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AN INVERSE SPECTRAL PROBLEM FOR LINEAR OPERATORS IN HILBERT SPACE

Introduction

In 1929 W.A. Ambarzumian published the following

Theorem A. ([1]). If the sequence of eigenvalues of the problem

$$(1) \qquad -y'' + qy = \lambda y, \; y'(0) = y'(\pi) = 0, \quad q \in C([0,\pi])$$

is identical with the sequence of eigenvalues of the problem

$$(2) \qquad -y'' = \lambda y, \; y'(0) = y'(\pi) = 0,$$

then $q(x) = 0$ for $x \in [0,\pi]$, that is to say the problems (1) and (2) are identical.

This theorem originated the so-called inverse spectral problem of the Sturm-Liouville type. This problem mainly consists in determining the dependence of the considering problem on the set of its eigenvalues. At present, this problem has the extensive bibliography concerning the differential operators.

Immediate generalization of Ambarzumian's theorem have been done by N. Kuznecov in paper [9] in 1962. In this paper Kuznecov considered the eigenvalue problem for the following equation

$$(3) \qquad -\Delta u + qu = \lambda u, \quad q \in C(\bar{\Omega})$$

with boundary condition

(4) $$\frac{du}{dn} = 0 \quad \text{on } \partial\Omega,$$

where Ω is a bounded domain in R^2 or R^3, with sufficiently smooth boundary $\partial\Omega$. In paper [9] Kuznecov proved the following

Theorem K. If 1^o $\sum_{n=1}^{\infty} (\lambda_n - \mu_n)$ is convergent where $\{\lambda_n\}$ denotes the sequence of eigenvalues of problem (3), (4) and $\{\mu_n\}$ denotes the sequence of eigenvalues of this problem with $q = 0$ in Ω,
2^o $\lambda_1 = \mu_1$, then $q = 0$ in Ω.

The next generalizations of Ambarzumian's theorem are contained in author's papers [2], [3], [5] and [6]. These generalizations go in three directions. First - generalization of the Kuznecov's result by reduction of the assumptions in Theorem K, in particular the assumption 1^o. In paper [6] the assumption 1^o of Theorem K is replaced by "$\sum_{n=1}^{\infty} \frac{1}{n} |\lambda_n - \mu_n|$ is convergent ...". Second - replacement the space R^2 or R^3 in Theorem K by the space R^m for any $m \in N$ (see [2] and [3]). Third - replacement the Neumann boundary condition by the third kind boundary condition [2] and the Laplace's operator Δ in equation (3) by any elliptic operator of second order with constant coefficients (see [3] and [6]).

The other direction of generalization of Theorem K lies in replacement the equation of second order by the equation of higher order. In author's paper [5] is considered equation

$$(-1)^m y^{(2m)} + qy = \lambda y \quad \text{in } (a,b), \quad m \geqslant 1,$$

with boundary conditions

$$\alpha_1 y^{(2\nu)}(a) - \alpha_2 y^{(2\nu+1)}(a) = 0, \quad \beta_1 y^{(2\nu)}(b) + \beta_2 y^{(2\nu+1)}(b) = 0,$$

where $\nu = 0,1,\dots,m-1$, α_1, α_2, β_1, β_2 are nonnegative constants fulfilling the condition $(\alpha_1^2+\alpha_2^2)(\beta_1^2+\beta_2^2) > 0$.

S. Postawa in paper [11] transfers author's results to the following problem

$$\Delta^2 u + qu = \lambda u \quad \text{in} \quad \Omega \subset R^m$$

$$\frac{du}{dn} = 0 \quad \text{and} \quad \frac{d(\Delta u)}{dn} = 0 \quad \text{on} \quad \partial\Omega.$$

The farthest generalizations of Theorem K, and thus the author's results, is contained in Pielichowski's paper [10]. The results of this paper concern the selfadjoint elliptic differential operator of order 2m with constant coefficients.

Quite other direction of generalizations of Ambarzumian's theorem is originated in author's paper [4]. In this paper are given some sufficient conditions concern the only first eigenvalue of problem (3), (4) in order the function $q = 0$ in Ω.

The purpose of this paper is the further generalization of the results of paper [4].

1. Let H be a real Hilbert space and let L be a linear self-adjoint operator defined in D_L dense in H. We assume that L is bounded bellow in D_L, such that L^{-1} exists and L^{-1} is a compact operator on H. Let $V : H \to H$ be a linear symmetric and bounded operator. We shall consider the linear eigenvalue problem with a parameter

$$(5) \qquad (L - tV)u = \lambda u,$$

where $\lambda \in R$ is the eigenvalue parameter and $t \in R$ is a parameter.

In the sequel we shall need the following result

T h e o r e m 1. For every $t \in R$ the problem (5) has the first eigenvalue $\lambda_1 = \lambda_1(t)$ such that the function $t \to \lambda_1(t)$ is continuous for $t \in R$ and it is differentiable for $t \in R$ except at most countably many points and

$$(6) \qquad \lambda_1'(t) = -(Vu_t, u_t),$$

where u_t is an eigenvector of problem (5) associated with the eigenvalue $\lambda_1(t)$, normalized by $\|u_t\| = 1$.

The proof of Theorem 1, in reality, is identical with the proofs of Lemmata 4 and 5 of paper [7], and is omitted.

In this paper [7], we have proved also that the function $t \to \lambda_1'(t)$ is a decreasing for $t \in R$ (cf. [7], Corollary 1). From this follows there exists at most one point $t_0 \in R$ such that

$$(7) \qquad \lambda_1'(t_0 - 0) \geq 0 \quad \text{and} \quad \lambda_1'(t_0 + 0) \leq 0,$$

unless $\lambda_1(t) = 0$ in some interval of R, i.e. $\lambda_1(t) = \text{const.}$ in this interval.

By (7) and by continuity the function $t \to \lambda_1(t)$, we have the following

Corollary 1. If the function $t \to \lambda_1(t)$ is not constant in any interval of R, then it cannot attain a minimum and it can attain his maximum at most in one point $t_0 \in R$.

We shall prove the following

Theorem 2. If the function $t \to \lambda_1(t)$ is constant in some interval $(\alpha,\beta) \subset R$, then it is constant in the whole R. It is possible if and only if $N_0 \subset \mathrm{Ker}V$, where N_0 denotes the eigenspace of the operator L corresponding to first eigenvalue $\lambda_1(0)$.

Proof. By assumption $\lambda_1(\alpha) = \lambda_1(\beta)$ and $\lambda_1'(\alpha) = \lambda_1'(\beta) = 0$. From this by (6) we get $(Vu_\alpha, u_\alpha) = (Vu_\beta, u_\beta) = 0$. Using the variational definition of eigenvalues of problem (5), we have (cf. for ex. [8])

$$\lambda_1(\alpha) = \min\left\{(L\varphi,\varphi) - \alpha(V\varphi,\varphi) \; : \; \varphi \in D_L, \; \|\varphi\| = 1\right\} = (Lu_\alpha, u_\alpha)$$

and

$$\lambda_1(\beta) = \min\left\{(L\varphi,\varphi) - \beta(V\varphi,\varphi) \; : \; \varphi \in D_L, \; \|\varphi\| = 1\right\} = (Lu_\beta, u_\beta).$$

Since $\lambda_1(\alpha) = \lambda_1(\beta)$ then $(Lu_\alpha, u_\alpha) = (Lu_\beta, u_\beta)$. This follows that $u_\beta \in N_\alpha$ and $u_\alpha \in N_\beta$, where N_α and N_β denote the eigenspaces of the operators $L - \alpha V$ and $L - \beta V$ corresponding to eigenvalues $\lambda_1(\alpha)$ and $\lambda_1(\beta)$, respectively. From this we get

$$(L - \alpha V)u_\alpha = \lambda_1(\alpha)u_\alpha$$
$$(L - \beta V)u_\alpha = \lambda_1(\alpha)u_\alpha,$$

therefore $(\alpha-\beta)Vu_\alpha = 0$. Since $\alpha \neq \beta$ then

(8) $$Vu_\alpha = 0.$$

From (8) we get that for every $t \in R$

(9) $$(L - tV)u_\alpha = \lambda_1(\alpha)u_\alpha.$$

The equality (9) denotes that $\lambda_1(\alpha)$ is an eigenvalue of problem (5) which corresponds the eigenvector u_α. For $t \in [\alpha, \beta]$ $\lambda_1(\alpha)$ is the first eigenvalue of problem (5), by definition. We shall prove that $\lambda_1(\alpha)$ is the first eigenvalue of problem (5) for every $t \in R$.

Suppose that $\varepsilon > 0$ is any number. Let $\lambda_1(\beta+\varepsilon) < \lambda_1(\alpha)$. Since $t \to \lambda_1(t)$ is continuous for $t \in R$, then $\lambda_1(\beta+\varepsilon) \to \lambda_1(\alpha)$ if $\varepsilon \searrow 0$. On the other hand $\lambda_1(\alpha)$ is isolated eigenvalue of $L - \alpha V$. We arrive a contradiction. This contradiction proves that the first eigenvalue of problem (5) is constant and $\lambda_1(t) = \lambda_1(\alpha)$ for every $t \in R$. In particular $\lambda_1(0) = \lambda_1(\alpha)$ and every $u_\alpha \in N_0$, i.e. $N_\alpha \subset N_0$ and vice versa $N_0 \subset N_\alpha$ or $N_0 = N_\alpha$. From this and from (8) we have

(10) $$N_0 \subset \mathrm{Ker}V.$$

Conversely, if (10) holds then each vector $u_0 \in N_0$ satisfies the equation (5) for $\lambda = \lambda_1(0)$ and $t \in R$, i.e. for any $t \in R$ $\lambda_1(t) = \lambda_1(0)$. The proof is complete.

C o r o l l a r y 2. If $(\mathrm{Ker}V) \cap N_0 = \{0\}$ or, in particular, $\mathrm{Ker}V = \{0\}$, then the function $t \to \lambda_1(t)$ is not constant in any interval of R, i.e. the assumption of Corollary 1 is satisfied.

2. In this section we shall formulate and prove some alternatives of the Ambarzumian's theorem.

T h e o r e m 3. If $(\mathrm{Ker}V) \cap N_0 = \{0\}$ and the operators L and V satisfy the assumptions of Introduction and if there exist real numbers $t_1, t_2, t_3 \in R$, $t_1 \neq t_2 \neq t_3$ such that

$$\lambda_1(t_1) = \lambda_1(t_2) = \lambda_1(t_3), \tag{11}$$

then the operator $V = 0$.

P r o o f . By Corollary 2, from assumptions of Theorem 3, follows that the function $t \to \lambda_1(t)$ is not constant in any interval of R. From this, by Corollary 1, this function cannot attain a minimum in R. Therefore this fact and continuity the function $t \to \lambda_1(t)$, contradicts equality (11). Theorem 3 is proved.

Theorem 3 may be formulated in the following form

T h e o r e m 3a. If the operators L and V satisfy the assumptions of Introduction, $(\mathrm{Ker}V) \cap N_0 = \{0\}$ and if the first eigenvalues of the problems

$$Lu = \lambda u, \quad (L + V)u = \lambda u, \quad (L - V)u = \lambda u$$

are equal, then V is the null operator on H.

T h e o r e m 4. Under assumptions of the Theorem 3a, if the operator V is non negative or non positive on H, i.e. for every $x \in H$ $(Vx,x) \geq 0$ or $(Vx,x) \leq 0$, and if the first eigenvalues of the problems

$$Lu = \lambda u, \quad (L - V)u = \lambda u \tag{12}$$

are equal, then V is the null operator on H.

P r o o f . From the assumption that V is non positive or non negative operator and from (6) follows that $t \to \lambda_1(t)$ is the monotonic function. Here $\lambda_1(t)$ is the first eigenvalue of problem (5) for fixed $t \in R$. On the other hand the first eigenvalues of problems (12) may be written as $\lambda_1(0)$ and $\lambda_1(1)$, respectively. The monotonicity of the function $t \to \lambda_1(t)$ and the assumption $\lambda_1(0) = \lambda_1(1)$ follow that $\lambda_1(t) = \mathrm{const.}$ in R. By Theorem 3, it is possible if and only if the operator V is the null operator on H.

R e m a r k 1. Theorems 3 and 4 are just the new versions of Kuznecov's or Ambarzumian's theorems and of the results of papers [2], [3], [5], [10], and [11]. The difference lies in that the assumptions of Theorems 3 and 4 concern the first eigenvalues of considerating problems, whereas the assumptions of previous theorems involve the whole sequences of eigenvalues of these problems.

R e m a r k 2. In Theorems 3 and 4 L and V are arbitrary operators in Hilbert space satisfying the assumptions of Introduction, whereas in previous theorems L is a special differential operator and V is a multiplication operator induced by a function q.

R e m a r k 3. Similar results to this paper are obtained in author's paper [4] for the special differential operator in bounded domain $\Omega \subset R^m$ and V the multiplication operator induced by a function $q \in C(\bar{\Omega})$.

R e m a r k 4. In the case when L is a differential-operator satisfying the assumptions of Introduction and V is a multiplication operator induced by a function q, the assumption $(\mathrm{Ker}V) \cap N_0 = \{0\}$ is satisfied automatically, unless $q = 0$.

3. In this section we shall consider two following problems

(13) $$Lv = \mu v$$

and

(14) $$(L + W)u = \vartheta u,$$

where L is the operator defined in Introduction and W is a linear selfadjoint and bounded operator on H. Let us denote by μ_1 and ϑ_1 the first eigenvalues of problems (13) and (14), respectively. Let us put

(15) $$c := \inf\{(Wx,x) : x \in H, \|x\| = 1\}.$$

We have the following

T h e o r e m 5. The first eigenvalue ϑ_1 of problem (14) satisfies the inequality

$$\mu_1 + c \leq \vartheta_1 \leq \mu_1 + \|W\|, \tag{16}$$

and $\vartheta_1 = \mu_1 + c$ or $\vartheta_1 = \mu_1 + \|W\|$ if and only if W is the multiplication operator induced by c or $\|W\|$, respectively.

P r o o f . The inequality (16) immediatelly follows from variational definition of eigenvalues of problem (5). Suppose that $\vartheta_1 = \mu_1 + c$, where c is defined by (15). Let us observe that (14) may be written in the form

$$[L - (c - W)]u = (\vartheta - c)u. \tag{17}$$

Problem (17) is the problem (5) with $V = c - W$, $\lambda = \vartheta - c$ and $t = 1$, whereas the problem (13) is it with $t = 0$. By definition the number c, the operator $V = c - W$ is non positive on H. On the other hand, by assumption, $\vartheta_1 - c = \lambda_1(1) = \mu_1 = \lambda_1(0)$. From this, by Theorem 4, we get $V = c - W = 0$, i.e. $W = c$.

If $\vartheta_1 = \mu_1 + \|W\|$, the proof is analogous. Theorem 5 is proved.

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