DEMONSTRATIO MATHEMATICA
Vol. XXII No 3 1989

Ali A. A. Daw

A FORMULA FOR THE NUMBER OF RETRACTS OF FINITE BOOLEAN ALGEBRAS

In this paper we will construct and count the retracts of a finite Boolean algebra. The construction is based on some of the already known facts from the theory of Boolean algebras. We denote a Boolean algebra by $\underline{A}$.

1. Let Δ be the (principal) ideal of $\underline{A}$ generated by an element $-E \in \underline{A}$. Then

$$\underline{A} \mid E \cong \underline{A} / \Delta .$$

The isomorphism is given by

$$h(A) = [A] \quad \text{for } A \in \underline{A} \mid E$$

(Sikorski, [1] pp. 30-31).

2. Let $h : \underline{A} \longrightarrow \underline{A}$ be an endomorphism, h is a retract if $h|h(\underline{A})$ = identity, i.e.

$$h(x) = x \quad \text{for all } x \in h(\underline{A}).$$

Since, $h(\underline{A}) \cong \underline{A} / \Delta$, we can find retracts as follows. We take Δ (generated by $-E \in \underline{A}$. We find $\underline{A} / \Delta$ and the isomorphism $h : \underline{A} \longrightarrow \underline{A} / \Delta$. We have

$$\underline{A} \mid E \cong \underline{A} / \Delta \quad \text{where } h(\underline{A}_1) = h(\underline{A}) .$$

Now let A_1 be a subalgebra of $\underline{A}$ with k elements such that,

$h(A_1) = \underline{A} / \Delta$. Then $\underline{A}_1 \cong \underline{A} / \Delta$ (both subalgebras have the same number of elements, so they are isomorphic).

Let $i : \underline{A} / \Delta \longrightarrow \underline{A}_1$ be this isomorphism. Then $i[x] = x$, $h_1 = i \circ h$ is a retract.

P r o o f. $h_1 : \underline{A} \longrightarrow \underline{A}_1 \subseteq \underline{A}$ is an endomorphism $h_1(\underline{A}) = \underline{A}_1$. For $x \in \underline{A}_1$ we have

$$h_1(x) = i \circ h(x) = i([x]) = x .$$

3. T h e o r e m. Let $\underline{A}$ be a finite Boolean algebra $\underline{A} = 2^n$, $n < \infty$. There are exactly n retractions onto two elements subalgebra $\underline{A}_0 = \{\emptyset, 1\}$.

P r o o f. Take $k = 1$ and let $E = \{a_1\}$, $-E = \{a_2, \ldots, a_n\}$. Then $\underline{A} \mid E = \{\emptyset, \{a_1\}\}$ and $\Delta = 2^{\{a_1, \ldots, a_n\}}$. Then $\underline{A} \mid E \cong \underline{A} / \Delta$ where the isomorphism is given by $h(A) = [A]$ for $A \in \underline{A} \mid E$ (Sikorski [1], pp. 30-31). We have

$[\emptyset] = \{\emptyset, \{a_2\}, \ldots, \{a_n\}, \ldots, \{a_2, \ldots, a_n\}\}$

$[a_1] = \{\{a_1\}, \{a_1, a_2\}, \ldots, \{a_1, a_2, \ldots, a_n\}\}$

$\underline{A}_1 = \{\emptyset, \{a_1, \ldots, a_n\}\}$, we find only one possibility. Hence the number of possible four selections of E is $\binom{n}{1}$. Hence, the number of possibilities is $\binom{n}{1} \cdot 1 = n$.

4. T h e o r e m. Let $\underline{A}$ be a finite Boolean algebra $|\underline{A}| = 2^n$, $n \geq 2$. There are exactly $\binom{n}{2} 2^{n-2}$ retraction onto four elements subalgebras.

P r o o f. Let $\underline{A} \cong 2^n$, $\underline{A} = 2^X$, where $X = \{ a_1, \ldots, a_n\}$ are atoms. Take an element $E \in \underline{A}$ and $-E$ complement of $E \in \underline{A}$. $E = \{a_1, a_2, \ldots, a_k\}$, $-E = \{a_{k+1}, \ldots, a_n\}$. Take $k = 2$. We have

$E = \{a_1, a_2\}$, $-E = \{a_3, a_4, \ldots, a_n\}$,

$\underline{A} \mid E = \{\emptyset, \{a_1\}, \{a_2\}, \{a_1, a_2\}\}$ and the principle ideals

$\Delta = \{\emptyset, \{a_3\}, \ldots, \{a_3, a_4, \ldots, a_n\}\}$

$\Delta = 2^{\{a_3, \ldots, a_n\}}$ and $\underline{A} \mid E \cong \underline{A} / \Delta$

$[\emptyset] = \{\emptyset, \{a_3\}, \ldots, \{a_n\}, \{a_3, \ldots, a_n\}\}$

$[a_1] = \{\{a_1\}, \{a_1, a_3\}, \{a_1, a_4\}, \ldots, \{a_1, a_3, \ldots, a_n\}\}$

$[a_2] = \{\{a_2\}, \{a_2, a_3\}, \ldots, \{a_2, a_3, \ldots, a_n\}\}$

$[a_1, a_2] = \{\{a_1, a_2\}, \{a_1, a_2, a_3\}, \ldots, \{a_1, a_2, a_3, \ldots, a_n\}\}$

$\underline{A}_1 = \{\emptyset, \{a_1, a_3\}, \{a_2, a_4, \ldots, a_n\}, \{a_1, a_2, \ldots, a_n\}\}$.

In general, we have

$\underline{A} = \{\emptyset, A, -A, \{a_1, \ldots, a_n\}\}$

where $A \in [a_1]$, $-A \in [a_2]$.

The number of elements in the class equals the number of all elements of Δ i.e. the number of all subsets of $\{a_3, \ldots, a_n\}$, that is 2^{n-2}. The number of possibilities for selection of E is $\binom{n}{2}$. Hence, the number of possibilities $\binom{n}{2} \cdot 2^{n-2}$.

5. D e f i n i t i o n. Let $P(n,k)$, $n = 1,2,\ldots$, $1 \le k \le n$ denote the number of partitions of $\{1,2,\ldots,n\}$ into k non-empty disjoint subsets (blocks) such that no member of these partitions is a subset of $\{k+1,\ldots,n\}$.

6. T h e o r e m. Let $\underline{A}$ be a finite Boolean algebra $|\underline{A}| = 2^n$. The number of retracts of $\underline{A}$ onto the 2^k - elements subalgebras of $\underline{A}$ $(1 \le k \le n)$ is exactly $\binom{n}{k} P(n,k)$. Consequently, the number of all retracts of $\underline{A}$ is

$$\sum_{k=1}^{n} \binom{n}{k} P(n,k) .$$

P r o o f. For k = 1 we have P(n,1) = 1. Hence the number of retracts of $\underline{A}$ onto two elements subalgebras is $\binom{n}{1}\cdot 1 = n$, is agreement with Theorem 3.

(P(n,1) = 1 because there is only one partition of {1,2,...,n} into 1 block - namely {1,2,...,n}; and this set is not a subset of {2,3,...,n}. For k=2 we have $P(n,2)= 2^{n-2}$ (in fact, there are 2^{n-2} partitions of {1,2,...,n} into two blocks in such a way that no member of this partition in a subset of {3,4,...,n}). One member of such a partition is of the form {1} ∪ A, where A is a subset of {3,4,...,n}, the other member of this partition is {2} ∪ B, where B = {3,4,...,n} - A). Since there are 2^{n-2} subsets of {3,4,...,n}, we obtain $P(n,2) = 2^{n-2}$ as above.

Hence the number of retracts of $A \cong 2^n$ onto the four elements subalgebras is $\binom{n}{2}\ 2^{n-2}$ as proved in Theorem 4.Hence Theorem 3 and 4 are special cases of Theorem 6.

We now prove Theorem 6 in general case for arbitrary $1\le k\le n$. We may assume that $\underline{A} = 2^X$ where $X = \{a_1,a_2,\dots,a_n\}$. To obtain retracts onto 2^k elements subalgebras we take e.g. $E = \{a_1,a_2,\dots,a_k\}$. We have $\binom{n}{k}$ such selections for possible E with k elements.

By the procedure explained previously we take

$$-E = \{a_{k+1},\dots,a_n\} ,$$

$$\Delta = 2^{-E} = \{\varnothing,\{a_{k+1}\},\dots,\{a_n\},\dots,\{a_{k+1},\dots,a_n\}\} .$$

Since $\underline{A} \mid E \cong \underline{A} / \Delta$ we have

$[\emptyset] = \{\emptyset, \{a_{k+1}\},\dots,\{a_n\},\dots,\{a_{k+1},\dots,a_n\}\}$

$[a_1] = \{\{a_1\}\ \{a_1 a_{k+1}\},\dots,\{a_1 a_{k+1},\dots,a_n\}\}$

$[a_2] = \{\{a_2\}\ \{a_2 a_{k+1}\},\dots,\{a_2 a_{k+1},\dots,a_n\}\}$

$\vdots$

$[a_i] = \{\{a_i\}\ \{a_i a_{k+1}\},\dots,\{a_i a_{k+1},\dots,a_n\}\}$

$\vdots$

$[a_k] = \{\{a_k\}\ \{a_k a_{k+1}\},\dots,\{a_k a_{k+1},\dots,a_n\}\}$

$\{a_1 a_2\} = \{\{a_1 a_2\}\ \{a_1 a_2 a_{k+1}\},\dots,\{a_1 a_2 a_{k+1},\dots,a_n\}\}$

$\vdots$

$[a_1 a_2,\dots,a_k] = \{\{a_1 a_2,\dots,a_k\},\dots,\{a_1,a_2,\dots,a_k a_{k+1},\dots,a_n\}\}.$

Now we have to select for each class

$$[\emptyset],\ [a_1],\ \dots,[a_k],\dots,[a_1,\dots,a_k]$$

exactly one element in such a way that the selected element form a 2^k element subalgebra of $\mathcal{A}$ (in this way we obtain an isomorphism

$i : \underline{A} / \Delta \longrightarrow \underline{A})$. Clearly we must have

$i([\emptyset]) = \emptyset = 0 \in \underline{A}$

$i([a_1,\dots,a_k]) = \{a_1,\dots,a_k,\ a_{k+1},\dots,a_n\} = 1 \in \underline{A}.$

First we make a selection from the k abstraction classes $[a_1],\dots,[a_k]$.

Let A_i be selected from $[a_i]$, $i= 1,2,\dots,k$. We show that all A_i are pairwise disjoint, $A_i \cap A_j = \emptyset$ for $i \neq j$. From the construction of $[a_j]$, $i = 1,2,\dots,n$ it is clear that no inclusion $A_i \subseteq A_j$ or $A_j \subseteq A_i$ is possible (no member of the class $[a_i]$ may be contained in a member of the classes $[a_j]$

- so we have $a_i \in A_i - A_j$ and $a_j \in A_j - A_i$).

Assume that we have

$$A_i \cap A_j \neq \emptyset .$$

Since $A_i - A_j \neq \emptyset$, $A_j - A_i \neq \emptyset$ we see that the subalgebra generated by the two elements A_i, A_j would contain at least three atoms (contained in $A_i - A_j \neq \emptyset$, $A_j - A_i \neq \emptyset$ and $A_i \cap A_j \neq \emptyset$).

Hence every pair of integer elements would give at least 3 atoms and one atom would belong to Δ. This means that the subalgebra generated by the elements $A_1, \dots, A_k$ would contain more than 2^k elements - a contradiction. Hence only selections $A_1, \dots, A_n$ with disjoint elements are possible.

(*) Let now $\{a_1 a_2, \dots, a_n\} = A_1 \cup A_2, \dots, \cup A_k$ be a partition of $\{a_1, \dots, a_n\}$ into k disjoint subsets such that no subset A_i is contained in $\{k+1, \dots, n\}$.

We show that every A_i belongs exactly to one class $[a_j]$. It suffices to show that for each A_i there is a_j such that $A_i \in [a_j]$ (the elements $A_1, \dots, A_n$ are disjoint, no two of them can belongs to the same class $[a_i]$). If A_i is a one-element set, this is clear because A_i must be then one of the set $\{a_1\}, \dots, \{a_k\}$. If A_i contains more than one element it must be the form

$$\{a_j\} \cup B \quad \text{where} \quad B \subseteq \{a_{k+1}, \dots, a_n\}$$
$$1 \leq j \leq k \qquad B \neq \emptyset$$

i.e. A_i must contain no more than one element from $\{a_1, \dots, a_k\}$.

In fact, if some A_i would be of the form e.g.

$$A_1 = \{a_1, a_2\} \cup B \qquad B \subseteq \{a_{k+1}, \ldots, a_n\}$$

then A_1 could not be a member of partitions satisfying the condition (*). In fact taking

$$\{a_1, \ldots, a_k, a_{k+1}, \ldots, a_n\} = (\{a_1 a_2\} \cup B) \cup (\{a_3\} \cup B_3) \cup$$
$$\cup (\{a_4\} \cup B_4) \cup \ldots \cup (\{a_k\} \cup B_k)$$

as possible partitions with $B_3, B_4, \ldots, B_k \subseteq \{a_{k+1}, \ldots, a_n\}$ we would obtain a partition into at most k-1 blocks, not into k blocks as required.

Hence A_1 is of the form $\{a_j\} \cup B$

$$B \subseteq \{a_{k+1}, \ldots, a_n\} \text{ and consequently } A_1 \in [a_j].$$

Hence for every partition $X = A_1 \cup \ldots \cup A_k$ satisfying the condition (*) we have (after suitable remembering)

$$A_1 = \{a_1\} \cup B_1$$
$$A_2 = \{a_2\} \cup B_2$$
$$\vdots$$
$$A_k = \{a_k\} \cup B_k$$

where $B_i \in \Delta$, $i = 1, 2, \ldots, k$.

Since $A_1, A_2, \ldots, A_k$ are disjoint, the subalgebra generated by them has exactly 2^k elements. We have exactly 2^k abstractions classes $\underline{A} / \Delta$. Every element of this subalgebra is of the form $A \cup B$ where $A \subseteq \{a_1, \ldots, a_k\}$, $B \subseteq \{a_{k+1}, \ldots, a_4\}$. Every element belongs to some abstraction class $\underline{A} / \Delta$, and no two elements may belong to the same class. In fact, if $A_1 \cup B_1$ and $A_2 \cup B_2$ are two differents elements, then $A_1 \neq A_2$

because this subalgebra isomorphic to $2^{\{a_1,\dots,a_k\}}$ and $A_1 \cup B_1 \in [A_1]$, $A_2 \cup B_2 \in [A_2]$ belong to a different classes.

Thus we have shown that for every partition of $\{1,\dots,n\}$ into k blocks such that no block is subset of $\{k+1,\dots,n\}$ we can have a subalgebra of $\underline{A}$ formed from elements selected from the abstraction classes $\underline{A} / \Delta$, and for different partition we obtain different subalgebras. Since every subalgebra defines a retract of $\underline{A}$ onto this subalgebra for a fixed E, we obtain P(n,k) retracts onto 2^k- elements subalgebras. This ends the proof of Theorem 6.

7. L e m m a. We have

$$P(n,k) = k^{n-k} .$$

P r o o f. We prove first the recurrence formula

$$P(n+1,k) = P(n,k)\cdot k .$$

In fact, from each partition T of the set

$$\{1,2,\dots,k,\ k+1,,\dots,n\}$$

into k non-empty disjoint subsets (satisfying the condition of Def. 5) we can obtain a partition of the set $\{1,2,\dots,n+1\}$ into k non-empty subsets by adding to a member of T the element n+1.

Of course, no member of this new partition is a subset of

$$\{k+1,\dots,n,\ n+1\} .$$

From each partition of $\{1,2,\dots,n\}$ (satisfying Def. 5) we obtain in this way k new partitions of

$$\{1,2,\dots,n+1\} .$$

Conversely, for partition T_1 of $\{1,\dots,n+1\}$ into k disjoint non-empty blocks such that no block is a subset of $\{k+1,\dots,n, n+1\}$ there is partition of $\{1,2,\dots,n\}$ (such that one block differs by $\{n+1\}$ from a block of T_1 (no block of T_1, contains only $\{n+1\}$).

Hence we obtain exactly

$$P(n+1,k) = P(n,k)\cdot k ,$$

$$P(n,k) = P(n-1,k)\cdot k .$$

Since $P(k,k) = 1$, we have by induction

$$P(n,k) = k^{n-k}.$$

8. C o r o l l a r y. The number of all retracts of $\underline{A} \cong 2^n$ onto its subalgebras is equal to

$$\sum_{k=1}^{m} \binom{n}{k} k^{n-k}.$$

This corollary follows directly from Theorem 6 and Lemma 7.

REFERENCES

[1] R. S i k o r s k i : Boolean algebras, 2nd ed. Springer Verlag, Berlin-Gottingen-Heidelberg-New York, 1964.

INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY OF WARSAW,
00-661 WARSZAWA

Received December 6, 1988.