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ON GENERALIZED FOLIATIONS WITH LOCALLY CONNECTED LEAVES

R. Sikorski in 1967 (see [2]) introduced the concept of a differential space (d.s.) being a generalization of C^∞-differentiable manifold. In the category of d.s. we have the concept of a generalized foliation being introduced by W. Waliszewski [6]. The same author in [7] proved that for any generalized foliation F on a d.s. M and for any point p of M there is an F-adapted chart on M at the point p. The aim of the present note is to prove the following

Theorem. If F is a foliation on M with locally connected leaves, then for any point p of M there exists an F-adapted chart ψ on M at the point p such that $\psi(p) = (p,p)$.

For some information about d.s. the reader is refered to [3], [4] and [5]. Especially simple generalized foliations are considered in [1].

Proof of Theorem. Let p be any point of M. By Lemma 1 in [7] we have a diffeomorphism $\varphi: M_U \to F(p)_V \times M_W$, $p \in W \subset U$, $p \in V$, $U \in \text{top}M$, $V \in \text{top}F(p)$, $\varphi(w) = (\varphi_1(w), w)$ for $w \in W$ such that every connected component of the set $U \cap \underline{L}$, where $L \in F$, is of the form $\varphi^{-1}[V \times \{b\}]$, where $b \in W$. Here $\underline{L}$ denotes the set of all points of a d.s. L. We have then $\varphi(p) = (\varphi_1(p), p)$ and $\varphi_1(p) \in V$. From local homogeneity of the leaf F(p) it follows that there exists a diffeomorphism $\delta: F(p)_A \to F(p)_B$ such that $p \in A \in \text{top}F(p)$, $\varphi_1(p) \in B \in \text{top}F(p)$ and $\delta(p) = \varphi_1(p)$. Let us set $A_0 = \delta^{-1}[B \cap V] \cap V$ and $B_0 = \delta[A_0]$.

We have then $p \in A_0 \subset V$, $A_0 \in \operatorname{top}F(p)$, $B_0 = B \cap V \cap \delta[A \cap V]$, $B_0 \in \operatorname{top}F(p)$, $\delta(p) = \varphi_1(p) \in B_0 \subset V$. We also have $\delta|A_0 : F(p)_{A_0} \longrightarrow F(p)_{B_0}$. Local connectivity of $F(p)$ yields the existence a set $A_1 \in \operatorname{top}F(p)$ being connected in $F(p)$ and such that $p \in A_1 \subset A_0$. Setting $B_1 = \delta[A_1]$ we get a diffeomorphism

$$(1) \qquad \delta|A_1 : F(p)_{A_1} \longrightarrow F(p)_{B_1}.$$

Hence, setting for $z \in B_1$

$$(2) \qquad \psi(z) = ((\delta|A_1)^{-1}(\varphi_1(z)), \varphi_2(z))$$

we get $\psi : M_{U_1} \longrightarrow F(p)_{A_1} \times M_W$, where $U_1 = \varphi_1^{-1}[B_1]$. Let $(t,w) \in A_1 \times W$ and $\psi(z) = (t,w)$. Then $z = \varphi^{-1}(\delta(t),w)$, $\delta(t) \in B_1 \subset \delta[A_0] = B_0 \subset V$, $\varphi(z) = (\delta(t),w)$, $\varphi_1(z) = (\delta|A_1)(t)$ and $\varphi_2(z) = w$. Hence it follows that $t = (\delta|A_1)^{-1}(\varphi_1(z))$. Consequently, $(t,w) = ((\delta|A_1)^{-1}(\varphi_1(z)), \varphi_2(z)) = \psi(z)$. Thus, we have a smooth one-one mapping ψ and its inverse one

$$(3) \qquad \psi^{-1} : F(p)_{A_1} \times M_W \longrightarrow M_{U_1}$$

defined by the formula $\psi^{-1}(t,w) = \varphi^{-1}(\delta(t),w)$ for $(t,w) \in A_1 \times M_W$. By smoothness of (1) and a diffeomorphism φ we get smoothness of (3). The mapping

$$(o) \qquad \psi : M_{U_1} \longrightarrow F(p)_{A_1} \times M_W$$

is then a diffeomorphism.

Let $L \in F$ and C be a connected component in M of the set $U_1 \cap \underline{L}$. We have $U_1 \cap \underline{L} \subset U \cap \underline{L}$. The connected component of the set $U \cap \underline{L}$ containing the set C we denote by C_0. We have then $C_0 = \varphi^{-1}[V \times \{b\}]$, $b \in W$. Thus $\varphi[C] \subset V \times \{b\}$. Notice that $\varphi_1^{-1}[B_1] = \varphi^{-1}[B_1 \times W]$. This yields that C is a connected component of the set $\varphi^{-1}[B_1 \times W] \cap \underline{L}$. Hence it follows that $C \subset \varphi^{-1}[B_1 \times W]$.

Thus $\varphi[C] \subset B_1\times W$. Consequently, $C\subset \varphi^{-1}[B_1\times\{b\}] = \varphi^{-1}[\delta[A_1]\times\{b\}]$. Let $u\in\varphi^{-1}[\delta[A_1]\times\{b\}]$. Then $\varphi(u)\in\delta[A_1]\times\{b\}$. This yields $\varphi_1(u) = \delta(t)$, $t\in A_1$, and $\varphi_2(u) = b$. Hence it follows that $t = (\delta|A_1)^{-1}(\varphi_1(u))$. So, $\psi(u) = ((\delta|A_1)^{-1}(\varphi_1(u)), \varphi_2(u)) = (t,b)\in A_1\times\{b\}$. Consequently, $\varphi^{-1}[\delta[A_1]\times\{b\}]\subset\psi^{-1}[A_1\times\{b\}]$. Now, let us take any $u\in\psi^{-1}[A_1\times\{b\}]$. Then $u\in U_1$ and $\psi(u)\in A_1\times\{b\}$. Thus, $u\in\varphi_1^{-1}[B_1]$ and $((\delta|A_1)^{-1}(\varphi_1(u)),\varphi_2(u))\in A_1\times\{b\}$. Hence $\varphi_1(u)\in B_1 = \delta[A_1]$ and $\varphi_2(u) = b$. Consequently, $\varphi(u) = (\varphi_1(u),\varphi_2(u))\in\delta[A_1]\times\{b\}$ and $u\in\varphi^{-1}[\delta[A_1]\times\{b\}]$. Therefore, we have

$$(4)\qquad \varphi^{-1}[\delta[A_1]\times\{b\}] = \psi^{-1}[A_1\times\{b\}]$$

and

$$(5)\qquad C\subset\psi^{-1}[A_1\times\{b\}].$$

Let us notice that $\psi^{-1}[A_1\times\{b\}]\subset\varphi^{-1}[V\times\{b\}] = C_0\subset U\cap\underline{L}$. Thus,

$$(6)\qquad \psi^{-1}[A_1\times\{b\}]\subset\underline{L}.$$

By (4) we have $\psi^{-1}[A_1\times\{b\}] = \varphi^{-1}[B_1\times\{b\}]$.

Taking any $u\in\varphi^{-1}[B_1\times\{b\}]$ we state that $\varphi_1(u)\in B_1$ or, equivalently, $u\in\varphi_1^{-1}[B_1] = U_1$; this yields

$$(7)\qquad \psi^{-1}[A_1\times\{b\}]\subset U_1.$$

From (6), (7) and (5) it follows that

$$C\subset\psi^{-1}[A_1\times\{b\}]\subset U_1\cap\underline{L}.$$

Connectivity of A_1 in $F(p)$ yields the same property of $\psi^{-1}[A_1\times\{b\}]$. Therefore

$$C = \psi^{-1}[A_1\times\{b\}].$$

From (2) it follows that for any $w \in W$ we have $\psi(w) = (\psi_1(w), \varphi_2(w))$. Moreover, $\varphi_2(w) = w$. Thus, $\psi(w) = (\psi_1(w), w)$ for $w \in W$. Also, by (2), we get $\psi(p) = ((\delta|A_1)^{-1}(\varphi_1(p)), p) = ((\delta|A_1)^{-1}(\delta(p)), p) = (p,p)$. What ends the proof.

R e m a r k 1. If in the definition of a generalized foliation we would not assume the local homogeneity of the family of all leaves but local homogeneity and connectedness of every leaf, then the thesis of Theorem would be satisfied.

R e m a r k 2. The theorem suggest also a slight modification of the definition of a generalized foliation in the category of d.s. Namely, a locally homogeneous and locally connected d.s. F such that $\underline{F} = \underline{M}$ could be regarded as a generalized foliation on M provided that for any $p \in \underline{M}$ there exists an F-adapted chart on M at the point p which sends p to (p,p). Then each leaf being, by definition, a connected component of F would be an open subspace of F.

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