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ON WAŻEWSKI'S METHOD OF SUCCESSIVE APPROXIMATIONS FOR CERTAIN BOUNDARY VALUE PROBLEMS

1. Introduction

We consider the boundary value problem (BVP) of the form

(1) $$x^{(n)} = f(t,x,x^{(1)},\dots,x^{(n-1)}),$$

(2) $$x^{(q)}(a_i) = C_i, \quad i = 1,\dots,k, \quad q = 0,1,\dots,q_i-1,$$

where $a \leqslant a_1 < \dots < a_k \leqslant b$, $2 \leqslant k \leqslant n$, $\sum_{i=1}^{k} q_i = n$, $f \in C(I\times R^n,R)$, $I = [a,b]$ and C_i are given constants. For the special forms of (1), (2), a great variety of existence and uniqueness theorems have been proved by many authors in diverse ways [1]-[18]. The object of this paper is to present a unified approach for treating the existence, uniqueness, error estimation and stability of the solutions of (1), (2) by reducing it into an equivalent integral equation and by using the general method of successive approximations based on the idea of Ważewski [18] (see, also [7], [10]). An interesting feature of our approach to the problem is that the proofs of our main results are much simpler and can be extended to the different types of boundary value problems treated before by using various methods (see [1]-[3], [12], [13], [15], [16]).

2. Statement of results

We say that x is a solution of (1), (2) if $x \in C^{(n)}(I,R)$ and satisfies (1), (2) on I. The set of all such solutions will be denoted by $C^*(I)$. For $x \in C^*(I)$, the equation $x^{(n)}(t) = y(t)$ with boundary conditions (2) is equivalent to the integral equation (see [15], p.538)

$$x(t) = w(t) + \int_a^b G(t,s)y(s)ds, \tag{3}$$

where $w(t)$ is the unique solution of the problem $x^{(n)}(t) = 0$, (2) and $G(t,s)$ is the Green function of this problem. By (3), we obtain

$$x^{(j)}(t) = w^{(j)}(t) + \int_a^b \frac{\partial^j}{\partial t^j} G(t,s)y(s)ds, \quad j = 1,\dots,n-1. \tag{4}$$

From (1), (3), (4) we see that the function y fulfils the equation

$$y(t) = f\Big(t, w(t) + \int_a^b G(t,s)y(s)ds, w^{(1)}(t) + \int_a^b \frac{\partial}{\partial t} G(t,s)y(s)ds, \dots$$
$$\dots, w^{(n-1)}(t) + \int_a^b \frac{\partial^{n-1}}{\partial t^{n-1}} G(t,s)y(s)ds\Big). \tag{5}$$

Conversely, if $y \in C(I,R)$ fulfils (5), then $x \in C^*(I)$ defined by (3) is a solution of (1), (2). By substituting in (5)

$$F(t,r_o,r_1,\dots,r_{n-1}) :=$$
$$:= f(t,w(t) + r_o, w^{(1)}(t) + r_1,\dots,w^{(n-1)}(t) + r_{n-1}),$$

we get an integral equation of the form

$$(6)\qquad y(t) = F\Big(t, \int_a^b G(t,s)y(s)ds, \int_a^b \frac{\partial}{\partial t} G(t,s)y(s)ds,\dots$$
$$\dots, \int_a^b \frac{\partial^{n-1}}{\partial t^{n-1}} G(t,s)y(s)ds\Big) = Ty(t)$$

equivalent to (1), (2).

We make the following hypotheses used throughout this paper.

(A_1) There exists a continuous function $g: I\times R_+^n \to R_+ = [0,\infty)$ nondecreasing with respect to the last n variables and such that $g(t,0,0,\dots,0) \equiv 0$.

(A_2) For $(t,r_0,r_1,\dots,r_{n-1})$, $(t,\bar r_0,\bar r_1,\dots,\bar r_{n-1}) \in I\times R^n$ there is
$|F(t,r_0,r_1,\dots,r_{n-1}) - F(t,\bar r_0,\bar r_1,\dots,\bar r_{n-1})| \leqslant$
$\leqslant g(t,|r_0-\bar r_0|,|r_1-\bar r_1|,\dots,|r_{n-1}-\bar r_{n-1}|)$.

(A_3) There exists a continuous function $\bar u: I \to R_+$ satisfying the inequality $M\bar u(t) + h(t) \leqslant \bar u(t)$, where the operator M is given by the relation

$$(7)\qquad Mu(t) = g\Big(t, \int_a^b |G(t,s)|\,u(s)ds, \int_a^b \Big|\frac{\partial}{\partial t} G(t,s)\Big|\,u(s)ds,\dots$$
$$\dots, \int_a^b \Big|\frac{\partial^{n-1}}{\partial t^{n-1}} G(t,s)\Big|\,u(s)ds\Big),$$

and

$$(8)\qquad h(t) = \sup_{a\leqslant\xi\leqslant t} |F(\xi,0,0,\dots,0)|.$$

(A_4) In the class of functions satisfying the condition $0 \leqslant u(t) \leqslant \bar u(t)$, $t\in I$, the function u, $u(t) = 0$ for $t \equiv I$, is the only measurable solution of the equation

$$(9)\qquad u(t) = Mu(t),\quad t\in I.$$

Define now the sequence $\{y_m\}$ by the relations

(10) $y_0(t) \equiv 0, \quad y_{m+1}(t) = Ty_m(t), \quad t \in I, \ m=0,1,2,\ldots$.

To prove the convergence of the sequence $\{y_m\}$ to the solution $\bar{y}$ of (6), we define the sequence $\{u_m\}$ by the relations

(11) $\bar{u}_0(t) = \bar{u}(t), \quad u_{m+1}(t) = Mu_m(t), \quad t \in I, \ m=0,1,2,\ldots,$

where the function $\bar{u}(t)$ is from hypothesis (A_3).

Theorem 1. By the hypotheses (A_1)-(A_4), there exists a solution $\bar{y} \in C(I,R)$ of the equation (6). The sequence (10) converges uniformly to $\bar{y}$ in I, and the estimations (cf. (11))

(12) $|\bar{y}(t) - y_m(t)| \leqslant u_m(t), \quad t \in I, \ m=0,1,2,\ldots,$

(13) $|\bar{y}(t)| \leqslant \bar{u}(t), \quad t \in I,$

hold. Moreover, the solution $\bar{y}$ of (6) is unique in the class of functions satisfying (13).

The next result gives conditions under which equation (6) has at most one solution; these conditions do not guarantee the existence of a solution of equation (6).

Theorem 2. Let hypotheses (A_1), (A_2) be fulfilled. If the function r, $r(t) \equiv 0$ for $t \in I$, is the nonnegative, finite and measurable solution of the inequality

(14) $r(t) \leqslant Mr(t), \quad t \in I,$

then the equation (6) has at most one solution in I.

In order to establish a theorem on the stability of the solution of (6), we consider the equation

$$(15)\quad z(t) = H\Big(t, \int_a^b G(t,s)z(s)ds, \int_a^b \frac{\partial}{\partial t} G(t,s)z(s)ds, \ldots$$

$$\ldots, \int_a^b \frac{\partial^{n-1}}{\partial t^{n-1}} G(t,s)z(s)ds\Big),$$

with $H \in C(I\times R^n, R)$.

T h e o r e m 3. Assume that the hypotheses (A_1), (A_2) hold and

(i) $\bar{y}$ and $\bar{z}$ are solutions of equations (6) and (15), respectively;

(ii) the sequence $\{v_m(t)\}$, $t \in I$, defined by the relations

$$(16)\quad \begin{cases} v_0(t) \geq |\bar{y}(t)| + |\bar{z}(t)|, \\ v_{m+1}(t) = Mv_m(t) + \bar{h}(t), \quad m = 0,1,2,\ldots, \end{cases}$$

where

$$(17)\quad \bar{h}(t) = |T\bar{z}(t) - \bar{z}(t)|,$$

has a limit $\bar{v}(t)$ for $t \in I$. Then

$$(18)\quad |\bar{y}(t) - \bar{z}(t)| \leq \bar{v}(t), \quad t \in I.$$

Theorems 1-3 can be very easily extended to the more general equation

$$(19)\quad x^{(n)} = f\Big(t,x,x^{(1)},\ldots,x^{(n-1)}, \int_a^t K\big[t,s,x,x^{(1)},\ldots,x^{(n-1)}\big]ds\Big),$$

with the boundary conditions (2) under some suitable assumptions on the functions involved in (19). In [11] Morchało has obtained some results on the existence and uniqueness of the solutions of (19), (2) with deviating arguments by using the Banach fixed point theorem (see also [14]). Here, our approach to the problem is different. We note that Theorems 1-3 can also be extended to (1), (2) and (19), (2) with deviating arguments as in [8], [9], [11] with suitable modifications.

3. Proofs of Theorems 1-3

First we prove the following lemma used in our further discussion.

Lemma 1. If the hypotheses (A_1)-(A_4) are satisfied, then

(20) $$0 \leq u_{m+1}(t) \leq u_m(t) \leq \bar{u}(t), \quad t \in I, \ m = 0,1,2,\ldots,$$

and $u_m \Rightarrow 0$ for $m \to \infty$, where $\Rightarrow$ denotes the uniform convergence in I.

Proof. The relation (20) follows by induction. Since the sequence of continuous functions u_m is nonincreasing and bounded below, it is convergent to a certain measurable function p such that $0 \leq p(t) \leq \bar{u}(t)$ for $t \in I$. By the Lebesgue theorem and the continuity of g, it follows that the function p satisfies equation (9) and, by assumption (A_4), we have $p(t) \equiv 0$, $t \in I$. The uniform convergence of $\{u_m\}$ in I follows from the Dini theorem. This completes the proof of Lemma 1.

Proof of Theorem 1. We first prove that the sequence (10) satisfies

(21) $$|y_m(t)| \leq \bar{u}(t) \quad t \in I, \quad m = 0,1,2,\ldots .$$

Obviously $|y_0(t)| \equiv 0 \leq \bar{u}(t)$, $t \in I$. Furthermore, supposing that (21) is true, by (A_1)-(A_4), we have

$$|y_{m+1}(t)| \leq M|y_m(t)| + h(t) \leq M\bar{u}(t) + h(t) \leq \bar{u}(t),$$

for $t \in I$. The relation (21) follows by induction. Next we prove that

(22) $$|y_{m+q}(t) - y_m(t)| \leq u_m(t), \quad t \in I, \ m,q = 0,1,2,\ldots .$$

By (21), we have

$$|y_q(t) - y_0(t)| = |y_q(t)| \leqslant \bar{u}(t) = u_0(t), \quad t\in I, \ q=0,1,2,\ldots .$$

Suppose that (22) is true for $m,q \geqslant 0$, then

$$|y_{m+q+1}(t) - y_{m+1}(t)| = |Ty_{m+q}(t) - Ty_m(t)| \leqslant$$

$$\leqslant M|y_{m+q}(t) - y_m(t)| \leqslant Mu_m(t) = u_{m+1}(t).$$

Now we obtain (22) by induction. By Lemma 1, $u_m(t) \Rightarrow 0$ in I, so from (22) we have $y_m \Rightarrow \bar{y}$ in I. The continuity of $\bar{y}$ follows from the uniform convergence of the sequence $\{y_m\}$ and from the continuity of all functions y_m. If $q \to \infty$, then (22) gives (12) and the estimation (13) is implied by (21). It is obvious that $\bar{y}$ is a solution of (6).

To prove the uniqueness of the solution $\bar{y}$ of (6), let us suppose that there exists another solution $\hat{y}$ of (6) such that $\bar{y}(t) \not\equiv \hat{y}(t)$ and $|\hat{y}(t)| \leqslant u(t)$ for $t \in I$. By induction we get $|\hat{y}(t)-y_m(t)| \leqslant u_m(t)$, $t \in I$, $m = 0,1,2,\ldots$, and hence it follows that $\bar{y}(t) \equiv \hat{y}(t)$, $t \in I$. This contradiction proves the uniqueness of $\bar{y}$ in the class of functions satisfying (13). This completes the proof of Theorem 1.

P r o o f of Theorem 2. Let us suppose that there exist two solutions $\bar{y}$ and $\hat{y}$ of equation (6) in I, $\bar{y}(t) \not\equiv \hat{y}(t)$, $t \in I$. Now, by hypotheses (A_1), (A_2), we have

$$(23) \qquad |\bar{y}(t) - \hat{y}(t)| \leqslant M|\bar{y}(t) - \hat{y}(t)|, \quad t \in I.$$

Putting $r(t) = |\bar{y}(t) - \hat{y}(t)|$, $t \in I$, in (23), we infer from (14) that $r(t) \equiv 0$ for $t \in I$, i.e. $\bar{y}(t) \equiv \hat{y}(t)$, $t \in I$. This contradiction completes the proof of Theorem 2.

P r o o f of Theorem 3. Letting

$$(24) \qquad v(t) = |\bar{y}(t) - \bar{z}(t)|, \quad t \in I,$$

we have

(25) $$v(t) \leq |T\bar{y}(t) - T\bar{z}(t)| + |T\bar{z}(t) - \bar{z}(t)| \leq$$
$$\leq M|\bar{y}(t) - \bar{z}(t)| + \bar{h}(t) = Mv(t) + \bar{h}(t).$$

From (24), (25) we deduce

(26) $$v(t) \leq |\bar{y}(t)| + |\bar{z}(t)| \leq v_0(t), \quad t \in I.$$

Now, by induction, we get

(27) $$v(t) \leq v_m(t), \quad t \in I, \ m=0,1,2,\dots .$$

Inequality (18) is implied by (27), as $m \to \infty$. This completes the proof of Theorem 3.

4. Further applications

First we shall consider the following BVP

(28) $$Lx = f(t,x),$$

(29) $$B_i x = C_i, \quad i = 1,\dots,m, \quad m \geq 2.$$

Here $f \in C(I\times R,R)$, C_i are given constants, the operators L, B_i are defined by the relations

$$Lx := x^{(m)}(t) + \sum_{j=1}^{m} p_j(t)\, x^{(m-j)}(t),$$

$$B_i x := \sum_{j=1}^{m} \alpha_{ij}\, x^{(j-1)}(a) + \sum_{j=1}^{m} \beta_{ij}\, x^{(j-1)}(b), \qquad i = 1,\dots,m,$$

where α_{ij}, β_{ij} are real constants and $p_j \in C(I,R)$. We notice that in recent papers [15], [16] V. Šeda has studied the existence and uniqueness of the solutions of (28), (29) by using certain fixed point theorems. For a function $x \in C^m(I,R)$ we define $y(t) = Lx(t)$, then (28), (29) is equivalent to the integral equation (see [16], p.538)

(30) $$y(t) = f\left(t, w(t) + \int_a^b G(t,s)y(s)ds\right),$$

where $w(t)$ is the unique solution of the problem $Lx = 0$, (29) and $G(t,s)$ is the Green function of this problem. The equation (30) can be considered as a special case of (5) and hence Theorems 1-3 can be applied.

We next consider the following BVP

(31) $$x' = A(t)x + f_0(t,x),$$

(32) $$Nx(a) + Qx(b) = 0,$$

where $f_0 \in C(I\times R^n, R^n)$, $A(t)$ is a continuous $n\times n$ matrix function defined on I, and N, Q are constant $n\times n$ matrices. For the detailed discussion on the various special forms of (31), (32) we refer the readers to the expositary paper by Conti [3] and some of the references given therein. For the function $x \in C^1(I,R^n)$, we define $y(t) = x'(t) - A(t)x(t)$, then (31), (32) is equivalent to (see [1], [3])

(33) $$y(t) = f_0\left(t, \int_a^b G(t,s)y(s)ds\right),$$

where

$$G(t,s) = \begin{cases} X(t,s) - X(t,a)\left[N+QX(b,a)\right]^{-1}NX(b,s), & a \leqslant s \leqslant t \leqslant b, \\ -X(t,a)\left[N+QX(b,a)\right]NX(b,s) & a \leqslant t \leqslant s \leqslant b, \end{cases}$$

and $X(t,s)$ is the Cauchy matrix for the linear equation $x'(t) = A(t)x(t)$ with $X(a,a) = I_0$ and I_0 is the unit matrix. Theorems 1-3 can be extended to the equation (33) with suitable modifications.

Finally we consider the following BVP arising in transport processes (see [2], [12], [13])

(34) $$\begin{cases} x' = f_1(t,x,y), & x(a) = x_a, \\ -y' = f_2(t,x,y), & y(b) = y_b, \end{cases}$$

where $f_1, f_2 \in C(I \times R^n \times R^n, R^n)$ and x_a, y_b are given vectors. Let $z, w \in C(I, R^n)$ and define $z = x'$, $w = y'$. It is easy to observe that the pair of functions (z,w) fulfils the equations

$$(35)\quad \begin{cases} z(t) = f_1\left(t, x_a + \int_a^t z(s)ds,\ y_b + \int_t^b w(s)ds\right), \\ \\ w(t) = f_2\left(t, x_a + \int_a^t z(s)ds,\ y_b + \int_t^b w(s)ds\right). \end{cases}$$

Theorems 1-3 can be extended to (35) under some appropriate assumptions on the functions involved in (34).

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