DEMONSTRATIO MATHEMATICA
Vol XXII No 4 1989

Bronisław Tembrowski

QUASI-TOPOLOGICAL BOOLEAN ALGEBRAS OF REFLEXIVE AND QUASI-TOPOLOGICAL SPACES

Introduction

Boolean algebras with operators and their relational counterparts were investigated by Jonsson and Tarski in [4]. They pointed out that many substantial algebraic properties can be interpreted in terms of relational systems. They showed among others that topological Boolean algebras (TBA's) are related to quasi-ordered sets and that cylindric algebras are connected with relational systems involving two equivalence relations. The Jonsson and Tarski ideas have been pursued later for studying of many other sorts of algebras. A one-to-one correspondence between well-founded algebras and well-founded binary relational systems has been proved by Goldblatt in [2]. The class of diagonalisable algebras and its connections with finite well-founded transitive relational systems was investigated by Bernardi in [1]. Pseudo-Boolean algebras and their connections with some partially ordered sets were considered by Grzegorczyk in [3].

The aim of this paper is to describe main relationships of totally complete atomic quasi-topological Boolean algebras (TCA-QTBA's) and some of their subclasses with total quasi-topological spaces (TQTS's) and reflexive relational systems (called here reflexive spaces for short). The motivation for studying of these classes of algebras are their applications to the semantics of the sentential calculus with identity (SCI, [4]) and to the semantics of some Lewis modal systems (cf. [6] and [8]).

The work consists of two sections. At the beginning of the first one basic definitions and properties concerning quasi-topological Boolean algebras (QTBA's), quasi-topological fields (QTF's) and quasi-topological spaces (QTS's) are given. Then we formulate representation theorems pertaining to QTBA's, QTF's and quotient QTBA's. The further part of the section deals with relationships between TCA-QTBA's, TQTS's and reflexive spaces. In the second section we consider relationships between normal TCA-QTBA's, totally complete atomic self-conjugate quasi-topological Boolean algebras (TCA-self-conjugate QTBA's), totally complete atomic self-dual algebras (TCA-self-dual algebras), totally complete atomic H-algebras (TCA-H-algebras) and their quasi-topological and relational counterparts. The section presents also an algebraic construction of normal TCA-QTBA's, a quasi-topological construction of strongly compact TQTS's and a relational construction of IP-reflexive spaces.

The author wishes to thank to Prof. C. Rauszer for many valuable remarks that were very helpful during preparation of this paper.

1. Quasi-topological Boolean algebras and their quasi-topological and reflexive spaces

The first part of the section is devoted to basic properties concerning QTS's, QTF's and QTBA's. The main results are representation theorems for QTBA's, QTF's, quotient QTBA's as well as theorems stating that TCA-QTBA's, TQTS's and reflexive spaces are in a one-to-one correspondence.

Let X be a non-empty set and let P(X) be the powerset of X. A unary operation $C: P(X) \longrightarrow P(X)$ is said to be a quasi-closure (Q-closure) if $C(Y \cup Z) = C(Y) \cup C(Z)$, $Y \subseteq C(Y)$ and $C(\emptyset) = \emptyset$ for every subset Y, Z of X. A pair $\langle X, C\rangle$ is referred to as a quasi-topological space (QTS) if X is a non-empty set and C is a Q-closure operation on $P(X)$. It is clear that every topological space is a QTS but not conversely. Any QTS$\langle X,C\rangle$ is topological iff the operation C is idempotent. A subset Y of X is called quasi-closed (Q-closed) provided that $C(Y) = Y$. With the help of standard arguments one shows that the class $\underline{C}(X)$ of all Q-closed subsets of X with respect to the set-theoretical union and intersection forms a distributive $\cap$-complete lattice. The smallest element in $\underline{C}(X)$ is $\emptyset$ and the greatest one is X. Let us denote by $\underline{C}_*(X)$ the class of all elements of the form $C(Y)$ for $Y \subseteq X$. Obviously elements of $\underline{C}_*(X)$ are not Q-closed in general. The class $\underline{C}_*(X)$ with respect to the set-theoretical union is a $\cup$-semilattice. Note that a QTS$\langle X,C\rangle$ becomes a topological space iff $\underline{C}(X) = \underline{C}_*(X)$.

Just as in topological spaces is any QTS $\langle X,C\rangle$ one may consider a dual operation to C. Namely a unary operation $I: P(X) \longrightarrow P(X)$ such that $I(Y) = -C(-Y)$ for every $Y \subseteq X$ will be called a quasi-interior (Q-interior). Making use a straightforward calculation one proves that $I(Y \cap Z) = I(Y) \cap I(Z)$, $I(Y) \subseteq Y$ and $I(X) = X$ for every $Y,Z \subseteq X$. A subset Y of X is called quasi-open (Q-open) if $I(Y) = Y$. A subset Y of X is said to be quasi-clopen (Q-clopen) if it is Q-closed and Q-open. The class $\underline{I}(X)$ of all Q-open subsets of X with respect to the set-theoretical union and intersection is a distributive $\cup$-complete lattice, while the class $\underline{I}_*(X)$ of all subsets $I(Y)$ in X for $Y \subseteq X$ with respect to the set-theoretical intersection is a $\cap$-semilattice. The complement of a Q-closed (Q-open) set is a Q-open (Q-closed) set. It is clear that every QTS can be described by a Q-closure, or equivalently, by a Q-interior operation. Observe that lattices $\underline{C}(X)$ and $\underline{I}(X)$ related to the same QTS $\langle X,C\rangle$ are dually isomorphic. In fact, the function $f: \underline{C}(X) \longrightarrow \underline{I}(X)$ defined by $f(Y) = -Y$ for every $Y \subseteq X$ is a bijection such that $f(Y \cup Z) = f(Y) \cap f(Z)$, $f(Y \cap Z) = f(Y) \cup f(Z)$, $f(\emptyset) = X$ and $f(X) = \emptyset$ for every $Y,Z \subseteq X$.

In our investigations an important role will play QTS's supplied with an additional property imposed upon closure operations. A QTS$\langle X,C\rangle$ is said to be <u>total</u> (TQTS) if $C(Y) = \bigcup_{y\in Y} C(\{y\})$ for any subset Y of X. In the subclass of TQTS's Q-closed, Q-open and Q-clopen subsets can be interpreted equivalently by means of C-decreasing, C-increasing and C-stable subsets, respectively. Let $T = \langle X,C\rangle$ be a TQTS. Then a subset Y in T is called C-decreasing provided that $x \in Y$ and $y \in C(\{x\})$ imply $y \in Y$ for every $x,y \in X$. A subset Z in T is called C-increasing if $x \in Z$ and $x \in C(\{y\})$ imply $y \in Z$ for every $x,y \in X$. Finally a subset W in T is called C-stable if $x \in W$ and $x\Delta y$ imply $y \in W$ for every $x,y \in X$, where Δ is a binary relation on X such that $x\Delta y$ iff $x \in C(\{y\})$ or $y \in C(\{x\})$ for every $x,y \in X$.

The relationships between C-decreasing, C-increasing, C-stable subsets and, respectively, Q-open, Q-closed and Q-clopen subsets in TQTS's are described in the following lemma.

L e m m a 1.1. Let $T = \langle X,C\rangle$ be a TQTS and let Y be a subset of X. Then the following conditions hold:

(i) Y is C-decreasing iff it is Q-closed,

(ii) Y is C-increasing iff it is Q-open,

(iii) Y is C-stable iff it is Q-clopen.

P r o o f of (i). If Y is a C-decreasing subset in T, then $y \in C(Y)$ implies $y \in Y$ for every $y \in X$, which means that $C(Y) \subseteq Y$. But $Y \subseteq C(Y)$, we get $C(Y) = Y$. Conversely, let Y be Q-closed and let $x \in Y$, $y \in C(\{x\})$ for every $x,y \in X$. Then, $y \in C(Y) = Y$. Thus Y is a C-decreasing subset in T.

P r o o f of (ii). Let Y be a C-increasing subset in T. Then $x \in Y$ and $x \in C(\{y\})$ imply $y \in Y$ for every $x,y \in X$. Since $x \in I(Y)$ iff for every $y \in X$ if $x \in C(\{y\})$, then $y \in Y$, we obtain $Y \subseteq I(Y)$. But $I(Y) \subseteq Y$, it follows that $I(Y) = Y$. The proof of the second part of (ii) is easy.

P r o o f of (iii). If Y is a C-stable subset in T, then $x \in Y$ and $x\Delta y$ imply $y \in Y$ for every $x,y \in X$. From this, $x \in Y$ and $y \in C(\{x\})$ imply $y \in Y$ as well as $x \in Y$ and $x \in C(\{y\})$

imply $y \in Y$ for every $x,y \in X$. Using (i) and (ii), Y is a Q-clopen subset in T. Conversely, let Y be a Q-clopen subset in T. Then by (i), $x \in Y$ and $y \in C(\{x\})$ imply $y \in Y$ for every $x,y \in X$. By (ii) we get in turn that $x \in Y$ and $x \in C(\{y\})$ imply $y \in Y$ for every $x,y \in X$. Consequently Y is a C-stable subset in T.

For an illustration let us consider now two simple examples of TQTS's.

Example 1.1. Let $\underline{G} = \langle G,o \rangle$ be a non-abelian group. Then $N_x = \{y \in G: [x,y] = e\}$ is a normalizer of an element $x \in G$, where e is the unit in $\underline{G}$ and $[x,y]$ is the commutator of elements $x,y \in G$. On the powerset $P(G)$ we define a unary operation C by the formula $C(Y) = \bigcup_{y \in Y} N_y$ for every $Y \subseteq G$. It is easy to check that C is a Q-closure. Hence $T_G = \langle G,C \rangle$ is a QTS connected with the group $\underline{G}$. Since $C(Y) = \bigcup_{y \in Y} C(\{y\})$, T_G is a TQTS. Note that T_G is not a topological space because for any element $x \in G$ such that $x \neq e$ we have $C(C(\{x\})) \neq$ $\neq C(\{x\})$. Therefore C is a Q-closure which is not a topological closure.

Example 1.2. Let Z be the set of all integers. On $P(Z)$ we define a unary operation C as follows: $C(Y) =$ $= \{x \in Z: x \in Y \text{ or } (x+m) \in Y\}$ for every $Y \subseteq Z$, where m is some fixed positive integer. It is a routine matter to verify that $T_m = \langle Z,C \rangle$ is a TQTS. Observe that T_m is not a topological space since for any one-element subset $\{y\} \subseteq Z$, we get $C(\{y\}) =$ $= \{y,y-m\} \neq C(C(\{y\})) = \{y, y-m, y-2m\}$.

It is easy to check that subsets in T_m of the form $\{y,y-m,y-2m,\dots\}$ for $y \in Z$ are C-decreasing ones. Hence, by Lemma 1.1, these sets are examples of Q-closed subsets in T_G. The only C-stable subsets in this space are $\emptyset$ and G. According to Lemma 1.1, $\emptyset$ and G are at the same time the only C-clopen subsets in T_G.

An algebra $\underline{F}(X) = \langle F(X),-,\cup,\cap,C \rangle$ is called a quasi-topological field (QTF) over a non-empty set X provided that $\langle F(X),-,\cup,\cap \rangle$ is a Boolean field over X and C is a Q-closure operation on $F(X)$. From this definition it follows that any topological field is a QTF but not conversely. QTF's are closely related to QTS's. In fact, if $T = \langle X, C\rangle$ is a QTS, then $\underline{F}_T = \langle P(X), -, \cup, \cap, C\rangle$ is obviously a QTF. Also the converse statement holds. If $\underline{F}(X) = \langle F(X), -, \cup, \cap, C\rangle$ is a QTF, then $T_{\underline{F}} = \langle X, C\rangle$ is a QTS determined by $\underline{F}(X)$. The QTF $\underline{F}_T$ will be called a standard QTF of T and the QTS $T_{\underline{F}}$ will be called a standard QTS of $\underline{F}$. Let T_G and T_m be QTS's considered in Examples 1.1 and 1.2. Then $\underline{F}_{T_G} = \langle P(G), -, \cup, \cap, C\rangle$ is the standard QTF of T_G and $\underline{F}_{T_m} = \langle P(Z), -, \cup, \cap, C\rangle$ is the standard QTF of T_m. Neither $\underline{F}_{T_G}$ nor $\underline{F}_{T_m}$ is a topological field since the closure operators in these fields do not possess the idempotent property. Any QTF $\underline{F}(X) = \langle F(X), -, \cap, \cup, C\rangle$ over a non-empty set X is said to be a totally complete atomic QTF (TCA-QTF) if the reduct $\langle \underline{F}(X), -, \cup, \cap\rangle$ is a complete atomic Boolean field and C is a total Q-closure. Clearly QTF's $\underline{F}_{T_G}$ and $\underline{F}_{T_m}$ are examples of TCA-QTF's.

Let us recall (cf. [9]) that an algebra $\underline{A} = \langle A, -, \cup, \cap, C\rangle$ is a quasi-topological Boolean algebra (QTBA) if $\langle A, -, \cup, \cap\rangle$ is a Boolean algebra and $C: A \to A$ is a Q-closure operation on A satisfying the following properties: $C(a \cup b) = C(a) \cup C(b)$, $a \leq C(a)$ and $C(0) = 0$ for every $a, b \in A$. Any QTBA $A = \langle \underline{A}, -, \cup, \cap, C\rangle$ will be called a totally complete atomic QTBA (TCA-QTBA) if its Boolean reduct $\langle A, -, \cup, \cap\rangle$ is complete atomic and C is a total Q-closure, i.e. $C(a) = \bigcup^{\underline{A}}\{C(x): x \leq a,\ x \in At(\underline{A})\}$, where $At(A)$ denotes the set of all atoms in $\underline{A}$. For any TCA-QTBA there exists a TCA-QTF that is isomorphic to $\underline{A}$. Indeed, let $A = \langle A, -, \cup, \cap, C\rangle$ be a TCA-QTBA. Then the Boolean reduct of $\underline{A}$ is isomorphic to the field $\langle P(At(\underline{A})), -, \cup, \cap\rangle$. This isomorphism yields the function $h: A \to P(At(\underline{A}))$ such that $h(a) = \{x \in At(\underline{A}): x \leq a\}$ for every $a \in A$. Defining on $P(At(A))$ an additional operation $C^*: P(At(\underline{A})) \to P(At(\underline{A}))$ by the formula

(1.0) $C^*(Y) = h(C(h^{-1}(Y)))$ for every $Y \subseteq At(\underline{A})$, we get a TCA-QTF $\underline{P}(At(\underline{A})) = \langle P(At(\underline{A})), -, \cup, \cap, C^*\rangle$ which is isomorphic to $\underline{A}$. Thus TCA-QTBA's are represented by means of TCA-QTF's.

Passing to the whole class of QTBA's we may also formulate the analogous representation for QTBA's by means of QTF's. This expresses the following lemma.

L e m m a 1.2. Every QTBA is isomorphic to some QTF.

P r o o f . Let $\underline{A} = \langle A, -, \cup, \cap, C \rangle$ be any QTBA and let $h(a)$ be the set of all ultrafilters in $\underline{A}$ containing an element $a \in A$. Moreover let $h(\underline{A}) = \{h(a) : a \in A\}$. As it is known (cf. [7]), $h(\underline{A})$ with respect to the set-theoretical operations $-, \cup, \cap$ is the Stone field of the Boolean reduct of $\underline{A}$. The function $a \mapsto h(a)$ is the Stone isomorphism from $\langle A, -, \cup, \cap \rangle$ onto $\langle h(\underline{A}), -, \cup, \cap \rangle$. Next let us define on $h(\underline{A})$ an additional unary operation C^* such that $C^*(h(a)) = h(C(a))$ for every $a \in A$. Then we obtain a QTF $\langle h(\underline{A}), -, \cup, \cap, C^* \rangle$ which is isomorphic to $\underline{A}$.

From the above lemma it is seen that the subclass of QTF's in the class of QTBA's plays a similar role to that of topological fields play in the class of TBA's.

The following lemma presents the extension property for QTBA s.

L e m m a 1.3. Let $\underline{B} = \langle B, -, \cup, \cap, C_B \rangle$ be a QTBA whose Boolean reduct $\underline{B}_B = \langle B, -, \cup, \cap \rangle$ is a Boolean subalgebra of a complete Boolean algebra $\underline{A}_B = \langle A, -, \cup, \cap \rangle$. Then there exists a Q-closure operation C_A on A such that $\underline{A} = \langle A, -, \cup, \cap, C_A \rangle$ is a complete QTBA and $C_A(b) = C_B(b)$ for every element $b \in B$.

P r o o f . Let us define on A a unary operation C_A as follows $C_A(x) = \bigcap \{C_B(y) : x \leqslant y\}$ for every $x \in A$. Then we get a complete QTBA $\underline{A} = \langle A, -, \cup, \cap, C_A \rangle$ such that $C_A(b) = C_B(b)$ for all $b \in B$.

The basic relationship between QTBA's, TCA-QTF's and TQTS's yields the following representation theorem.

T h e o r e m 1.1. For any QTBA $\underline{A}$ there exists a TQTS $T_{\underline{A}}$ such that $\underline{A}$ is isomorphic to some subalgebra of the standard TCA-QTF $A_{T_{\underline{A}}}$ of $T_{\underline{A}}$.

P r o o f . Let $\underline{A} = \langle A, -, \cup, \cap, C_A \rangle$ be any QTBA and let $X(\underline{A})$ be the set of all ultrafilters in $\underline{A}$. Then by Lemma 1.2, $\underline{A}$ is isomorphic to the QTF $\underline{h}(\underline{A}) = \{h(\underline{A}), -, \cup, \cap, C^*_A\}$, where

$h(a) = \{x \in X(\underline{A}): a \in x\}$ and $C^*_A(h(a)) = h(C_A(a))$ for every $a \in A$. The Boolean reduct of $\underline{h}(\underline{A})$ is a subfield of the complete and atomic field $\langle P(X(\underline{A})),-,\cup,\cap\rangle$ of all subsets of $X(\underline{A})$. According to Lemma 1.3, the operation C on $P(X(\underline{A}))$ such that

$$(1.1) \qquad C(Y) = \bigcup_{y \in Y} \bigcup_{a \in y} h(C_A(a)) \quad \text{for every} \quad Y \in P(X(\underline{A}))$$

is a Q-closure satisfying $C(Y) = C^*_A(Y)$ for all $Y \in h(\underline{A})$. It is easy to see that C is also a total Q-closure. Hence, $\underline{P}(X(\underline{A})) = \langle P(X(\underline{A})),-,\cup,\cap,C\rangle$ is the standard TCA-QTF of the TQTS $T_{\underline{A}} = \langle X(\underline{A}),C\rangle$. Since h is a monomorphism from $\underline{A}$ into $\underline{P}(X(\underline{A}))$, the image $\underline{h}(\underline{A})$ is a subalgebra of $\underline{P}(X(\underline{A}))$. Thus $\underline{A}$ is isomorphic to $\underline{h}(\underline{A})$ which is a subalgebra of the TCA-QTF $\underline{P}(X(\underline{A}))$.

Now we will present a quasi-topological version of the topological McKinsey-Tarski theorem given in [5] (Theorem 2.5).

T h e o r e m 1.2. For every complete atomic QTF $\underline{F}_1$ there exists some infinite complete atomic QTF $\underline{F}_2$ such that $\underline{F}_1$ is isomorphic to a subfield of $\underline{F}_2$.

P r o o f . Let $\underline{F}_1 = \langle P(X),-,\cup,\cap,C_X\rangle$ be a complete atomic QTF over a non-empty set X and let $\bar{f}$ be a function on X with infinite values $\bar{f}(x)$, $x \in X$ such that $\bar{f}(x) \cap \bar{f}(y) = \emptyset$ whenever $\{x\} \cap \{y\} = \emptyset$ for every $x,y \in X$. Denote by $Y = \bigcup \{\bar{f}(x): x \in X\}$. Then $\bar{f}: X \to P(Y)$ can be extented to a Boolean homomorphism $f: P(X) \to P(Y)$ such that $f(X') = \bigcup \{\bar{f}(x): x \in X'\}$ for every $X' \subseteq X$. Now let us consider the function $g: P(Y) \to P(X)$ induced by $\bar{f}$ and defined by the formula $g(Y') = \bar{f}^{-1}(Y')$ for every $Y' \subseteq Y$. Then $g(Y) = X$ and $g(F) = \emptyset$ whenever F is a finite subset of Y. Note that $(g \circ f)(X') = g(f(X')) = i_X(X') = X'$ for all $X' \in P(X)$. Since the unary operation $C_Y: P(Y) \to P(Y)$ such that $C_Y(Y') = Y' \cup f(C_X(g(Y')))$ for every $Y' \in P(Y)$ is a Q-closure, $\underline{F}_2 = \langle P(Y),-,\cup,\cap,C_Y\rangle$ is a complete atomic QTF. It is easy to verify that the function f preserves all Boolean operations and fulfils the condition $f(C_X(X')) = f(X') \cup f(C_X(g(f(X')))) = C_Y(f(X'))$ for every $X' \in P(X)$. Hence f is a homomorphism from $\underline{F}_1$ to $\underline{F}_2$. Since $\ker(f) = \{\emptyset\}$, f is a monomorphism.

Therefore the image $f(\underline{F}_1)$ is a quasi-topological subfield of $\underline{F}_2$. Consequently, $\underline{F}_1$ is isomorphic to $f(\underline{F}_1)$ which is a quasi-topological subfield of the infinite complete atomic QTF $\underline{F}_2$.

The next theorem shows that quotient QTBA's (modulo I-filters) can be represented by means of QTF's constructed on closed subsets of the Stone topological spaces.

T h e o r e m 1.3. Any quotient QTBA $\underline{A}|\nabla$, where ∇ is an I-filter in a QTBA $\underline{A}$ is isomorphic to a quasi-topological subfield of some TCA-QTF built up on a closed subspace of the Stone space of the Boolean reduct of $\underline{A}$.

P r o o f . Let $\underline{A} = \langle A,-,\cup,\cap,C_A\rangle$ be a QTBA and let ∇ be an I-filter in $\underline{A}$. Then the relation $\tilde{\nabla}$ defined by the formula:

$$(1.2) \qquad a \tilde{\nabla} b \quad \text{iff} \quad I(a \div b) \in \nabla \quad \text{for all} \quad a,b \in A$$

is a congruence of $\underline{A}$, where $a \div b$ is the symmetric co-difference of elements $a,b \in A$ (cf. [9]). Hence, $\underline{A}|\nabla$ is the quotient QTBA of $\underline{A}$ modulo ∇. Let us assign to ∇ the subset $F = \bigcap\{h(a) : a \in \nabla\}$ in the Stone space $X(\underline{A})$ of the Boolean reduct of $\underline{A}$. As it is known, F is a closed subset in $X(\underline{A})$ and forms the topological subspace of $X(\underline{A})$ with the induced topology. The set $X(\underline{A})$ together with a Q-closure defined by (1.1) is a TQTS. Also the closed subset F with the operation C_F such that $C_F(Y \cap F) = C(Y) \cap F$ for every subset Y of $X(\underline{A})$ is a TQTS. Obviously $\underline{P}(F) = \langle P(F),-,\cup,\cap,C_F\rangle$ is a TCA-QTF. Let us take into account the mapping $h_F : A \twoheadrightarrow P(F)$ defined by $h_F(a) = h(a) \cap F$ for every $a \in A$. Clearly h_F is a homomorphism from $\underline{A}$ to $\underline{P}(F)$. Since $[a]_\nabla = [b]_\nabla$ implies $h_F(a) = h_F(b)$ for every $a,b \in A$, h_F induces a homomorphism $f : \underline{A}|\Delta \twoheadrightarrow \underline{P}(F)$ such that $f([a]_\nabla) = h_F(a)$ for every $a \in A$. But since $\ker(f) = \{\nabla\}$, f is a monomorphism. Then the image $f(\underline{A}|\Delta)$ is a quasi-topological subfield of $\underline{P}(F)$. Hence $\underline{A}|\nabla$ is isomorphic to $f(\underline{A}|\nabla) = \underline{h}_F(\underline{A})$. Thus we have shown that $\underline{A}|\nabla$ is isomorphic to $\underline{h}_F(\underline{A})$ which is a quasi-topological subfield of the TCA-QTF $\underline{P}(F)$ constructed on the closed subspace F of the Stone space $X(\underline{A})$.

Before we pass to examine relationships of TQTS's to TCA-QTBA's, let us make the following definition. If $T_1 = \langle X_1, C_1\rangle$ and $T_2 = \langle X_2, C_2\rangle$ are two TQTS's, then any function $f: X_1 \to X_2$ is called an isomorphism from T_1 onto T_2 ($T_1 \simeq T_2$) provided that f is a bijection and for every $Y \subseteq X_1$, $f(C_1(Y)) = C_2(f(Y))$.

L e m m a 1.4. Let T be a TQTS and let $\underline{A}_T$ be its standard TCA-QTF. Then there exists a TQTS $T_{\underline{A}_T}$ corresponding to $\underline{A}_T$ such that $T \simeq T_{\underline{A}_T}$.

P r o o f . If $T = \langle X, C\rangle$ is a TQTS, then $\underline{A}_T = \langle P(X), -, \cup, \cap, C\rangle$ is the standard TCA-QTF of T. Denote by $\bar{X} = \{\{x\}: x \in X\}$ the set of all atoms in $\underline{A}_T$. Next let us define on $\bar{X}$ the Q-closure operation C^* by the formula

$$(1.3)\qquad C^*(\bar{Y}) = \bigcup_{\{y\}\in\bar{Y}} C^*(\{y\}) \text{ for every } \bar{Y} \subseteq \bar{X}, \text{ where}$$
$$C^*(\{y\}) = \{\{x\} \in \bar{X}: x \in C(\{y\})\}.$$

Then $T_{\underline{A}_T} = \langle \bar{X}, C^*\rangle$ is a TQTS. It is not hard to verify that the function $f: X \to \bar{X}$ such that $f(x) = \{x\}$ for every $x \in X$ is an isomorphism from T onto $T_{\underline{A}_T}$.

L e m m a 1.5. Let $\underline{A} = \langle A, -, \cup, \cap, C_A\rangle$ be a TCA-QTBA. Then there exists a TQTS $T_{\underline{A}}$ determined by $\underline{A}$ such that the standard TCA-QTF $\underline{A}_{T_{\underline{A}}}$ of $T_{\underline{A}}$ is isomorphic to $\underline{A}$.

P r o o f . $\underline{A}$ TCA-QTBA $A = \langle A, -, \cup, \cap, C_A\rangle$ determines a TQTS $T_{\underline{A}} = \langle At(\underline{A}), C\rangle$, where $At(\underline{A})$ denotes as usual the set of all atoms in $\underline{A}$, $C(Y) = \bigcup\{C(\{y\}): y \in Y\}$ for every $Y \subseteq At(\underline{A})$ and $C(\{y\}) = \{x \in At(\underline{A}): x \leqslant C_A(y)\}$ for every $y \in At(\underline{A})$. The space $T_{\underline{A}}$ determines in turn the standard TCA-QTF $\underline{A}_{T_{\underline{A}}} = \langle P(At(\underline{A})), -, \cup, \cap, C\rangle$. As it is well known (cf. [7]) the function $a \mapsto h(a) = \{x \in At(\underline{A}): x \leqslant a\}$ is a Boolean isomorphism from $\underline{A}$ onto $\underline{A}_{T_{\underline{A}}}$. Furthermore, h satisfies the condition $h(C_A(b)) = C(h(b))$ for every $b \in A$. Indeed, $h(C_A(b)) =$

$= h\left(\bigcup_{y\leq b}^{\underline{A}} C_A(y)\right) = \bigcup_{y\leq b}\{x \in At(\underline{A}):\ x \leq C_A(y)\} = \bigcup_{y\leq b} C(\{y\}) =$

$= \bigcup_{y\in h(b)} C(h(y)) = C(h(b))$. Thus $\underline{A} \cong \underline{A}_{T_{\underline{A}}}$.

Immediately from Lemmas 1.4 and 1.5 one derives the following result establishing connections between classes of TCA-QTBA's and TQTS's.

T h e o r e m 1.4. There exists a one-to-one correspondence between TCA-QTBA's and TQTS's.

Applying the Jonsson-Tarski ideas of interpreting algebraic notions by means of concepts of relational system (cf. [4]), we will consider now connections of this sort for the class of TCA-QTBA's. Any pair $\underline{X} = \langle X,R\rangle$ will be called a reflexive space if R is a binary reflexive relation on a non-empty set X. Reflexive spaces are closely related to TCA-QTBA's. In fact, if $\underline{X} = \langle X,R\rangle$ is a given reflexive space, then the unary operation $C_R: P(X) \longrightarrow P(X)$ defined by the formula

$$C_R(Y) = \bigcup_{y\in Y} R(y) \quad \text{for any} \quad Y \subseteq X, \tag{1.4}$$

where $R(y) = \{x \in X:\ xRy\}$ is a total Q-closure operation. Hence, $\underline{A}_{\underline{X}} = \langle P(X),-,\cap,\cup,C_R\rangle$ is a TCA-QTF and $T_{\underline{X}} = \langle X,C_R\rangle$ is a TQTS. Any TCA-QTF $\underline{A}_{\underline{X}}$ as well as any TQTS $T_{\underline{X}}$ obtained in this manner will be called, respectively, a standard TCA-QTF and standard TQTS determined by $\underline{X}$. Let $T_G = \langle G,C\rangle$ be a TQTS considered in Example 1.1. Then it is determined by the reflexive space $\underline{X}_G = \langle G,R\rangle$ in which R is defined by the formula $x\ R\ y$ iff $x \circ y = y \circ x$ for every $x,y \in G$. Indeed, $C(Y) = \bigcup_{y\in Y} N_y = \bigcup_{y\in Y} R(y) = C_R(Y)$ for every $Y \subseteq G$. Likewise one shows that the TQTS $T_m : \langle Z,C\rangle$ from Example 1.2 is also determined by some reflexive space. To see that, let us define a binary relation R on Z by the formula $x\ R\ y$ iff $x = y$ or $y = x + m$ for every $x,y \in Z$. A straightforward calculation shows that R is reflexive and such that $C(Y) = \bigcup_{y\in Y} R(y) = C_R(Y)$ for every $Y \subseteq Z$, that is, $C = C_R$. So, T_m is determined by the reflexive space $\underline{X}_{T_m} = \langle Z,R\rangle$.

Recall now the well known definition of an isomorphism between two relational systems. If $\underline{X}_1 = \langle X_1,R_1\rangle$ and $\underline{X}_2 = \langle X_2,R_2\rangle$ are any reflexive spaces, then a function $f: X_1 \to X_2$ is said to be an isomorphism from $\underline{X}_1$ onto $\underline{X}_2$ provided that f is a bijection and it satisfies the condition: $x\ R_1\ x'$ iff $f(x)\ R_2\ f(x')$ for every elements $x,x' \in X_1$. Two reflexive spaces $\underline{X}_1$ and $\underline{X}_2$ are isomorphic if there exists an isomorphism from $\underline{X}_1$ onto $\underline{X}_2$ (in symbols $\underline{X}_1 \simeq \underline{X}_2$).

The next two lemmas describe main connections between TCA-QTBA's and reflexive spaces.

L e m m a 1.6. Let $\underline{X} = \langle X,R\rangle$ be a reflexive space. Then the standard TCA-QTF $\underline{A}_{\underline{X}}$ determines some reflexive space $\underline{X}_{\underline{A}_X}$ which is isomorphic to $\underline{X}$.

P r o o f . On the set of atoms $At(\underline{A}_{\underline{X}})$ of the algebra $\underline{A}_{\underline{X}}$ we define a binary relation R_{C_R} by the formula:

(1.5) $\{x\}R_{C_R}\{y\}$ if $\{x\} \subseteq C_R(\{y\})$ for every $\{x\},\{y\} \in At(\underline{A}_{\underline{X}})$.

Then $\underline{X}_{\underline{A}_X} = \langle At(\underline{A}_{\underline{X}}),R_{C_R}\rangle$ is a reflexive space such that $\underline{X} \simeq \underline{X}_{\underline{A}_X}$.

L e m m a 1.7. Let $\underline{A} = \langle A,-,\cup,\cap,C_A\rangle$ be a TCA-QTBA. Then it determines a reflexive space $\underline{X}_{\underline{A}}$ such that the standard TCA-QTF $\underline{A}_{\underline{X}_A}$ of $\underline{X}_{\underline{A}}$ is isomorphic to $\underline{A}$.

P r o o f . On the set of all atoms $At(\underline{A})$ of $\underline{A}$ we define a binary relation $R_{\underline{A}}$ as follows:

(1.6) $x\ R_{\underline{A}}\ y$ iff $x \leq C_{\underline{A}}(y)$ for every $x,y \in At(\underline{A})$.

Then $\underline{X}_{\underline{A}} = \langle At(\underline{A}),R_{\underline{A}}\rangle$ is a reflexive space. This space determines in turn the standard TCA-QTF $\underline{A}_{\underline{X}_A} = \langle P(At(\underline{A})),-,\cup,\cap,C_{R_A}\rangle$, where C_{R_A} is defined by (1.4). On the other hand, using Lemma 1.2, $\underline{A}$ is isomorphic to the TCA-QTF $\underline{h}(\underline{A}) = \langle h(\underline{A}),-,\cup,\cap,C^*_{\underline{A}}\rangle$.

But since $C^*_{\underline{A}}(Y) = C_{R_{\underline{A}}}(Y)$ for all $Y \subseteq At(\underline{A})$, we get finally that $\underline{A}_{\underline{X}_A} \cong \underline{A}$.

By virtue of Lemmas 1.6, 1.7 and Theorem 1.4 we obtain the following theorem describing fundamental relationships between TCA-QTBA's, TQTS's and reflexive spaces.

T h e o r e m 1.5. The classes of TCA-QTBA's, TQTS s and reflexive spaces are in a one-to-one correspondence.

2. Some subclasses of TCA-QTBA's and their quasi-topological and relational counterparts

This section deals with normal TCA-QTBA's, self-conjugate TCA-QTBA's, TCA-self-dual algebras, TCA-H-algebras and their relations to dual quasi-topological and reflexive spaces. Let us recall (cf. [10]) that a QTBA $\underline{A} = \langle A,-,\cup,\cap,C\rangle$ is said to be normal if it contains a normal ultrafilter $\nabla(C(a) \notin \nabla$ iff $a = 0$ for all $a \in A)$. Obviously not every TCA-QTBA is normal. For instance the standard algebra $\underline{A}_{T_{\bar{G}}}$ of the TQTS T_G from Example 1.1 is a normal TCA-QTBA, whereas the standard algebra $\underline{A}_{T_m}$ of the TQTS T_m from Example 1.2 is a non-normal TCA-QTBA since it does not possess any normal ultrafilter. It is worth to emphasize here that the subclass of normal QTBA's has important applications to the semantics of the SCI_Q (the quasi-topological strengthening of the SCI). The class of H-algebras presents a special subclass of QTBA's and consists of all those QTBA's in which every ultrafilter is normal. A QTBA $\underline{A} = \langle A,-,\cup,\cap,C\rangle$ is referred to as self-conjugate if its Q-closure operation satisfies the following condition:

(2.1) $C(a) \leqslant -b$ iff $C(b) \leqslant -a$ for every $a,b \in A$.

The concept of a self-conjugate operation was introduced by Jonsson and Tarski in [4]. We apply it here to the class of QTBA's. Any TBA $\underline{A} = \langle A,-,\cup,\cap,C\rangle$ is called a self-dual algebra if

(2.2) $C(a) \cap C(-C(a)) = 0$ for every $a \in A$.

The class of self-dual algebras finds applications to the semantics of the SCI and it relates both to the WH-theory as well as to the modal system S_5 (cf. [8]). It is not difficult to verify that any QTBA $\underline{A}$ is a self-dual algebra iff the set $\underline{C}(\underline{A}) = \{C(a): a \in A\}$ with respect to the operations $-, \cup, \cap$ in $\underline{A}$ restricted to $\underline{C}(\underline{A})$ is a Boolean subalgebra of the Boolean reduct of $\underline{A}$.

Let $T = \langle X, C \rangle$ be a QTS. Then T is said to be strongly compact if $\bigcap_{i \in I} (Y_i) \neq \emptyset$ for any indexed family $(Y_i)_{i \in I}$ of non-empty subsets of X. From this definition it is seen that the property of a strong compactness for QTS's is a quasi-topological generalization of the well-known concept of the strong compactness for topological spaces. It turns out that the subclass of strongly compact TQTS's presents the quasi-topological counterpart of the subclass of normal TCA-QTBA's. This is shown in the next two lemmas and theorem.

L e m m a 2.1. If $T = \langle X, C \rangle$ is a strongly compact TQTS, then the standard TCA-QTF $\underline{A}_T$ is normal and it determines a strongly compact TQTS $T_{\underline{A}_T}$ which is isomorphic to T.

P r o o f . Let $T = \langle X, C \rangle$ be a strongly compact TQTS. Then its standard TCA-QTF $\underline{A}_T = \langle P(X), -, \cup, \cap, C \rangle$ is normal. To see that let us consider the set $D = \{C(Y): Y \neq \emptyset, Y \subseteq X\}$. Note that this set has the intersection property (i.e. intersections of elements belonging to D are non-empty). Hence D generates a proper filter which can be extended to an ultrafilter U_D. A simple calculation shows that U_D is a normal ultrafilter. Therefore $\underline{A}_T$ is a normal TCA-QTF. This algebra determines in turn a TQTS $T_{\underline{A}_T} = \langle \bar{X}, C^* \rangle$, where C^* is a total Q-closure defined by (1.3). By virtue of Lemma 1.4, $T_{\underline{A}_T}$ is isomorphic to T. Since T is a strongly compact TQTS and isomorphisms preserve the property of a strong compactness for TQTS's, it follows that $T_{\underline{A}_T}$ is strongly compact such that $T_{\underline{A}_T} \simeq T$.

L e m m a 2.2. Let $\underline{A} = \langle A,-,\cup,\cap,C_A\rangle$ be any normal TCA-QTBA. Then the standard TQTS $T_{\underline{A}}$ is strongly compact and it determines a normal TCA-QTF $\underline{A}_{T_{\underline{A}}}$ that is isomorphic to $\underline{A}$.

P r o o f . If $\underline{A} = \langle A,-,\cup,\cap,C_A\rangle$ is a normal TCA-QTBA, then there exists a normal ultrafilter ∇ in $\underline{A}$. This implies that the set $D = \{C_A(a)\colon a \neq 0,\ a \in A\}$ possesses the intersection property, i.e. $\bigcap_{i\in I} C_A(a_i) \neq 0$ for every indexed set $(a_i)_{i\in I}$ of non-zero elements of $\underline{A}$. Hence the standard TQTS $T_{\underline{A}} = \langle At(\underline{A}),C\rangle$, where C is defined by (1.0) is strongly compact. Indeed, from the intersection property of the set D it follows that $\bigcap_{i\in I} C(\{y_i\}) \neq \emptyset$ for any indexed set $(\{y_j\})_{i\in I}$ of singletons in $At(\underline{A})$. Consequently, $\bigcap_{i\in I} C(Y_i) =$ $= \bigcap_{i\in I} \bigcup_{y\in Y_i} C(\{y\}) = \bigcup_{\alpha\in Y} \bigcap_{i\in I} C(\{\alpha(i)\}) \neq \emptyset$ for any indexed family $(Y_i)_{i\in I}$ of non-empty subsets of $At(\underline{A})$, where $Y = \prod_{i\in I} Y_i$. So, $T_{\underline{A}}$ is strongly compact. By Lemma 1.5, $\underline{A}$ is isomorphic to the TCA-QTF $\underline{A}_{T_{\underline{A}}} = \langle P(At(\underline{A})),-,\cup,\cap,C\rangle$. Since $\underline{A}$ is normal and isomorphisms preserve the normality of QTBA's, it follows that $\underline{A}_{T_{\underline{A}}}$ is a normal TCA-QTF that is isomorphic to $\underline{A}$.

T h e o r e m 2.1. There exists a one-to-one correspondence between normal TCA-QTBA's and strongly compact TQTS's.

P r o o f . By Lemmas 2.1 and 2.2.

To establish relational counterparts of normal TCA-QTBA's we introduce the concept of an IP-reflexive space. If $\underline{X} = \langle X,R\rangle$ is a reflexive space, then $\underline{X}$ will be called an IP-reflexive space (a space with the intersection property) provided that $\bigcap_{i\in I} R(x_i) \neq \emptyset$ for every indexed family $(R(x_i))_{i\in I}$. Observe that the standard TCA-QTF $\underline{A}_{\underline{X}} = \langle P(X),-,\cup,\cap,C_R\rangle$ of any IP-reflexive space $\underline{X} = \langle X,R\rangle$ is normal. In fact, note that the set $D(\underline{A}_{\underline{X}}) = \{C(Y)\colon Y \neq \emptyset,\ Y \subseteq X\}$ possesses the intersection property because $\bigcap_{i\in I} C(Y_i) = \bigcap_{i\in I} \bigcup_{y\in Y_i} R(y) = \bigcup_{\alpha\in Y} \bigcap_{i\in I} R(\alpha(i)) \neq \emptyset$

for any indexed set $(C(Y_i))_{i\in I}$ in $D(\underline{A}_X)$, where $Y = \prod_{i\in I} Y_i$. This means that $D(\underline{A}_X)$ generates a proper filter in $\underline{A}_X$ extendable to an ultrafilter $U(D(\underline{A}_X))$. A straightforward computation proves that $U(D(\underline{A}_X))$ is a normal ultrafilter in $\underline{A}_X$. Thus $\underline{A}_X$ is a normal TCA-QTF. By virtue of these observations, by the fact that isomorphisms of reflexive spaces preserve the IP-property of reflexive spaces and by Lemma 1.6 we obtain the following lemma.

L e m m a 2.3. For every IP-reflexive space $\underline{X}$ the standard TCA-QTF $\underline{A}_X$ is normal and it determines an IP-reflexive space $\underline{X}_{\underline{A}_X}$ that is isomorphic to $\underline{X}$.

The next lemma shows that every normal TCA-QTBA is determined by some IP-reflexive space.

L e m m a 2.4. If $\underline{A} = \langle A,-,\cup,\cap,C_A\rangle$ is a normal TCA-QTBA, then its standard IP-reflexive space $\underline{X}_{\underline{A}}$ determines a normal TCA-QTF $\underline{A}_{\underline{X}_{\underline{A}}}$ such that $\underline{A} \simeq \underline{A}_{\underline{X}_{\underline{A}}}$.

P r o o f . Since $\underline{A}$ is normal, the set $D(\underline{A}) = \{C_A(a): a \neq 0,\ a \in A\}$ has the intersection property. Hence, in the standard reflexive space $\underline{X}_{\underline{A}} = \langle At(\underline{A}), R_{\underline{A}}\rangle$ we have,
$\bigcap_{i\in I} R_{\underline{A}}(y_i) = \{x \in At(\underline{A}):\ x \leqslant \bigcap_{i\in I} C_A(y_i)\} \neq \emptyset$ for any family $(R_{\underline{A}}(y_i))_{i\in I}$. Therefore $\underline{X}_{\underline{A}}$ is an IP-reflexive space. Since an isomorphic image of any normal QTBA also belongs to the class of QTBA's, by Lemma 1.7, $\underline{A}$ is isomorphic to the standard normal TCA-QTF $\underline{A}_{\underline{X}_{\underline{A}}} = \langle P(At(\underline{A})),-,\cup,\cap,C_{R_{\underline{A}}}\rangle$ determined by $\underline{X}_{\underline{A}}$.

The main relationships between normal TCA-QTBA's, strongly compact TQTS's and IP-reflexive spaces establishes the following theorem.

T h e o r e m 2.2. The classes of normal TCA-QTBA's, strongly compact TQTS's and IP-reflexive spaces are in a one-to-one correspondence.

P r o o f . By Lemmas 2.3, 2.4 and by Theorem 2.1.

Dual relational counterparts of self-conjugate TCA-QTBA's are reflexive spaces supplied additionaly with the symmetric property. The spaces of this sort will be called tolerance spaces. One can easily show that for every tolerance space $\underline{X} = \langle X,R\rangle$ its standard TCA-QTF $\underline{A}_{\underline{X}} = \langle P(X),-,\cup,\cap,C_R\rangle$ is self-conjugate and it determines in turn a tolerance space $\underline{X}_{\underline{A}_{\underline{X}}}$ such that $\underline{X} \simeq \underline{X}_{\underline{A}_{\underline{X}}}$. Also one proves that starting with any self-conjugate TCA-QTBA $\underline{A} = \langle A,-,\cup,\cap,C_A\rangle$ we obtain the tolerance space $\underline{X}_{\underline{A}} = \langle At(\underline{A}),R_{\underline{A}}\rangle$ determining a self-conjugate TCA-QTF $\underline{A}_{\underline{X}_{\underline{A}}} = \langle P(At(\underline{A})),-,\cup,\cap,C_{R_{\underline{A}}}\rangle$ such that $\underline{A} \simeq \underline{A}_{\underline{X}_{\underline{A}}}$. Therefore the classes of self-conjugate TCA-QTBA's and tolerance spaces are in a one-to-one correspondence.

In order to get a quasi-topological representation of self-conjugate TCA-QTBA's we will distinguish in the class of TQTS's a new subclass. Namely any TQTS $T = \langle X,C\rangle$ will be called C-symmetric if for every $Y,Z \subseteq X$ the following condition holds:

$$(2.3) \qquad C(Y) \cap Z = \emptyset \quad \text{iff} \quad C(Z) \cap Y = \emptyset.$$

Let $T = \langle X,C\rangle$ be any C-symmetric TQTS. Then $\underline{A}_T = \langle P(X),-,\cup,\cap,C\rangle$ is a self-conjugate TCA-QTF and its standard TQTS $T_{\underline{A}_T} = \langle \bar{X},C^*\rangle$ (see Lemma 1.4) is a C-symmetric TQTS which is isomorphic to the space T. Moreover to any self-conjugate TCA-QTBA $\underline{A} = \langle A,-,\cup,\cap,C_A\rangle$ is assigned $T_{\underline{A}} = \langle At(\underline{A}),C\rangle$ which is a C-symmetric TQTS (see Lemma 1.5) determining the standard self-conjugate TCA-QTF $\underline{A}_{T_{\underline{A}}} = \langle P(At(\underline{A})),-,\cup,\cap,C\rangle$ such that $\underline{A} \simeq \underline{A}_{T_{\underline{A}}}$. Putting all these facts together we get the following theorem characterizing self-conjugate TCA-QTBA's in terms of quasi-topological and relational spaces.

T h e o r e m 2.3. The classes of self-conjugate TCA-QTBA's, tolerance spaces and C-symmetric TQTS's are in a one-to-one correspondence.

Observe that TCA-self-dual algebras form a subclass in the class of all self-conjugate TCA-QTBA's. In fact, let $\underline{A} = \langle A,-,\cup,\cap,C\rangle$ be a TCA-self-dual algebra. Then $C(a) \leqslant -b$

implies $b \leqslant I(-a)$ for all $a,b \in A$. From this $C(b) \leqslant I(-a) \leqslant -a$ for every $a,b \in \underline{A}$. Similarly one proves the second part of (2.1). But since every TCA-TBA is a TCA-QTBA, $\underline{A}$ is a self--conjugate TCA-QTBA.

The class of TCA-self-dual algebras is related to total partitional topological spaces. Any total topological space $T = \langle X,C \rangle$ is said to be partitional if $I(C(Y)) = C(Y)$ for every $Y \subseteq X$. One can verify that any total topological space $\langle X,C \rangle$ is partitional iff $\underline{C}(X) = \underline{I}(X)$. Furthermore in every total partitional topological space $T = \langle X,C \rangle$ the following conditions: $x \in C(\{y\})$ iff $y \in C(\{x\})$ and $C(\{x\}) \cap C(\{y\}) \neq \emptyset$ implies $C(\{x\}) = C(\{y\})$ are satisfied for all $x,y \in X$. Consequently the family $(C(\{y\}))_{y \in X}$ is a partition of T. Clearly every total partitional topological space is a C-symmetric TQTS but not conversely. A straightforward calculation shows that the standard TCA-QTF $\underline{A}_T = \langle P(X),-,\cup,\cap,C \rangle$ of a total partitional topological space $T = \langle X,C \rangle$ is a TCA-self-dual algebra. Also every TCA-self-dual algebra $\underline{A} = \langle A,-,\cup,\cap,C_A \rangle$ determines $T_{\underline{A}} = \langle At(\underline{A}),C \rangle$ which is a total partitional topological space. With the help of easy arguments one shows finally that the class of TCA-self-dual algebras and total partitional topological spaces are in a one-to-one correspondence.

It turns out that relational binary systems connected with TCA-self-dual algebras are equivalential ones. To see that, let us suppose that $\underline{A} = \langle A,-,\cup,\cap,C_A \rangle$ is any TCA-self--dual algebra. Then $\underline{X}_{\underline{A}} = \langle At(\underline{A}),R_{\underline{A}} \rangle$ is a quasi-ordered set because $C_{\underline{A}}$ is idempotent. From the fact that $x \leqslant C(y)$ iff $y \leqslant C(x)$ holds for all atoms $x,y \in At(\underline{A})$ it follows that $R_{\underline{A}}$ is symmetric. So, $\underline{X}_{\underline{A}}$ is an equivalential relational system. Conversely, if $\underline{X} = \langle X,R \rangle$ is any equivalential relational system, then by the reflexivity and transitivity of R, $\underline{A}_{\underline{X}} = \langle P(X),-,\cup,\cap,C_R \rangle$ is a TCA-TBA. In view of the symmetry of R, $C_R(I_R(Y)) = \bigcup\{R(y) : R(y) \subseteq Y\} = I_R(Y)$ for every $Y \subseteq X$. Hence $\underline{A}_{\underline{X}}$ is a TCA-self-dual algebra. In the light of these results we may derive the following topological and relational representation theorem for TCA-self-dual algebras.

T h e o r e m 2.3. There exists a one-to-one correspondence between TCA-self-dual algebras, total partitional topological spaces and equivalential relational systems.

Now we proceed to consider the subclass of TCA-H-algebras and their relational and topological counterparts. Any TQTS $T = \langle X,C \rangle$ is called a total H-topological space if $C(Y) = X$ whenever $Y \neq \emptyset$ and $C(Y) = \emptyset$ otherwise for every $Y \subseteq X$. For any TCA-H-algebra $\underline{A} = \langle A,-,\cup,\cap,C_A \rangle$ the standard topological space $T_{\underline{A}} = \langle At(\underline{A}),C \rangle$ is a total H-topological space, while the standard algebra $\underline{A}_T$ corresponding to a total H-topological space $T = \langle X,C \rangle$ is a TCA-H-algebra. On the other hand, relational objects corresponding to TCA-H-algebras are universal relational systems, i.e. the systems in which binary relations are universal ones. Using similar reasoning to that in considered above classes of algebras one may show the following theorem.

T h e o r e m 2.4. The classes of TCA-H-algebras, total H-topological spaces and universal binary relational systems are in a one-to-one correspondence.

The class of TCA-self-dual algebras is not a subclass of normal TCA-QTBA's. The next theorem presents a characterization of TCA-H-algebras by means of relational systems as well as points out that normal TCA-self-dual algebras coincide with the subclass of TCA-H-algebras.

T h e o r e m 2.5. Let $\underline{X} = \langle X,R \rangle$ be a reflexive space and let $\underline{A}_{\underline{X}} = \langle P(X),-,\cup,\cap,C_R \rangle$ be its standard TCA-QTF. Then the following conditions are equivalent:

(i) $\underline{X}$ is an universal relational system,

(ii) $\underline{A}_{\underline{X}}$ is a normal TCA-self-dual algebra,

(iii) $\underline{A}_{\underline{X}}$ is a TCA-H-algebra.

P r o o f . In view of Theorems 2.2 and 2.4, the condition (i) implies (ii). If (ii) is satisfied, then there exists a normal ultrafilter ∇ in $\underline{A}_{\underline{X}}$. Hence, $C(Y) \in \nabla$ iff $I(C(Y)) \in \nabla$ iff $C(Y) = X$ whenever Y is any nonempty subset of X. In the case that $Y = \emptyset$, we get obviously $C(Y) = \emptyset$.

Therefore $\underline{A}_{\underline{X}}$ is a TCA-H-algebra. The implication (iii) $\Rightarrow$ (i) is true by Theorem 2.4.

Applying Theorems 2.2, 2.4 once again and performing easy calculations one obtains the dual topological characterization of the subclass of normal TCA-self-dual algebras. This is shown in the following theorem.

T h e o r e m 2.5. Let $T = \langle X,C\rangle$ be a TQTS and let $\underline{A}_T = \langle P(X),-,\cup,\cap,C\rangle$ be its standard algebra. Then the following conditions are equivalent:

(i) T is a total H-topological space,

(ii) $\underline{A}_T$ is a normal TCA-self-dual algebra,

(iii) $\underline{A}_T$ is a TCA-H-algebra.

Observe that TCA-self-dual algebras are those TCA-self-conjugate QTBA's in which Q-closure operators become usual topological ones. In fact, if $\underline{A} = \langle A,-,\cup,\cap,C_A\rangle$ is a TCA-self-conjugate QTBA in which C_A is idempotent, then $C_A(C_A(a)) \cap -C_A(a) = 0$ for all $a \in A$. Using formula (2.1), we get $C_A(a) \cap C_A(-C_A(a)) = 0$ for every $a \in A$. Therefore $\underline{A}$ is a TCA-self-dual algebra. From this result and from Theorem 2.3 we infer that any C-symmetric TQTS $T = \langle X,C\rangle$ is a total partitional topological space iff the operator C is idempotent.

A reflexive space $\underline{X} = \langle X,R\rangle$ is said to be trivial if R is the identity relation on X, i.e. $x\,R\,y$ iff $x = y$ for every $x,y \in X$. Any TQTS $T = \langle X,C\rangle$ is referred to as discrete provided that $C(Y) = Y$ for all subsets $Y \subseteq X$. Any QTBA in which all elements are Q-closed will be regarded as a Boolean algebra. Easy observations show that any TCA-QTBA $\underline{A}$ is a complete atomic Boolean algebra iff $\underline{X}_A$ is trivial iff T_A is discrete.

Now we will show that normal TCA-QTBA's, strongly compact TQTS's and IP-reflexive spaces can be obtained constructively in a simple way from TCA-QTBA's, TQTS's and reflexive spaces, respectively.

Let $\underline{A} = \langle A,-,\cup,\cap,C\rangle$ be any TCA-QTBA and let $\underline{2} = \langle\{0,1\},-,\cup,\cap\rangle$ be a two-element Boolean algebra. Defining on the Cartesian product $A^* = A \times \{0,1\}$ the Boolean operations componentwise and a unary operation C^* by the formula:

$$C^*(a,x) = \begin{cases} (C(a),1) & \text{if } (a,x) \neq (0,0) \\ (0,0) & \text{if } (a,x) = (0,0) \end{cases}$$

for all $(a,x) \in A^*$, we get a TCA-QTBA $\underline{A}^* = \langle A, -, \cup, \cap, C^* \rangle$. One may verify that $\underline{A}^*$ is a normal TCA-QTBA. This algebra will be called an algebraic normal product of the algebra $\underline{A}$.

To construct a strongly compact TQTS, let us take any TQTS $T = \langle X,C \rangle$. Furthermore, let $X^{(1)} = X \cup \{x_1\}$, where $x_1 \notin X$. Next let us define on $X^{(1)}$ a Q-closure operation $C^{(1)}$ by the formula $C^{(1)}(Y) = C(Y) \cup \{x_1\}$ for every non-empty subset Y of $X^{(1)}$ and $C^{(1)}(\emptyset) = \emptyset$. Then $T^{(1)} = \langle X^{(1)}, C^{(1)} \rangle$ is a strongly compact TQTS. In fact, if $(Y_i)_{i \in I}$ is any indexed family of non-empty subsets of $X^{(1)}$, then $\bigcap_{i \in I} C^{(1)}(Y_i) = \bigcap_{i \in I} C(Y_i) \cup \cup \{x_1\} \neq \emptyset$. The space $T^{(1)}$ will be called a one-point compactification of the TQTS T.

Finally let $\underline{X} = \langle X,R \rangle$ be a given reflexive space. On the set $X^{(1)} = X \cup \{x_1\}$, where $x_1 \notin X$ we define a new relation $R^{(1)}$ such that $R^{(1)} = R \cup \{(x_1,y) : y \in X^{(1)}\}$. Clearly $R^{(1)}$ is reflexive. Hence, $\underline{X}^{(1)} = \langle X^{(1)}, R^{(1)} \rangle$ is a reflexive space. Since $\bigcap_{i \in I} R^{(1)}(z_i) = \bigcap_{i \in I} R(z_i) \cup \{x_1\} \neq \emptyset$ for any indexed family $(R^{(1)}(z_i))_{i \in I}$, it follows that $\underline{X}^{(1)}$ is an IP-reflexive space. This space will be called a one-point IP-closure of $\underline{X}$.

With each TQTS $T = \langle X,C \rangle$ we may associate two normal TCA-QTBA's of the form $\underline{A}_T^*$ and $\underline{A}_T(1)$. The first one is the algebraic normal product of the standard algebra $\underline{A}_T$ of T, while the second one is obtained as the standard algebra of the one-point compactification $T^{(1)}$ of the space T. Both these algebras are normal and have been constructed from the same TQTS T. The connections between them establishes the following theorem.

Theorem 2.6. Let $T = \langle X,C \rangle$ be a TQTS. Then $\underline{A}_T^* \simeq \underline{A}_T(1)$.

P r o o f . If $T = \langle X,C\rangle$ is a given TQTS, then $\underline{A}_T^* = \langle P(X)^*,-,\cup,\cap,C^*\rangle$ is the algebraic normal product of the standard algebra $\underline{A}_T$ of T, where

$$C^*(Y,w) = \begin{cases} (C(Y),1) & \text{if } (Y,w) \neq (\emptyset,0) \\ (\emptyset,0) & \text{if } (Y,w) = (\emptyset,0) \end{cases}$$

for all $(Y,w) \in P(X)^*$. On the other hand, for T one can construct a strongly compact TQTS $T^{(1)}$ which is a one-point strong compactification of T. By Theorem 2.1, $\underline{A}_T(1) = \langle P(X^{(1)}),-,\cup,\cap,C^{(1)}\rangle$ is a normal TCA-QTF, where $C^{(1)}(Y) = C(Y)\cup\{x_1\}$ for every non-empty subset Y of $X^{(1)}$ and $C^{(1)}(\emptyset) = \emptyset$. It is easy to check that the function $g: P(X)^* \to P(X^{(1)})$ defined by $g(Y,w) = Y\cup f(w)$ for every $(Y,w)\in P(X)^*$ is an isomorphism from $\underline{A}_T^*$ onto $\underline{A}_T(1)$, where f is a Boolean isomorphism from $\underline{2}$ onto the two-element field $P(\{x_1\})$ such that $f(0) = \emptyset$ and $f(1) = \{x_1\}$.

Now let us start with a given reflexive space $\underline{X} = \langle X,R\rangle$. Then we can also construct two normal TCA-QTBA's of the form $\underline{A}_{\underline{X}}^*$ and $\underline{A}_{\underline{X}}(1)$ by applying the algebraic normal product and one-point IP-closure of $\underline{X}$, respectively. Indeed, $\underline{A}_{\underline{X}}^* = \langle P(X)^*,-,\cup,\cap,C_R^*\rangle$ is the normal algebraic product of the standard algebra $\underline{A}_{\underline{X}}$ of $\underline{X}$, where $C_R^*(Y,w) = (C_R(Y),1)$ whenever $(Y,w) \neq (\emptyset,0)$ and $C_R^*(Y,w) = (\emptyset,0)$ whenever $(Y,w) = (\emptyset,0)$ for every $(Y,w)\in P(X)^*$. Clearly, $\underline{A}_{\underline{X}}(1) = \langle P(X^{(1)}),-,\cup,\cap,C_R(1)\rangle$ is the standard algebra of the one-point IP-closure $\underline{X}^{(1)}$ of $\underline{X}$, where $C_R(1)\ (Y) = \bigcup_{y\in Y} R^{(1)}(y)$ for every $Y\subseteq X^{(1)}$. A similar calculation to that in the above theorem shows that $\underline{A}_{\underline{X}}^* \simeq \underline{A}_{\underline{X}}(1)$.

REFERENCES

[1] C. B e r n a r d i : On the equational class of diagonalisable algebras, Studia Logica 34 (1975), 322-331.

[2] R. Goldblatt: An algebraic study of well-foundedness, Studia Logica 44 (1985), 423-437.

[3] A. Grzegorczyk: Some relational systems on the associated topological spaces, Fund. Math. 60 (1967), 223-231.

[4] B. Jonsson, A. Tarski: Boolean algebras with operators, part I, Amer. J. Math. 73 (1951), 891-939.

[5] J. McKinsey, A. Tarski: The algebra of topology, Ann. Math. 45 (1944), 141-191.

[6] E. Lemmon: An extension algebra and the modal system T, Notre Dame J. Formal Logic 1 (1960), 3-12.

[7] R. Sikorski: Boolean Algebras, Berlin-Göttingen-Heidelberg-New York 1964.

[8] R. Suszko: Abolition of the Fregean axiom, Lecture Notes in Math. 453 (1975), 169-239.

[9] B. Tembrowski: On some class of Boolean algebras with an additional binary operation, Demonstratio Math. 15 (1982), 189-205.

[10] B. Tembrowski: Normal B_3-algebras, Demonstratio Math. 16 (1983), 601-617.

DEPARTMENT OF MATHEMATICS, PEDAGOGICAL UNIVERSITY,
SIEDLCE, POLAND

Received January 12, 1988.