(1)}{g},\underset{(1)}{S})_{abcd} - \frac{n-p}{2F}Q(\underset{(1)}{g},T)_{abcd},$$

$$(16) \quad Q(g,S)_{a\alpha\beta b} = \underset{(1)}{g}_{ab}\underset{(2)}{S}_{\alpha\beta} - \left(\frac{1}{2F}\left(tr(T) + \frac{n-p-1}{2F}\Delta_1 F\right)\underset{(1)}{g}_{ab} + \right.$$
$$\left. + \underset{(1)}{S}_{ab} - \frac{n-p}{2F}T_{ab}\right) g_{\alpha\beta},$$

$$(17) \qquad Q(g,S)_{\alpha\beta\gamma\delta} = F\,Q(\underset{(2)}{g},\underset{(2)}{S})_{\alpha\beta\gamma\delta}.$$

T h e o r e m 1. Let $(M_i, \underset{(i)}{g})$, $i = 1,2$, be Riemannian manifolds and F a smooth positive function on M_1. For the manifold $(M_1\times M_2, \underset{(1)}{g} \oplus F\underset{(2)}{g})$ the condition $\tilde{R}\cdot S = L\,Q(g,S)$ holds if and only if the following relations are satisfied

$$(18) \quad (\underset{(1)}{\tilde{R}}\cdot\underset{(1)}{S})_{abcd} - LQ(\underset{(1)}{g},\underset{(1)}{S})_{abcd} = \frac{n-p}{F}\left((\underset{(1)}{R}\cdot H)_{abcd} - LQ(\underset{(1)}{g},H)_{abcd}\right),$$

(19) $$H_{ab}(\underset{(2)}{S}_{\alpha\beta} - \frac{1}{2F}(\mathrm{tr}(T) + \frac{n-p-1}{2F}\Delta_1 F)g_{\alpha\beta}) = $$
$$= H_{cb}\left(\underset{(1)}{S}{}^{c}{}_{a} - \frac{n-p}{2F}T^{c}{}_{a}\right)g_{\alpha\beta},$$

(20) $$(\underset{(2)(2)}{\tilde{R}\cdot S})_{\alpha\beta\gamma\delta} = \left(LF + \frac{1}{4F}\Delta_1 F\right)Q(\underset{(2)(2)}{g,S})_{\alpha\beta\gamma\delta},$$

where H is the tensor field of type (0,2) with local components

(21) $$H_{ab} = \frac{1}{2}T_{ab} + FL\underset{(1)}{g}_{ab}.$$

P r o o f . Combining the relations (12)-(14) with the relations (15)-(17) and (21) we obtain our assertion.

As an immediate consequence of Theorem 1 we get

C o r o l l a r y 1. Let $(M_i, \underset{(i)}{g})$ ($\dim M_i \geq 2$, $i = 1,2$) be Einstein manifolds and F a smooth positive function on M_1. For the manifold $(M_1 \times M_2, \underset{(1)}{g} \oplus F\underset{(2)}{g})$ the condition $\tilde{R}\cdot S = LQ(g,S)$ holds if and only if the following relations are satisfied

(22) $$(\underset{(1)}{\tilde{R}}\cdot H)_{abcd} = LQ(\underset{(1)}{g},H)_{abcd}$$

and

(23) $$\frac{F}{n-p}\left(\frac{1}{p}\underset{(1)}{K} - \frac{1}{(n-p)F}\underset{(2)}{K} + (n-p)L + \frac{1}{2F}(\mathrm{tr}(T) + \frac{n-p-1}{2F}\Delta_1 F)\right)H_{ab} = $$
$$= H_{ac}H^{c}{}_{b}.$$

R e m a r k 1. It is clear that if in the relation (20) $\underset{(2)}{S} \neq \frac{1}{n-p}\underset{(2)}{K}\underset{(1)}{g}$ on some subset $V'' \subset M_2$ then $LF + \frac{1}{4F}\Delta_1 F\Big|_{M_1\times\{u\}} = \mathrm{const}$ for any $u \in V''$.

3. Examples

We denote by $S^p(\rho) = \left\{x \in E^{p+1} : \langle x,x\rangle = \rho^2,\ \rho > 0\right\}$ the p-dimensional ($p \geq 2$) sphere of radius ρ centered at the origin of an Euclidean space E^{p+1} with usual scalar product $\langle\cdot,\cdot\rangle$. Let $e = (e^1,\dots,e^{p+1})$ be a fixed unit vector in E^{p+1}. Define a function Φ in E^{p+1} by

$$\Phi(x) = \langle x,e\rangle = \sum_{k=1}^{k=p+1} x^k e^k ,$$

where $x = (x^1,\dots,x^{p+1})$ and denote by

$$f = \Phi\Big|_{S^p(\rho)} \tag{24}$$

the restriction of Φ to $S^p(\rho)$. Further, denote by $\underset{(1)}{g}$ the standard metric tensor of $S^p(\rho)$ induced from $\langle\cdot,\cdot\rangle$.

L e m m a 1. ([5, Lemma 3]). (i) Let $\{U;u^a\}$ be a chart of $(S^p(\rho),\underset{(1)}{g})$ such that the function f is different from zero at each point of U. Then the function $F = f^2$ satisfies on U the following equalities

$$\text{(a)}\quad T_{ab} = -\frac{2F}{\rho^2} g_{ab}, \qquad \text{(b)}\quad \frac{1}{4F}\Delta_1 F = 1 - \frac{F}{\rho^2} . \tag{25}$$

(ii) Let $F = (f + k)^2$, where $k > \rho$ is a constant. If $\{U;u^a\}$ is a chart of $(S^p(\rho),\underset{(1)}{g})$, then the function F satisfies on U the following equalities

$$\text{(a)}\quad T_{ab} = -\frac{2}{\rho^2} f(f + k) g_{ab}, \qquad \text{(b)}\quad \frac{1}{4F}\Delta_1 F = 1 - \frac{f^2}{\rho^2} . \tag{26}$$

T h e o r e m 2. Let $\{U;u^a\}$ be a chart of $(S^p(\rho),\underset{(1)}{g})$ such that the function F defined in Lemma 1(i) is different from zero at each point of U and let $(M_2,\underset{(2)}{g})$ be an $(n-p)$-dimensional, $n-p \geqslant 2$, Riemannian manifold.

(i) The manifold $(U\times M_2,\underset{(1)}{g} \oplus F\underset{(2)}{g})$ is Ricci-pseudo-symmetric if and only if the condition

$$\underset{(2)}{\tilde{R}} \cdot \underset{(2)}{S} = Q(\underset{(2)}{g}, \underset{(2)}{S}) \tag{27}$$

holds on $(M_2, \underset{(2)}{g})$.

(ii) Let $(M_2, \underset{(2)}{g})$ be additionally of constant curvature and assume that

$$A = K - (n-p)(n-p-1). \tag{28}$$

If $A \neq 0$ then $(U \times M_2, \underset{(1)}{g} \oplus F\underset{(2)}{g})$ is non Ricci-semi-symmetric Ricci-pseudo-symmetric manifold. If $A = 0$ then $(U \times M_2, \underset{(1)}{g} \oplus F\underset{(2)}{g})$ is an Einstein manifold.

P r o o f . The assertion (i) is an immediate consequence of Lemma 1(i) and Theorem 1, where we suppose $L = \frac{1}{\rho^2}$. The manifold $(U \times M_2, \underset{(1)}{g} \oplus F\underset{(2)}{g})$ defined in (ii), in view of (i), is Ricci-pseudo-symmetric. The equality (13), by (25) and (28), turns into

$$(\tilde{R} \cdot S)_{a\alpha\beta b} = \frac{AL}{n-p} \underset{(1)}{g}_{ab} \underset{(2)}{g}_{\alpha\beta}.$$

So, if $A \neq 0$, the manifold $(U \times M_2, \underset{(1)}{g} \oplus F\underset{(2)}{g})$ is non Ricci-semi-symmetric. Further, applying the relations (25) and $A = 0$ in (8), (9) and (11) we get $S = \frac{K}{n} g$, which completes the proof.

In Examples 1 and 2 we state manifolds fulfilling the condition (27).

E x a m p l e 1. Let I be an open interval of the real line with the metric $\underset{(1)}{g}$, $\underset{(1)}{g}_{11} = -1$, and $F(t) = \exp(2t)$, $t \in I$. If $(M_3, \underset{(3)}{g})$, $\dim M_3 \geqslant 3$, is an Einstein manifold then the manifold $(M_2, \underset{(2)}{g}) = (I \times M_3, \underset{(1)}{g} \oplus F\underset{(3)}{g})$ satisfies (27) (cf. [8, Example 2]). If $(M_3, \underset{(3)}{g})$, $\dim M_3 \geqslant 3$, is a non Einstein Ricci-semi-symmetric manifold, then the manifold $(M_2, \underset{(2)}{g}) = (I \times M_3, \underset{(1)}{g} \oplus F\underset{(3)}{g})$ satisfies also (27) (cf. [8, Example 3]).

E x a m p l e 2. The manifold $(U \times M_2, \underset{(1)}{g} \oplus F\underset{(2)}{g})$ defined in Theorem 1 (ii) with $A \neq 0$ and $\rho = 1$ satisfies (27).

T h e o r e m 2. Let $(M_2, \underset{(2)}{g})$, $\dim M_2 \geq 2$, be an Einstein manifold and let F be the function defined in Lemma 1 (ii) on a sphere $(S^p(\rho), \underset{(1)}{g})$, $p \geq 2$. Then the manifold $(S^p(\rho) \times M_2, \underset{(1)}{g} \oplus F\underset{(2)}{g})$ is Ricci-pseudo-symmetric and non Ricci-semi-symmetric.

P r o o f . The Ricci-pseudo-symmetry follows immediately from Lemma 1 (ii) and Corollary 1, where we suppose $L = \frac{f}{f+k}\frac{1}{\rho^2}$. Applying the formulas (26) and (28) into (13) we obtain

$$(\tilde{R} \cdot S)_{a\alpha\beta b} = \frac{1}{\rho^2}\frac{f}{f+k}\left(\frac{A}{n-p} - \frac{k}{\rho^2}\left((n-2)f + k(p-1)\right)\right)\underset{(1)}{g_{ab}}\,\underset{(2)}{g_{\alpha\beta}},$$

which completes the proof.

C o r o l l a r y 2. Let $(M_2, \underset{(2)}{g})$, $\dim M_2 \geq 2$, be a compact Einstein manifold and let F be the function defined in Lemma 1 (ii) on a sphere $(S^p(\rho), \underset{(1)}{g})$, $p \geq 2$. The manifold $(S^p(\rho) \times M_2, \underset{(1)}{g} \oplus F\underset{(2)}{g})$ is a compact non Ricci-semi-symmetric Ricci-pseudo-symmetric manifold.

An example of a compact Ricci-pseudo-symmetric manifold is given also in [8, Remark 3.4].

T h e o r e m 3. Let $(M_2, \underset{(2)}{g})$, $\dim M_2 \geq 2$, be an Einstein manifold and let F be a function on an Euclidean space E^p, $p \geq 2$, defined by the formula $F(x) = \frac{1}{4}\left(\sum_{a=1}^{a=p} (x^a)^2 + k\right)^2$, where $x = (x^1, \ldots, x^p) \in E^p$, $\underset{(1)}{g}$ is the standard metric of E^p and k is a positive constant. Then the manifold $(E^p \times M_2, \underset{(1)}{g} \oplus F\underset{(2)}{g})$ is Ricci-pseudo-symmetric and non Ricci-semi-symmetric.

P r o o f . It is easy to verify that the following equations

$$(29)\ (a)\quad T_{ab} = 2(F)^{\frac{1}{2}} \underset{(1)}{g}_{ab}, \qquad (b)\quad \frac{1}{4F}\Delta_1 F = 2F^{\frac{1}{2}} - k,$$

hold on E^p (see [5, Theorem 8]). Corollary 1 (take $L = -F^{\left(-\frac{1}{2}\right)}$) implies that the manofild $(E^p \times M_2, \underset{(1)}{g} \oplus F\underset{(2)}{g})$ is Ricci-pseudo-symmetric. Applying the formula (29) into (13) we obtain

$$(\tilde{R}\cdot S)_{a\alpha\beta b} = \frac{1}{2} F^{-\frac{1}{2}}\Big((n-2)\big(2(F)^{\frac{1}{2}} - k\big) - (n-2p)k - \frac{2}{n-p} \underset{(2)}{K}\Big)\underset{(1)}{g}_{ab}\underset{(2)}{g}_{\alpha\beta},$$

which completes the proof.

In the above described examples of warped product Ricci-pseudo-symmetric manifolds $(M_1 \times M_2, \underset{(1)}{g} \oplus F\underset{(2)}{g})$ the manifold $(M_1, \underset{(1)}{g})$ is a manifold of constant curvature. We give now an example of a warped product Ricci-pseudo-symmetric manifold for which the manifold $(M_1, \underset{(1)}{g})$ is Ricci-pseudo-symmetric and not of constant curvature.

E x a m p l e 3. Let I be an open interval of the real line considered with its standard metric $\tilde{g}$, $\tilde{g}_{11} = \varepsilon$, $\varepsilon \in \{-1,1\}$, $\tilde{F}$ a function on I defined by $\tilde{F}(x^1) = \exp(b\, x^1)$, $x^1 \in I$, $b \in \mathbf{R} - \{0\}$ and $(M_3, \underset{(3)}{g})$, $\dim M_3 = p-1 \geq 3$, a not of constant curvature Einstein manifold with non zero scalar curvature. Then the manifold $(M_1, \underset{(1)}{g}) = (I \times M_3, \tilde{g} \oplus F\underset{(3)}{g})$ satisfies the conditions $\underset{(1)}{S} \neq \frac{1}{p} \underset{(1)}{K}\,\underset{(1)}{g}$ and

$$(30)\qquad \underset{(1)}{\tilde{R}}\cdot \underset{(1)}{S} = L\,Q(\underset{(1)}{g}, \underset{(1)}{S})$$

with

$$(31)\qquad L = -\frac{\varepsilon}{4} b^2$$

[8, Corollary 3.2]. Further, let $f = \varepsilon \partial_1 F^{\frac{1}{2}}$ and v be a covector field of local components $v_1 = \tilde{F}^{\frac{1}{2}}$, $v_2 = \ldots = v_p = 0$. The covector field v and the function f satisfy on $(M_1, \underset{(1)}{g})$ the equality ([10, p.145])

$$(32) \qquad \underset{(1)}{\nabla} v = f \underset{(1)}{g} .$$

Moreover, the relation

$$(33) \qquad df = -L v$$

holds on $(M_1, \underset{(1)}{g})$ ([8, Corollary 2.4]). From the last equation, by covariant differentiation and making use of (31) and (32) we get

$$(34) \qquad \underset{(1)}{\nabla}{}^2 f = -L f \underset{(1)}{g} .$$

Putting $F = f^2$ and using (34), (33) and (31) we can easily verify that the following relations are satisfied on $(M_1, \underset{(1)}{g})$

$$(35) \qquad \frac{1}{2}\left(\underset{(1)}{\nabla}{}^2 - \frac{1}{2F}\, dF \otimes dF\right) + L\, F \underset{(1)}{g} = 0$$

and

$$(36) \qquad L\, F + \frac{1}{4F} \nabla_1 F = 0.$$

To obtain our example we consider two cases. (i) Let $(M_2, \underset{(2)}{g})$, $\dim M_2 \geq 2$, be an Einstein manifold. Then the manifold $(M_1 \times M_2, \underset{(1)}{g} \oplus F \underset{(2)}{g})$ satisfies (2). In fact, in virtue of (35), (10), (21) and (30), the relations (18)-(20) are fulfilled. Thus, by Theorem 1, we obtain (2). (ii) Let $(M_2, \underset{(2)}{g})$, $\dim M_2 \geq 3$, be a non Einstein Ricci-semi-symmetric manifold. Then the manifold $(M_1 \times M_2, \underset{(1)}{g} \oplus F \underset{(2)}{g})$ satisfies (2). In the same way, as in (i), we can prove that the relations (18) and (19) are fulfilled. The relation (20) is a consequence of (36) and the equation $\underset{(2)}{\tilde{R}}\cdot\underset{(2)}{S} = 0$. Thus, by Theorem 1, (2) holds on $(M_1\times M_2, \underset{(1)}{g} \oplus F\underset{(2)}{g})$.

The final remark concernes of totally umbilical submanifolds.

R e m a r k 2. In view of [10, Theorem 1] (see also [1, Theorems 1 and 2]), examples 2 and 3 of [8] as well as examples of Ricci-pseudo-symmetric manifolds obtained in this paper, give rise to examples of Ricci-pseudo-symmetric totally umbilical submanifolds of Ricci-pseudo-symmetric manifolds.

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Received March 5, 1988.