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3-STRUCTURE GENERATED ON M × R BY A MANIFOLD WITH THE 3-STRUCTURE

Introduction

This paper deals with the problem reverse to that one discussed in [1].

The Riemannian manifold M^{4n-1} with the $3-\varepsilon$ almost contact structure $\left\{\underset{\alpha}{F}, \underset{\alpha}{\omega}, \underset{\alpha}{\eta}\right\}$ satisfying certain conditions is considered. Using this 3-structure, we are able to define on the manifold $M^{4n} = M^{4n-1}\times R$ three tensor fields $\underset{\alpha}{\tilde{F}}$, which satisfy the conditions introduced in [1]. Moreover, there is given an example of 3-structures and conditions of integrability and completeness of adequate structures.

1. The fundamental notations

Let M^{4n} be 4n-dimensional differential manifold on which the 3 tensor-fields $\underset{\alpha}{\tilde{F}}$ $(\alpha = 1,2,3)$ of the type (1,1) are defined satisfying the conditions

$$(1)\qquad \underset{\alpha}{\tilde{F}}\circ\underset{\alpha}{\tilde{F}} = \underset{\alpha}{\tilde{F}}^2 = \underset{\alpha}{\varepsilon}\tilde{I}, \qquad \underset{\alpha}{\tilde{F}}\circ\underset{\beta}{\tilde{F}} = \underset{\alpha\beta}{\varepsilon}\underset{\gamma}{\tilde{F}},$$

where $\underset{\alpha}{\varepsilon} = \pm 1$, $\underset{\alpha\beta}{\varepsilon} = \pm 1$, $\alpha \neq \beta \neq \gamma \neq \alpha$, $\tilde{I}$ - means the identity mapping on TM^{4n} and the coefficients $\underset{\alpha}{\varepsilon}, \underset{\alpha\beta}{\varepsilon}$ satisfy the following identities

$$\varepsilon_\alpha \varepsilon_{\beta\gamma} = \varepsilon_{\alpha\beta} \varepsilon_\gamma, \quad \varepsilon_{\alpha\beta} \varepsilon_{\alpha\gamma} = \varepsilon_{\beta\alpha} \varepsilon_{\gamma\alpha} = \varepsilon_\alpha \tag{2}$$

for $\alpha \neq \beta \neq \gamma \neq \alpha$.

The three-tensor fields $\tilde{F}_\alpha$, $\alpha = 1,2,3$ satisfying the conditions (1) are called the generalized 3-structure or briefly 3-structure on the manifold M^{4n} and designated $\{\tilde{F}_\alpha\}$.

We denote by $\Gamma(M^{4n})$ the modul of vector fields on a differentiable manifold M^{4n}.

On M^{4n} there exists the metric $\tilde{g}$ satisfying the condition:

$$\tilde{g}(\tilde{F}_\alpha \tilde{X}, \tilde{F}_\alpha \tilde{Y}) = \tilde{g}(\tilde{X},\tilde{Y}) \quad \text{for} \quad \alpha = 1,2,3 \tag{3}$$

and for any vector fields $\tilde{X}$, $\tilde{Y}$ on M^{4n}, i.e. $\tilde{X},\tilde{Y} \in \Gamma(M^{4n})$ (Theorem 1, [1]).

Let M^{4n-1} be a smooth, oriented hypersurface immersed in M^{4n}. We assume that there exists a smooth vector field N normal to M^{4n-1} with respect to the metric $\tilde{g}$ and $\tilde{g}(N,N) = 1$. Then for an arbitrary vector field $\tilde{X} \in \Gamma(M^{4n})$ we have the decomposition

$$\tilde{F}_\alpha \tilde{X} = F_\alpha \tilde{X} + \varepsilon_\alpha \omega_\alpha(\tilde{X})N, \qquad \alpha = 1,2,3, \tag{4}$$

where F_α denotes the tensor field of the type (1,1), $F_\alpha \tilde{X} \in \Gamma(M^{4n-1})$, ω_α-tensor field of the type (0,1), ([1]).

We introduce the notations

$$\eta_\alpha = F_\alpha N \in \Gamma(M^{4n-1}), \quad \lambda_\alpha = \omega_\alpha(N) \in R. \tag{5}$$

In particular we have

$$\tilde{F}_\alpha N = \eta_\alpha + \varepsilon_\alpha \lambda_\alpha N. \tag{6}$$

With respect to (4) we get

$$\tilde{F}_\alpha X = F_\alpha X + \varepsilon_\alpha \omega_\alpha(X)N \quad \text{for} \quad X \in \Gamma(M^{4n-1}). \tag{7}$$

In this way the 3-structure $\{\underset{\alpha}{\tilde{F}}\}$ on the manifold M^{4n} induces on an oriented hypersurface M^{4n-1} the tensor fields $\underset{\alpha}{F}$, $\underset{\alpha}{\omega}$, $\underset{\alpha}{\eta}$, $\alpha = 1,2,3$ satisfying the following conditions (Theorem 2, [1]):

$$
(8)\qquad \begin{cases}
\underset{\alpha}{F}^2 = \underset{\alpha}{\varepsilon}(I - \underset{\alpha}{\omega}\otimes\underset{\alpha}{\eta}),\\
\underset{\alpha}{\omega}\circ\underset{\alpha}{F} = -\underset{\alpha}{\varepsilon}\,\underset{\alpha}{\lambda}\,\underset{\alpha}{\omega},\\
\underset{\alpha}{F}\,\underset{\alpha}{\eta} = -\underset{\alpha}{\varepsilon}\,\underset{\alpha}{\lambda}\,\underset{\alpha}{\eta},\\
\underset{\alpha}{\omega}(\underset{\alpha}{\eta}) = 1 - \underset{\alpha}{\varepsilon}\,\underset{\alpha}{\lambda}^2,\\
\underset{\alpha}{F}\circ\underset{\beta}{F} = \underset{\alpha\beta\gamma}{\varepsilon}\,\underset{\gamma}{F} - \underset{\beta}{\varepsilon}\,\underset{\beta}{\omega}\otimes\underset{\alpha}{\eta},\\
\underset{\alpha}{\omega}\circ\underset{\beta}{F} = \underset{\beta\gamma}{\varepsilon}\,\underset{\gamma}{\omega} - \underset{\beta}{\varepsilon}\,\underset{\alpha}{\lambda}\,\underset{\beta}{\omega},\\
\underset{\alpha}{F}\,\underset{\beta}{\eta} = \underset{\alpha\beta}{\varepsilon}\,\underset{\gamma}{\eta} - \underset{\beta}{\varepsilon}\,\underset{\beta}{\lambda}\,\underset{\alpha}{\eta},\\
\underset{\alpha}{\omega}(\underset{\beta}{\eta}) = \underset{\beta\gamma}{\varepsilon}\,\underset{\gamma}{\lambda} - \underset{\beta}{\varepsilon}\,\underset{\alpha}{\lambda}\,\underset{\beta}{\lambda},
\end{cases}
$$

$\alpha \neq \beta \neq \gamma \neq \alpha$ and where I denotes the identity mapping on TM^{4n-1}.

This tensor fields are called the 3-ε almost contact structure on the manifold M^{4n} and denoted $\{\underset{\alpha}{F},\underset{\alpha}{\omega},\underset{\alpha}{\eta}\}$.

On the hypersurface M^{4n-1} we introduce the metric g induced by $\tilde{g}$ as follows

$$
(9)\qquad g(X,Y) = \tilde{g}(X,Y) \quad \text{for} \quad X,Y \in \Gamma(M^{4n-1}).
$$

For the metric g and the tensor fields $\underset{\alpha}{F}$, $\underset{\alpha}{\omega}$, $\underset{\alpha}{\eta}$ we have

$$
(10)\qquad g(X,\underset{\alpha}{\eta}) = \underset{\alpha}{\omega}(X), \quad g(\underset{\alpha}{F}X, \underset{\alpha}{F}Y) = g(X,Y) - \underset{\alpha}{\omega}(X)\underset{\alpha}{\omega}(Y)
$$

for arbitrary $X,Y \in \Gamma(M^{4n-1})$ (Theorem 3, [1]).

With respect to (1), (3) and (7) we obtain

1) If $\underset{\alpha}{\varepsilon} = -1$, then $\underset{\alpha}{\lambda} = 0$ and $\tilde{g}(N,\underset{\alpha}{\tilde{F}}(N)) = 0$.

2) If $\underset{\alpha}{\varepsilon} = 1$, then $\tilde{g}(N,\underset{\alpha}{\tilde{F}}(N)) = \underset{\alpha}{\lambda}$.

2. The 3-structure on $M^{4n} = M^{4n-1} \times R$

Let M^{4n-1} be a smooth Riemannian manifold with the 3-structure $\left\{\underset{\alpha}{F}, \underset{\alpha}{\omega}, \underset{\alpha}{\eta}\right\}$ and metric g, where $\underset{\alpha}{F}, \underset{\alpha}{\omega}, \underset{\alpha}{\eta}$ are the tensor fields on M^{4n-1} of the types (1,1), (0,1), (1,0) respectively, satisfying the conditions (8) and a Riemannian metric g on M^{4n-1} satisfies the conditions (10).

Let us consider a vector field $X \in \Gamma(M^{4n-1})$ and a vector field $a \frac{d}{dt} \in \Gamma(R)$. Then $\tilde{X} = X + a \frac{d}{dt} \in \Gamma(M^{4n})$ and we put

$$(11) \qquad \underset{\alpha}{\tilde{F}}(\tilde{X}) = \underset{\alpha}{F}X + a\underset{\alpha}{\eta} + \underset{\alpha}{\varepsilon}\,(\underset{\alpha}{\omega}(X) + \underset{\alpha}{\lambda} a)\,\frac{d}{dt} \,.$$

We know, that $T_{(p,t)}M^{4n} = T_pM \oplus T_tR$ and for any vector $w \in T_{(p,t)}M^{4n}$ there exists the vector field $X + a\frac{d}{dt} \in \Gamma(M^{4n})$ such that $X(p) + a(t)\frac{d}{dt} = w$. So the formula (11) defines the tensor field of the type (1,1) on M^{4n}.

T h e o r e m 1. The 3-ε almost contact structure $\left\{\underset{\alpha}{F}, \underset{\alpha}{\omega}, \underset{\alpha}{\eta}\right\}$ on M^{4n-1} generates the 3-structure $\left\{\underset{\alpha}{\tilde{F}}\right\}$ on M^{4n} satysfying

$$(1\ ) \qquad \underset{\alpha}{\tilde{F}}{}^2 = \underset{\alpha}{\varepsilon}\tilde{I}, \quad \underset{\alpha}{\tilde{F}} \circ \underset{\beta}{\tilde{F}} = \underset{\alpha\beta\gamma}{\varepsilon}\,\underset{\gamma}{\tilde{F}}, \quad \alpha \neq \beta \neq \gamma \neq \alpha,$$

where $\tilde{I}$ denotes the identity mapping on $\Gamma(M^{4n})$.

P r o o f . For $\tilde{X} \in \Gamma(M^{4n})$ we have

$$\begin{aligned} \underset{\alpha}{\tilde{F}}{}^2(\tilde{X}) &= \underset{\alpha}{\tilde{F}}(\underset{\alpha}{F}X + a\underset{\alpha}{\eta} + \underset{\alpha}{\varepsilon}\,(\underset{\alpha}{\omega}(X) + \underset{\alpha}{\lambda} a)\,\frac{d}{dt}) = \\ &= \underset{\alpha}{F}{}^2X + a\underset{\alpha}{F}\underset{\alpha}{\eta} + \underset{\alpha}{\varepsilon}\,(\underset{\alpha}{\omega}(X) + \underset{\alpha}{\lambda} a)\,\underset{\alpha}{\eta} + \\ &+ \underset{\alpha}{\varepsilon}\left[(\underset{\alpha}{\omega} \circ \underset{\alpha}{F})(X) + a\underset{\alpha}{\omega}(\underset{\alpha}{\eta}) + \underset{\alpha\alpha}{\varepsilon}\,\underset{\alpha}{\lambda}(\underset{\alpha}{\omega}(X) + \underset{\alpha}{\lambda} a)\right]\frac{d}{dt} \,. \end{aligned}$$

Then according to (8) we have

$$\underset{\alpha}{\tilde{F}}{}^2(\tilde{X}) = \underset{\alpha}{\varepsilon}\tilde{X}.$$

Similarly for $\alpha \neq \beta$ we obtain

$$(\underset{\alpha}{\tilde{F}} \circ \underset{\beta}{\tilde{F}})(\tilde{X}) = \underset{\alpha}{\tilde{F}}(\underset{\beta}{F}X + a\underset{\beta}{\eta} + \underset{\beta}{\varepsilon}\left[\underset{\beta}{\omega}(X) + \underset{\beta}{\lambda}a\right]\frac{d}{dt}) =$$

$$= (\underset{\alpha}{F} \circ \underset{\beta}{F})(X) + a\underset{\alpha}{F}(\underset{\beta}{\eta}) + \underset{\beta}{\varepsilon}(\underset{\beta}{\omega} \otimes \underset{\alpha}{\eta})(X) + \underset{\beta}{\varepsilon}\underset{\beta}{\lambda}a\underset{\alpha}{\eta} +$$

$$+ \underset{\alpha}{\varepsilon}\left[(\underset{\alpha}{\omega} \circ \underset{\beta}{F})(X) + a\underset{\alpha}{\omega}(\underset{\beta}{\eta}) + \underset{\beta}{\varepsilon}\underset{\alpha}{\lambda}\left[\underset{\beta}{\omega}(X) + \underset{\beta}{\lambda}a\right]\right]\frac{d}{dt} .$$

Then according to (8) we have

$$(\underset{\alpha}{\tilde{F}} \circ \underset{\beta}{\tilde{F}})(\tilde{X}) = \underset{\alpha\beta\gamma}{\varepsilon}\underset{\gamma}{\tilde{F}}(\tilde{X}) \quad \text{for} \quad \alpha \neq \beta \neq \gamma \neq \alpha.$$

We define the Riemannian metric $\tilde{g}$ on M^{4n} as follows

$$(12) \qquad \begin{cases} \tilde{g}(X,Y) = g(X,Y) \\ \tilde{g}(X, \frac{d}{dt}) = 0 \\ \tilde{g}(\frac{d}{dt}, \frac{d}{dt}) = 1 \end{cases}$$

for $X,Y \in \Gamma(M^{4n-1})$.

T h e o r e m 2. If the metric g on M^{4n-1} satisfies the conditions (10), then the metric $\tilde{g}$ on M^{4n} defined by conditions (12) is compatible with the 3-structure $\{\underset{\alpha}{\tilde{F}}\}$ on TM^{4n}, i.e. satisfies the condition

$$(3^*) \qquad \tilde{g}(\underset{\alpha}{\tilde{F}}\tilde{X}, \underset{\alpha}{\tilde{F}}\tilde{Y}) = \tilde{g}(\tilde{X},\tilde{Y})$$

for each $\tilde{X},\tilde{Y} \in \Gamma(M^{4n})$.

P r o o f . From (12) it follows that

$$\tilde{g}(\tilde{X},\tilde{Y}) = g(X,Y) + ab$$

for $\tilde{X} = X + a\frac{d}{dt}$, $\tilde{Y} = Y + b\frac{d}{dt}$, $X,Y \in \Gamma(M^{4n-1})$.

Then according to (10) and (8) we have

$$\tilde{g}(\underset{\alpha}{\tilde{F}}\tilde{X},\underset{\alpha}{\tilde{F}}\tilde{Y}) = g(\underset{\alpha}{F}X, \underset{\alpha}{F}Y) + bg(\underset{\alpha}{F}X, \underset{\alpha}{\eta}) + ag(\underset{\alpha}{\eta}, \underset{\alpha}{F}Y) +$$

$$+ abg(\underset{\alpha}{\eta},\underset{\alpha}{\eta}) + (\underset{\alpha}{\omega}(X) + a\underset{\alpha}{\lambda})(\underset{\alpha}{\omega}(Y) + b\underset{\alpha}{\lambda}) = g(X,Y) - \underset{\alpha}{\omega}(X)\underset{\alpha}{\omega}(Y) +$$

$$-\underset{\alpha}{\varepsilon}\, b\underset{\alpha\alpha}{\lambda\omega}(X) - \underset{\alpha}{\varepsilon}\, a\underset{\alpha\alpha}{\lambda\omega}(Y) + ab(1 - \underset{\alpha\alpha}{\varepsilon\lambda^2}) + \underset{\alpha}{\omega}(X)\underset{\alpha}{\omega}(Y) +$$

$$+ a\underset{\alpha\alpha}{\lambda\omega}(Y) + b\underset{\alpha\alpha}{\lambda\omega}(X) + ab\underset{\alpha}{\lambda^2} = \tilde{g}(X,Y) +$$

$$+ (1 - \underset{\alpha}{\varepsilon})\underset{\alpha}{\lambda}\left[b\underset{\alpha}{\omega}(X) + a\underset{\alpha}{\omega}(Y) + ab\underset{\alpha}{\lambda}\right].$$

Then if $\underset{\alpha}{\varepsilon} = 1$ or $\underset{\alpha}{\varepsilon} = -1$ (then $\underset{\alpha}{\lambda} = 0$) we obtain (3^*).

E x a m p l e . Let us take the following matrices of 4^{th} degree:

$$A'_1 = \begin{bmatrix} -\frac{2}{3} & 0 & \frac{\sqrt{2}}{3} & \frac{\sqrt{3}}{3} \\ 0 & 1 & 0 & 0 \\ \frac{\sqrt{2}}{3} & 0 & -\frac{1}{3} & \frac{\sqrt{6}}{3} \\ \frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{6}}{3} & 0 \end{bmatrix},$$

$$A'_2 = \begin{bmatrix} \frac{2}{3} & -\frac{\sqrt{6}}{6} & \frac{\sqrt{2}}{6} & \frac{\sqrt{3}}{3} \\ -\frac{\sqrt{6}}{6} & 0 & -\frac{\sqrt{3}}{3} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{6} & -\frac{\sqrt{3}}{3} & -\frac{2}{3} & -\frac{\sqrt{6}}{6} \\ \frac{\sqrt{3}}{3} & \frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} & 0 \end{bmatrix},$$

$$A'_3 = \begin{bmatrix} 0 & \frac{\sqrt{6}}{6} & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{3}}{3} \\ -\frac{\sqrt{6}}{6} & 0 & -\frac{\sqrt{3}}{3} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{6}}{6} \\ \frac{\sqrt{3}}{3} & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} & 0 \end{bmatrix}.$$

These matrices satisfy the conditions

$$(13)\quad \begin{cases} A_1'A_1' = I, \quad A_2'A_2' = I, \quad A_3'A_3' = -I \\ A_1'A_2' = -A_2'A_1' = A_3', \quad A_1'A_3' = -A_3'A_1' = A_2' \\ A_2'A_3' = -A_3'A_2' = -A_1', \end{cases}$$

where I is the identity matrix of 4^{th} degree. Crossing out the first line and the first column in each matrix we obtain the matrices:

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{3} & \frac{\sqrt{6}}{3} \\ 0 & \frac{\sqrt{6}}{3} & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & -\frac{\sqrt{3}}{3} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{3}}{3} & -\frac{2}{3} & -\frac{\sqrt{6}}{6} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} & 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & -\frac{\sqrt{3}}{3} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{6}}{6} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} & 0 \end{bmatrix},$$

Let

$$B_\alpha = \begin{bmatrix} A_\alpha' & 0 \\ 0 & A_\alpha \end{bmatrix} \quad \text{for} \quad \alpha = 1,2,3.$$

If $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^7}$ denotes a natural basis of $\Gamma(R^7)$, then we can define three tensor fields $\underset{\alpha}{F}$ of the type (1,1) having matrices $\underset{\alpha}{B}$ as matrices representations in this basis. If $X = x^1 \frac{\partial}{\partial x^1} + \ldots + x^7 \frac{\partial}{\partial x^7}$, then we define the forms:

$$\omega_1(X) = -\frac{\sqrt{6}}{9}x^6 - \frac{1}{3}x^7,$$

$$\omega_2(X) = \frac{\sqrt{2}}{6}x^5 - \frac{\sqrt{6}}{18}x^6 - \frac{1}{3}x^7,$$

$$\omega_3(X) = \frac{\sqrt{2}}{6}x^5 - \frac{\sqrt{6}}{6}x^6 - \frac{1}{3}x^7$$

and let

$$\underset{1}{\eta} = \left(0,\ 0,\ 0,\ 0,\ 0,\ -\frac{\sqrt{6}}{3},\ -1\right),$$

$$\underset{2}{\eta} = \left(0,\ 0,\ 0,\ 0,\ \frac{\sqrt{2}}{2},\ -\frac{\sqrt{6}}{6},\ -1\right),$$

$$\underset{3}{\eta} = \left(0,\ 0,\ 0,\ 0,\ \frac{\sqrt{2}}{2},\ -\frac{\sqrt{6}}{2},\ -1\right),$$

$$\underset{1}{\varepsilon} = \underset{2}{\varepsilon} = 1, \quad \underset{3}{\varepsilon} = -1, \quad \underset{12}{\varepsilon} = -\underset{21}{\varepsilon} = 1,$$

$$\underset{13}{\varepsilon} = -\underset{31}{\varepsilon} = 1, \qquad \underset{23}{\varepsilon} = -\underset{32}{\varepsilon} = -1,$$

$$\underset{1}{\lambda} = -\frac{2}{3}, \quad \underset{2}{\lambda} = \frac{2}{3}, \quad \underset{3}{\lambda} = 0.$$

Then conditions (8) are satisfied.

Let the metric g in R^+ be introduced by

$$g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = 0 \quad \text{for} \quad i \neq j,\ i,j = 1,2,\ldots,7,$$

$$g\left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^1}\right) = g\left(\frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^2}\right) = g\left(\frac{\partial}{\partial x^3}, \frac{\partial}{\partial x^3}\right) = g\left(\frac{\partial}{\partial x^4}, \frac{\partial}{\partial x^4}\right) = 1,$$

$$g\left(\frac{\partial}{\partial x^5}, \frac{\partial}{\partial x^5}\right) = g\left(\frac{\partial}{\partial x^6}, \frac{\partial}{\partial x^6}\right) = g\left(\frac{\partial}{\partial x^7}, \frac{\partial}{\partial x^7}\right) = \frac{1}{3}.$$

Let $R^8 = R^7 \times R$. Then we can define the (1,1) tensors $\underset{\alpha}{\tilde{F}}$ having as matrices representations in the basis $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^8}$ matrices

$$\underset{\alpha}{\tilde{B}} = \begin{bmatrix} \underset{\alpha}{A'} & 0 \\ 0 & \underset{\alpha}{\tilde{A}} \end{bmatrix},$$

where

$$\tilde{A}_\alpha = \begin{bmatrix} A_\alpha & \eta_\alpha \\ \omega_\alpha & \lambda_\alpha \end{bmatrix}$$

Hence, we have

$$\tilde{B}_1 = \begin{bmatrix} -\frac{2}{3} & 0 & \frac{\sqrt{2}}{3} & \frac{\sqrt{3}}{3} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\sqrt{2}}{3} & 0 & -\frac{1}{3} & \frac{\sqrt{6}}{3} & 0 & 0 & 0 & 0 \\ \frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{6}}{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & \frac{\sqrt{6}}{3} & -\frac{\sqrt{6}}{3} \\ 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{6}}{3} & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{6}}{9} & -\frac{1}{3} & -\frac{2}{3} \end{bmatrix},$$

$$\tilde{B}_2 = \begin{bmatrix} \frac{2}{3} & -\frac{\sqrt{6}}{6} & \frac{\sqrt{2}}{6} & \frac{\sqrt{3}}{3} & 0 & 0 & 0 & 0 \\ -\frac{\sqrt{6}}{6} & 0 & -\frac{\sqrt{3}}{3} & \frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 \\ \frac{\sqrt{2}}{6} & -\frac{\sqrt{3}}{3} & -\frac{2}{3} & -\frac{\sqrt{6}}{6} & 0 & 0 & 0 & 0 \\ \frac{\sqrt{3}}{3} & \frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{3} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{3} & -\frac{2}{3} & -\frac{\sqrt{6}}{6} & -\frac{\sqrt{6}}{6} \\ 0 & 0 & 0 & 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} & 0 & -1 \\ 0 & 0 & 0 & 0 & \frac{\sqrt{2}}{6} & -\frac{\sqrt{6}}{18} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix},$$

$$\tilde{B}_3 = \begin{bmatrix} 0 & \frac{\sqrt{6}}{6} & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{3}}{3} & 0 & 0 & 0 & 0 \\ -\frac{\sqrt{6}}{6} & 0 & -\frac{\sqrt{3}}{3} & \frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{6}}{6} & 0 & 0 & 0 & 0 \\ \frac{\sqrt{3}}{3} & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{3} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{6}}{6} & -\frac{\sqrt{6}}{2} \\ 0 & 0 & 0 & 0 & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} & 0 & -1 \\ 0 & 0 & 0 & 0 & \frac{\sqrt{2}}{6} & -\frac{\sqrt{6}}{6} & -\frac{1}{3} & 0 \end{bmatrix}.$$

Then

$$\tilde{B}_1\tilde{B}_1 = I, \quad \tilde{B}_2\tilde{B}_2 = I, \quad \tilde{B}_3\tilde{B}_3 = -I, \quad \tilde{B}_1\tilde{B}_2 = -\tilde{B}_2\tilde{B}_1 = \tilde{B}_3,$$

$$\tilde{B}_1\tilde{B}_3 = -\tilde{B}_3\tilde{B}_1 = \tilde{B}_2, \quad \tilde{B}_2\tilde{B}_3 = -\tilde{B}_3\tilde{B}_2 = -\tilde{B}_1$$

Hence, the tensors $\tilde{F}_\alpha$ satisfy conditions (1).

The metric $\tilde{g}$ defined by the equalities

$$\tilde{g}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right), \quad i,j = 1,\ldots,7$$

$$\tilde{g}\left(\frac{\partial}{\partial x^8}, \frac{\partial}{\partial x^8}\right) = 1, \quad \tilde{g}\left(\frac{\partial}{\partial x^8}, \frac{\partial}{\partial x^i}\right) = 0$$

satisfies conditions (3).

Generalization of this example to the space R^{4n-1} can be made replacing the matrices $\underset{\alpha}{A}'$ of the fourth degree by the matrices of $4(n-1)^{th}$ degree written in a cage form as follows

$$\underset{\alpha}{A}' = \begin{bmatrix} \underset{\alpha}{A}'^{1} & & & \\ & \underset{\alpha}{A}'^{2} & & \\ & & \ddots & \\ & & & \underset{\alpha}{A}'^{n-1} \end{bmatrix},$$

where $\underset{\alpha}{A}'^{i}$ are the matrices of the fourth degree and satisfying the conditions (13) for each $i = 1,2,\ldots,n-1$. Further construction is made as above.

3. The integrability conditions

Let us consider the integrability conditions for 3-structure $\{\underset{\alpha}{\tilde{F}}\}$.

D e f i n i t i o n . The 3-structure $\{\underset{\alpha}{\tilde{F}}\}$ on M^{4n} is α-integrable if the Nijenhuis tensor fields $\underset{\alpha}{\tilde{N}}$ disappears, where

$$\underset{\alpha}{\tilde{N}}(\tilde{X},\tilde{Y}) = \underset{\alpha\alpha}{\tilde{N}}(\tilde{X},\tilde{Y}) = 2\left\{\left[\underset{\alpha}{\tilde{F}}\tilde{X},\underset{\alpha}{\tilde{F}}\tilde{Y}\right] - \underset{\alpha}{\tilde{F}}\left[\underset{\alpha}{\tilde{F}}\tilde{X},\tilde{Y}\right] - \underset{\alpha}{\tilde{F}}\left[\tilde{X},\underset{\alpha}{\tilde{F}}\tilde{Y}\right] + \underset{\alpha}{\tilde{F}}\circ\underset{\alpha}{\tilde{F}}\left[\tilde{X},\tilde{Y}\right]\right.$$

for $\tilde{X},\tilde{Y}\in\Gamma(M^{4n})$.

D e f i n i t i o n . The 3-structure $\{\underset{\alpha}{\tilde{F}}\}$ on M^{4n} is integrable if all Nijenhuis tensor fields $\underset{\alpha\beta}{\tilde{N}}$ disappears where

$$\underset{\alpha\beta}{\tilde{N}}(\tilde{X},\tilde{Y}) = \left[\underset{\alpha}{\tilde{F}}\tilde{X},\underset{\beta}{\tilde{F}}\tilde{Y}\right] + \left[\underset{\beta}{\tilde{F}}\tilde{X},\underset{\alpha}{\tilde{F}}\tilde{Y}\right] + (\underset{\alpha\beta}{\tilde{F}\tilde{F}} + \underset{\beta\alpha}{\tilde{F}\tilde{F}})\left[\tilde{X},\tilde{Y}\right] - \underset{\alpha}{\tilde{F}}\left[\underset{\beta}{\tilde{F}}\tilde{X},\tilde{Y}\right] -$$
$$- \underset{\alpha}{\tilde{F}}\left[\tilde{X},\underset{\beta}{\tilde{F}}\tilde{Y}\right] - \underset{\beta}{\tilde{F}}\left[\underset{\alpha}{\tilde{F}}\tilde{X},\tilde{Y}\right] - \underset{\beta}{\tilde{F}}\left[\tilde{X},\underset{\alpha}{\tilde{F}}\tilde{Y}\right]$$

for $\tilde{X},\tilde{Y}\in\Gamma(M^{4n})$.

Let $\underset{\alpha}{N}$ and $\underset{\alpha\beta}{N}$ denote the Nijenhuis tensors corresponding to the 3-ε almost contact structure $\{\underset{\alpha}{F},\underset{\alpha}{\omega},\underset{\alpha}{\eta}\}$ on M^{4n-1}. However, let $\underset{\alpha}{\overset{\circ}{N}}$ and $\underset{\alpha\beta}{\overset{\circ}{N}}$ are components of the corresponding Nijenhuis tensors $\underset{\alpha}{\tilde{N}}$ and $\underset{\alpha\beta}{\tilde{N}}$ on M^{4n-1}. Hence

$$\tilde{\underset{\alpha}{N}}(X,Y) = \overset{\circ}{\underset{\alpha}{N}}(X,Y) + \underset{\alpha}{a}(X,Y)$$

$$\tilde{\underset{\alpha\beta}{N}}(X,Y) = \overset{\circ}{\underset{\alpha\beta}{N}}(X,Y) + \underset{\alpha\beta}{a}(X,Y)$$

for $X,Y \in \Gamma(M^{4n-1})$.

Theorem 3. Between the tensor fields $\overset{\circ}{\underset{\alpha}{N}}$, $\underset{\alpha}{N}$ and $\overset{\circ}{\underset{\alpha\beta}{N}}$, $\underset{\alpha\beta}{N}$ on M^{4n-1} we have the following relations:

$$(14)\qquad \begin{cases} \overset{\circ}{\underset{\alpha}{N}}(X,Y) = \underset{\alpha}{N}(X,Y) - 2\underset{\alpha}{\varepsilon} d\underset{\alpha}{\omega}(X,Y)\underset{\alpha}{\eta}, \\ \overset{\circ}{\underset{\alpha\beta}{N}}(X,Y) = \underset{\alpha\beta}{N}(X,Y) - \underset{\alpha}{\varepsilon} d\underset{\alpha}{\omega}(X,Y)\underset{\beta}{\eta} - \underset{\beta}{\varepsilon} d\underset{\beta}{\omega}(X,Y)\underset{\alpha}{\eta}, \end{cases}$$

where

$$d\underset{\alpha}{\omega}(X,Y) = \partial_X\underset{\alpha}{\omega}(Y) - \partial_Y\underset{\alpha}{\omega}(X) - \underset{\alpha}{\omega}([X,Y]).$$

Proof. We have

$$\begin{aligned}\tilde{\underset{\alpha}{N}}(X,Y) &= 2\Big\{\big[\underset{\alpha}{F}X,\underset{\alpha}{F}Y\big] + \underset{\alpha}{\varepsilon}\partial_{\underset{\alpha}{F}X}\underset{\alpha}{\omega}(Y)\frac{d}{dt} - \underset{\alpha}{\varepsilon}\partial_{\underset{\alpha}{F}Y}\underset{\alpha}{\omega}(X)\frac{d}{dt} + \\ &+ (\underset{\alpha}{F}\circ\underset{\alpha}{F})[X,Y] + \underset{\alpha}{\varepsilon}\underset{\alpha}{\omega}([X,Y])\underset{\alpha}{\eta} - \underset{\alpha}{F}\big[\underset{\alpha}{F}X,Y\big] - \underset{\alpha}{\varepsilon}\underset{\alpha}{\omega}(\big[\underset{\alpha}{F}X,Y\big])\frac{d}{dt} + \\ &+ \underset{\alpha}{\varepsilon}\partial_Y\underset{\alpha}{\omega}(X)\Big(\underset{\alpha}{\eta} + \underset{\alpha}{\varepsilon}\underset{\alpha}{\lambda}\frac{d}{dt}\Big) - \underset{\alpha}{F}\big[X,\underset{\alpha}{F}Y\big] - \underset{\alpha}{\varepsilon}\underset{\alpha}{\omega}(\big[X,\underset{\alpha}{F}Y\big])\frac{d}{dt} - \\ &- \underset{\alpha}{\varepsilon}\partial_X\underset{\alpha}{\omega}(Y)\Big(\underset{\alpha}{\eta} + \underset{\alpha}{\varepsilon}\underset{\alpha}{\lambda}\frac{d}{dt}\Big)\Big\} = \underset{\alpha}{N}(X,Y) - 2\underset{\alpha}{\varepsilon}\Big\{\partial_X\underset{\alpha}{\omega}(Y) - \partial_Y\underset{\alpha}{\omega}(X) - \\ &- \underset{\alpha}{\omega}([X,Y])\Big\}\underset{\alpha}{\eta} + 2\underset{\alpha}{\varepsilon}\Big\{\partial_{\underset{\alpha}{F}X}\underset{\alpha}{\omega}(Y) - \partial_{\underset{\alpha}{F}Y}\underset{\alpha}{\omega}(X) - \underset{\alpha}{\omega}(\big[\underset{\alpha}{F}X,Y\big]) - \\ &- \underset{\alpha}{\omega}(\big[X,\underset{\alpha}{F}Y\big]) + \underset{\alpha}{\varepsilon}\underset{\alpha}{\lambda}\partial_Y\underset{\alpha}{\omega}(X) - \underset{\alpha}{\varepsilon}\underset{\alpha}{\lambda}\partial_X\underset{\alpha}{\omega}(Y)\Big\}\frac{d}{dt}.\end{aligned}$$

Hence

$$\overset{\circ}{\underset{\alpha}{N}}(X,Y) = \underset{\alpha}{N}(X,Y) - 2\underset{\alpha}{\varepsilon} d\underset{\alpha}{\omega}(X,Y)$$

and

$$\begin{aligned}\underset{\alpha}{a}(X,Y) &= 2\underset{\alpha}{\varepsilon}\Big\{\partial_{\underset{\alpha}{F}X}\underset{\alpha}{\omega}(Y) - \partial_{\underset{\alpha}{F}Y}\underset{\alpha}{\omega}(X) - \underset{\alpha}{\omega}(\big[\underset{\alpha}{F}X,Y\big]) - \underset{\alpha}{\omega}(\big[X,\underset{\alpha}{F}Y\big]) + \\ &+ \underset{\alpha}{\varepsilon}\underset{\alpha}{\lambda}\partial_Y\underset{\alpha}{\omega}(X) - \underset{\alpha}{\varepsilon}\underset{\alpha}{\lambda}\partial_X\underset{\alpha}{\omega}(Y)\Big\}\frac{d}{dt},\end{aligned}$$

$$\begin{aligned}
\underset{\alpha\beta}{\tilde N}(X,Y) &= \Big[\underset{\alpha}{F}X,\underset{\beta}{F}Y\Big] + \underset{\beta}{\varepsilon}\,\partial_{\underset{\alpha}{F}X}\underset{\beta}{\omega}(Y)\frac{d}{dt} - \underset{\alpha}{\varepsilon}\,\partial_{\underset{\beta}{F}Y}\underset{\alpha}{\omega}(X)\frac{d}{dt} + \Big[\underset{\beta}{F}X,\underset{\alpha}{F}Y\Big] + \\
&+ \underset{\alpha}{\varepsilon}\,\partial_{\underset{\beta}{F}X}\underset{\alpha}{\omega}(Y)\frac{d}{dt} - \underset{\beta}{\varepsilon}\,\partial_{\underset{\alpha}{F}Y}\underset{\beta}{\omega}(X)\frac{d}{dt} + (\underset{\alpha\beta}{\varepsilon} + \underset{\beta\alpha}{\varepsilon})\Big(\underset{\gamma}{F}[X,Y] + \\
&+ \underset{\gamma}{\varepsilon}\,\underset{\gamma}{\omega}([X,Y])\frac{d}{dt}\Big) - \underset{\alpha}{\tilde F}\Big(\Big[\underset{\beta}{F}X,Y\Big] - \underset{\beta}{\varepsilon}\,\partial_Y\underset{\beta}{\omega}(X)\frac{d}{dt}\Big) - \underset{\alpha}{\tilde F}\Big(\Big[X,\underset{\beta}{F}Y\Big] + \\
&+ \underset{\beta}{\varepsilon}\,\partial_X\underset{\beta}{\omega}(Y)\frac{d}{dt}\Big) - \underset{\beta}{\tilde F}\Big(\Big[\underset{\alpha}{F}X,Y\Big] - \underset{\alpha}{\varepsilon}\,\partial_Y\underset{\alpha}{\omega}(X)\frac{d}{dt}\Big) - \underset{\beta}{\tilde F}\Big(\Big[X,\underset{\alpha}{F}Y\Big] + \\
&+ \underset{\alpha}{\varepsilon}\,\partial_X\underset{\alpha}{\omega}(Y)\frac{d}{dt}\Big) = \underset{\alpha\beta}{N}(X,Y) - \underset{\alpha}{\varepsilon}\Big\{\partial_X\underset{\alpha}{\omega}(Y) - \partial_Y\underset{\alpha}{\omega}(X) - \\
&- \underset{\alpha}{\omega}([X,Y])\Big\}\underset{\beta}{\eta} + \underset{\beta}{\varepsilon}\Big\{\partial_X\underset{\beta}{\omega}(Y) - \partial_Y\underset{\beta}{\omega}(X) - \underset{\beta}{\omega}([X,Y])\Big\}\underset{\alpha}{\eta} + \\
&+ \Big\{\underset{\alpha}{\varepsilon}(\partial_{\underset{\beta}{F}X}\underset{\alpha}{\omega}(Y) - \partial_{\underset{\beta}{F}Y}\underset{\alpha}{\omega}(X) - \underset{\alpha}{\omega}(\Big[\underset{\beta}{F}X,Y\Big]) - \underset{\alpha}{\omega}(\Big[X,\underset{\beta}{F}Y\Big])) + \\
&+ \underset{\beta}{\varepsilon}(\partial_{\underset{\alpha}{F}X}\underset{\beta}{\omega}(Y) - \partial_{\underset{\alpha}{F}Y}\underset{\beta}{\omega}(X) - \underset{\beta}{\omega}(\Big[\underset{\alpha}{F}X,Y\Big]) - \underset{\beta}{\omega}(\Big[X,\underset{\alpha}{F}Y\Big])) + \\
&+ (\underset{\alpha\beta}{\varepsilon} + \underset{\beta\alpha}{\varepsilon})\underset{\gamma}{\varepsilon}\,\underset{\gamma}{\omega}([X,Y]) + \underset{\alpha}{\varepsilon}\,\underset{\beta}{\varepsilon}(\underset{\alpha}{\lambda}\partial_Y\underset{\beta}{\omega}(X) - \underset{\alpha}{\lambda}\partial_X\underset{\beta}{\omega}(Y) + \\
&+ \underset{\beta}{\lambda}\partial_Y\underset{\alpha}{\omega}(X) - \underset{\beta}{\lambda}\partial_X\underset{\alpha}{\omega}(Y))\Big\}\frac{d}{dt}\,.
\end{aligned}$$

Hence we obtain

$$\underset{\alpha\beta}{\overset{\circ}{N}}(X,Y) = \underset{\alpha\beta}{N}(X,Y) - \underset{\alpha}{\varepsilon}\,d\underset{\alpha}{\omega}(X,Y)\,\underset{\beta}{\eta} - \underset{\beta}{\varepsilon}\,d\underset{\beta}{\omega}(X,Y)\,\underset{\alpha}{\eta},$$

and

$$\begin{aligned}
\underset{\alpha\beta}{a}(X,Y) &= \Big\{\underset{\alpha}{\varepsilon}(\partial_{\underset{\beta}{F}X}\underset{\alpha}{\omega}(Y) - \partial_{\underset{\beta}{F}Y}\underset{\alpha}{\omega}(X) - \underset{\alpha}{\omega}(\Big[\underset{\beta}{F}X,Y\Big]) - \\
&- \underset{\alpha}{\omega}(\Big[X,\underset{\beta}{F}Y\Big])) + \underset{\beta}{\varepsilon}(\partial_{\underset{\alpha}{F}X}\underset{\beta}{\omega}(Y) - \partial_{\underset{\alpha}{F}Y}\underset{\beta}{\omega}(X) - \underset{\beta}{\omega}(\Big[\underset{\alpha}{F}X,Y\Big]) - \\
&- \underset{\beta}{\omega}(\Big[X,\underset{\alpha}{F}Y\Big])) + (\underset{\alpha\beta}{\varepsilon} + \underset{\beta\alpha}{\varepsilon})\underset{\gamma}{\varepsilon}\,\underset{\gamma}{\omega}([X,Y]) + \underset{\alpha}{\varepsilon}\,\underset{\beta}{\varepsilon}(\underset{\alpha}{\lambda}\partial_Y\underset{\beta}{\omega}(X) - \\
&- \underset{\alpha}{\lambda}\partial_X\underset{\beta}{\omega}(Y) + \underset{\beta}{\lambda}\partial_Y\underset{\alpha}{\omega}(X) - \underset{\beta}{\lambda}\partial_X\underset{\alpha}{\omega}(Y))\Big\}\frac{d}{dt}\,.
\end{aligned}$$

T h e o r e m 4. If the induced 3-structure $\{\underset{\alpha}{\tilde F}\}$ is α-integrable, then

$$\underset{\alpha}{N}(X,Y) = 2\varepsilon \, d\underset{\alpha}{\omega}(X,Y)\underset{\alpha}{\eta}.$$

Thus it results in the case of 3-structure completeness of $\left\{\underset{\alpha}{F},\underset{\alpha}{\omega},\underset{\alpha}{\eta}\right\}$ on M^{4n-1} with regard to each operator:

$$\underset{\alpha}{N} = 2\varepsilon \, d\underset{\alpha}{\omega} \otimes \underset{\alpha}{\eta}.$$

Here completeness means that the value of Nijenhuis tensors for each operator are collinear to the corresponding $\underset{\alpha}{\eta}$ fields independent of the choice of the vector fields $X,Y \in \Gamma(M^{4n-1})$.

T h e o r e m 5. If induced 3-structure $\left\{\underset{\alpha}{\tilde{F}}\right\}$ is α-integrable then

$$\underset{\alpha\beta}{N}(X,Y) = \varepsilon \, d\underset{\alpha}{\omega}(X,Y)\underset{\beta}{\eta} + \varepsilon \, d\underset{\beta}{\omega}(X,Y)\underset{\alpha}{\eta}.$$

Hence, in the case of the completeness of the 3-structure $\left\{\underset{\alpha}{F},\underset{\alpha}{\omega},\underset{\alpha}{\eta}\right\}$ on M^{4n-1}

$$\underset{\alpha}{N} = 2\varepsilon \, d\underset{\alpha}{\omega}\otimes\underset{\alpha}{\eta}, \qquad \underset{\alpha\beta}{N} = \varepsilon \, d\underset{\alpha}{\omega}\otimes\underset{\beta}{\eta} + \varepsilon \, d\underset{\beta}{\omega}\otimes\underset{\alpha}{\eta}.$$

That is to say that Nijenhuis tensor values are included in the space spanned by $\underset{1}{\eta}, \underset{2}{\eta}, \underset{3}{\eta}$ independent of the choice of the fields $X,Y \in \Gamma(M^{4n-1})$.

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