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VARIATIONAL FORMULATION FOR A HOMOGENEOUS DIRICHLET PROBLEM FOR A PARABOLIC EQUATION

This work deals with a variational formulation in a closed subspace of the Sobolev-type space for the parabolic equation with the homogeneous Dirichlet condition. A variational principle is constructed i.e. the functional such that its critical point corresponds to a solution of the system of the initial and some terminal-boundary value problem.

1. Preliminaries

The parabolic equation is considered in a bounded domain $Q := \Omega \times (0,T)$ where $T > 0$ and Ω is an open bounded domain of $\mathbb{R}^m$ with a Lipschitz boundary $\partial\Omega$ (in the sense of Nečas [5]). We denote the parts of boundary ∂Q by $\Omega_0 := \Omega \times \{0\}$, $\Omega_T := \Omega \times \{T\}$, $\Gamma := \partial\Omega \times (0,T)$. We define a Sobolev-type space

$$H := \{ v \in L^2(Q) : D_t v,\ D_i v,\ D_i D_t v \in L^2(Q) \text{ for } i = 1,\dots,m \}$$

where D_i, D_t are distributional derivatives. This space is the Hilbert space with the scalar product given by

$$(1.1)\quad (u,v)_H := \int_Q [uv + D_t u\, D_t v + D_i u\, D_i v + D_i D_t u\, D_i D_t v]\, dx\, dt$$

and the corresponding norm $\|\cdot\|_H$. Here and below we shall use the Einstein convention for the sum

$$D_i u\, D_i v := \sum_{i=1}^{m} D_i u\, D_i v \quad \text{and} \quad (D_i u)^2 := D_i u\, D_i u.$$

For a formal description of the boundary condition on Γ we introduce a closed linear subspace V of H.

D e f i n i t i o n. Let $C^{\infty}_{0x}(\bar{Q})$ be the set of all functions infinitely differentiable in $\bar{Q}$ which vanish in some neighbourhood of $\bar{\Gamma}$. Then V is the closure of $C^{\infty}_{0x}(\bar{Q})$ in H. Under the assumption on the boundary $\partial\Omega$ the measure γ on Γ and the space $L^2(\Gamma)$ are well defined and the following lemma is true [5].

L e m m a 1. There are the linear and continuous operators of trace :

Tr_0: H $\longrightarrow H^1(\Omega)$ Tr_T: H $\longrightarrow H^1(\Omega)$ Tr_Γ : H $\longrightarrow L^2(\Gamma)$. ∎

We remark that there exists $Tr_\Gamma(D_t v) \in L^2(\Gamma)$ the trace of the derivative $D_t v$ on $L^2(\Gamma)$. From the definition of the space V there follows

L e m m a 2. For all $v \in V$, $Tr_0 v$, $Tr_T v \in H^1_0(\Omega)$ and $Tr_\Gamma v = Tr_\Gamma(D_t v) = 0$. ∎

Since for $\varphi \in C^{\infty}_{0x}(\bar{Q})$ $\quad \varphi(x,t) = \varphi(x,0) + \int_0^t \frac{\partial\varphi}{\partial t}(x,s)ds$

then

$$(1.2) \quad v(x,t) = Tr_0 v(x) + \int_0^t D_t v(x,s)ds \quad \text{a.e. in } \bar{Q} \quad \forall\, v \in V.$$

Let $k \geq 0$ be a constant. Let $a, a^* : V \times V \longrightarrow \mathbb{R}$ be bilinear forms given by

$$(1.3)\qquad a(u,v) := \int_Q [D_t u\, D_t v + D_i u\, D_i D_t v + kuD_t v]\, dxdt + \int_\Omega [D_i u|_0\, D_i v|_0 + ku|_0\, v|_0]\, dx$$

$$(1.4)\qquad a^*(u,v) := \int_Q [D_t u\, D_t v - D_i u\, D_i D_t v - kuD_t v]\, dxdt + \int_\Omega [D_i u|_T\, D_i v|_T + ku|_T\, v|_T]\, dx$$

where $v|_0$, $D_i v|_0$ and $v|_T$, $D_i v|_T$ are corresponding traces.

By integrating (1.3) by parts we get that these forms are adjoint i.e.

$$(1.5)\qquad \forall\ u,v \in V \quad a(u,v) = a^*(v,u)$$

and also that the bilinear form $e : V \times V \longrightarrow \mathbb{R}$ given by

$$(1.6)\qquad e(u,v) := \frac{1}{2}\,[a(u,v) + a^*(u,v)]$$

is symmetric. Its integral shape is the following

$$(1.7)\quad e(u,v) = \int_Q D_t u\, D_t v\, dxdt + \frac{1}{2}\int_\Omega [D_i u|_0 D_i v|_0 + ku|_0 v|_0] dx + \frac{1}{2}\int_\Omega [D_i u|_T\, D_i v|_T + ku|_T\, v|_T]\, dx\ .$$

L e m m a 3. The form $e : V \times V \longrightarrow \mathbb{R}$ is strictly positive

$$\forall\ v \in V \quad e(v,v) \geq 0 \quad \text{and} \quad e(v,v) = 0 \iff v = 0$$

P r o o f. Since k is a nonnegative constant $e(v,v) \geq 0$. Let $e(v,v) = 0$, then from equality $\int_Q (D_t v)^2\, dxdt = 0$ we have $D_t v = 0$ in $L^2(Q)$. By Fridrich's inequality [5]

$$\exists\, C > 0 \quad \forall\, w \in H_0^1(\Omega) \quad \|w\|_{H_0^1(\Omega)} \le C \sum_{i=1}^{m} \|D_i w\|_{L^2(\Omega)}$$

and from $\int_\Omega (D_i v_{|0})^2 dx = 0$ we get that $Tr_0 v = 0$.

Finally by (1.2) $v = 0$. ∎

This lemma permits us to introduce a new norm in the space V.

D e f i n i t i o n. The function $\|v\|_V := \sqrt{e(v,v)}$ is the norm in the space V and $e(\cdot,\cdot)$ is a scalar product in V.

The space $\bar{V}$ is a completion of V in the norm $\|\cdot\|_V$ and it is a Hilbert space with this product.

By $V(\Delta)$ we denote the space of all functions $v \in V$ such that $D_i D_i v \in L^2(Q)$.

We remark that from the continuity of the operators of trace Tr_0, Tr_T in $(V, \|\cdot\|_H)$ we have the following estimates

$$\|v\|_V \le C\|v\|_H \tag{1.8}$$

and a following lemma.

L e m m a 4. The space $V(\Delta)$ is dense in V.

P r o o f. From the density of $C_{0x}^\infty(\bar{Q})$ in $(V, \|\cdot\|_H)$ and from the inequality (1.8) we obtain that $C_{0x}^\infty(\bar{Q})$ is dense in $(V, \|v\|_V)$. Since $C_{0x}^\infty(\bar{Q}) \subset V(\Delta)$ we get our lemma. ∎

The bilinear forms a, a^* have the following continuity conditions in this new norm.

L e m m a 5. For every $v \in V(\Delta)$ there exist constants $C_1, C_2 > 0$ depending on v such that for all $u \in V$

$$|a(u,v)| \le C_1(v)\ \|u\|_V \tag{1.9}$$

$$|a^*(u,v)| \le C_2(v)\ \|u\|_V . \tag{1.10}$$

P r o o f. Integrating by parts (1.4) we obtain

$$a(u,v) = a^{*}(v,u) = \int_Q [D_t v\, D_t u + D_1 D_1 v\, D_t u - kvD_t u]\, dxdt + $$

$$+ \int_\Omega [D_1 v_{|T}\, D_1 u_{|T} + kv_{|T}\, u_{|T}]\, dx$$

and from Schwartz's inequality

$$|a(u,v)| \le \int_Q |D_t v + D_1 D_1 v - kv|\,|D_t u|\, dxdt +$$

$$+ \int_\Omega [\,|D_1 v_{|T}|\,|D_1 u_{|T}| + k|v_{|T}|\,|u_{|T}|\,]\, dx \le C_1(v)\|u\|_V .$$

Proof of the inequality (1.10) is similar. ∎

2. Formulation of the variational problem

For $f_Q \in L^2(\Omega)$ and $f_0 \in H^1_0(\Omega)$ we define a linear functional $f : V \longrightarrow \mathbb{R}$ by

$$\langle f,v\rangle := \int_Q f_Q D_t v\, dxdt + \int_Q [D_1 f_0 D_1 v_{|0} + kf_0 v_{|0}]\, dx. \tag{2.1}$$

It is obvious that

$$|\langle f,v\rangle| \le C\, \|v\|_V \qquad \forall\, v \in V. \tag{2.2}$$

We state our initial-boundary value problem in the following variational formulation :

$$\left[\begin{array}{l} \text{Find } u \in V \text{ such that :} \\ a(u,v) = \langle f,v\rangle \qquad \forall\, v \in V . \end{array}\right. \tag{2.3}$$

We denote this problem by VP.

The equality : $a(v,v) = e(v,v) = \|v\|^2_V$ implies that this problem has at least one solution. The above formulation is justified by the following

T h e o r e m 1. Let $u \in V$. Then u is a solution of VP iff it is a solution of the following IBV (initial-boundary value) problem

$$\text{(2.4a)} \qquad D_t u - D_i D_i u + ku = f_Q \qquad \text{in } L^2(Q) \,,$$

$$\text{(2.4b)} \qquad Tr_0 u = f_0 \qquad \text{in } H_0^1(\Omega) \,,$$

$$\text{(2.4c)} \qquad Tr_\Gamma u = 0 \qquad \text{in } L^2(\Gamma) \,.$$

P r o o f. ($\Longrightarrow$) Let $u \in V$ be a solution of VP. Identity (2.3) with

$v(x,t) = \int_0^{t'} \varphi(x,s)ds$ for any function $\varphi \in C_0^\infty(Q)$ yields

$$\text{(2.5)} \qquad \int_Q [D_t u \, \varphi + D_i u \, D_i \varphi + ku \, \varphi] \, dxdt = \int_Q f_Q \varphi \, dxdt \,.$$

By definition of the distributional derivative D_i we obtain (2.4a) in the space of distribution $\mathcal{D}'(Q)$. Since $f_Q \in L^2(Q)$ the left side of (2.4a) is a regular distribution. So $D_i D_i u \in L^2(Q)$ and (2.4a) is the equality in $L^2(Q)$. Now we take v such that

$$v(x,t) = \varphi_0(x) + \int_0^t \varphi(x,s)ds \text{ for any } \varphi_0 \in C_0^\infty(\Omega) \text{ and } \varphi \in C_0^\infty(Q).$$

Because (2.5) is still valid for all such v we get the following weak elliptic equation in $H_0^1(\Omega)$

$$\int_\Omega [D_i(u_{|0} - f_0) D_i \varphi_0 + k(u_{|0} - f_0)\varphi_0] dx = 0 \qquad \forall \, \varphi_0 \in C_0^\infty(\Omega)$$

which has a unique trivial solution. Hence $u_{|0} = f_0$ in $H_0^1(\Omega)$. Finally we recall that (2.4c) is valid for all elements $v \in V$ (cf (1.2)).

The reverse implication ($\Leftarrow$) is obvious. Indeed, multiplying (2.4a) by $D_t v$ for any $v \in V$ and integrating by parts we obtain (2.5) with $\varphi = D_t v$. From (2.4b) we have for all $v \in V$

$$\int_\Omega [D_1 u|_0 D_1 v|_0 + ku|_0 v|_0]\, dx = \int_\Omega [D_1 f_0 D_1 v|_0 + kf_0 v|_0] dx$$

and finally equality (2.3) holds. ∎

C o r o l l a r y. $D_1 D_1 u \in L^2(Q)$ is a necessary condition for the existence of the solution of the VP.

We associate the adjoint variational problem to VP denoted by VP* using the "mirror method" (cf [1], [4]).

$$(2.6)\qquad \left[\begin{array}{l} \text{Find } u \in V \text{ such that :} \\ a^*(u,v) = \langle g, v\rangle \quad \forall\, v \in V . \end{array}\right.$$

where $g : V \longrightarrow \mathbb{R}$ is a linear functional given by

$$(2.7)\qquad \langle g,v\rangle := \int_Q g_Q D_t v\, dxdt + \int_\Omega [D_1 g_T D_1 v|_T + kg_T v|_T] dx \quad \forall\, v \in V$$

for $g_Q \in L^2(Q)$ and $g_T \in H_0^1(\Omega)$. This VP* is equivalent to the following terminal-boundary value problem

$$(2.8a)\qquad D_t u + D_1 D_1 u + ku = g_Q \quad \text{in } L^2(Q) ,$$

$$(2.8b)\qquad Tr_T u = g_T \quad \text{in } H_0^1(\Omega) ,$$

$$(2.8c)\qquad Tr_\Gamma u = 0 \quad \text{in } L^2(\Gamma) .$$

3. Existence and uniqueness of a solution of the generalized variational problem

Lemma 4 permits us to define a generalized variational problem. For $v \in V(\Delta)$ we define the linear functional

$P_v : V \longrightarrow \mathbb{R}$ by

$$(3.1) \qquad \langle P_v, u\rangle := a(u,v) \qquad \forall\, u \in V .$$

Because it is continuous in $(V, \|\cdot\|_V)$ (cf (1.9)) there exists only one extension of P_v to $\overline{V}$ which we denote by $\overline{P}_v$. Similarly, there exists the extension $\overline{f}$ of the continuous functional f (2.1-2).

D e f i n i t i o n. The generalized variational problem denoted by GVP is

$$(3.2) \qquad \left[\begin{array}{l} \text{Find } u \in \overline{V} \text{ such that :} \\ \langle \overline{P}_v, u\rangle = \langle \overline{f}, v\rangle \qquad \forall\, v \in V(\Delta) . \end{array}\right.$$

We remark that if the solution u of GVP belongs to V it is a solution of VP in the sense of (2.3).

We shall state a certain general version of the Lax-Milgram theorem. Similar theorem can be found in [2], [3].

L e m m a 5. Let X be a normed space with the scalar product $[\cdot,\cdot]$ and the Hilbert space Y be the completion of X in the norm $\|\cdot\|$. If $F : Y \longrightarrow \mathbb{R}$ is a linear and continuous functional and $A : Y \times X \longrightarrow \mathbb{R}$ is a bilinear form satisfying the following conditions

$$(3.3) \qquad \forall\, v \in X \;\; \exists\, C > 0 \;\; \forall\, u \in Y \qquad |A(u,v)| \le C(v)\|u\|$$

$$(3.4) \qquad \exists\, \alpha > 0 \;\; \forall\, v \in X \qquad A(v,v) \ge \alpha\|v\|^2$$

then a variational problem

$$(3.5) \qquad \left[\begin{array}{l} \text{Find } u \in Y \text{ such that :} \\ A(u,v) = \langle F, v\rangle \qquad \forall\, v \in X , \end{array}\right.$$

has exactly one solution.

P r o o f. From the above conditions and the Riesz theorem the identity $A(u,v) = [u,Sv]$ defines the linear mapping $S : X \longrightarrow Y$ which is one to one onto $D := S(X)$. Indeed, if $Sv = 0$ then $0 = [v,Sv] = A(v,v) \ge \alpha\|v\|^2$ implies $v = 0$. The inequality

$$(3.6) \qquad \alpha\|v\|^2 \le A(v,v) = [v,Sv] \le \|v\|\|Sv\|$$

gives that there exists the invers operator $S^{-1} : D \longrightarrow X$ such that $\forall\ w \in D \quad \|S^{-1}w\| \le \frac{1}{\alpha}\|w\|$.

Because the functional F is continuous the linear functional $G : D \longrightarrow \mathbb{R}$ defined as $G := F \circ S^{-1}$ is also continuous, indeed for all $w \in D$ we have

$$(3.7) \qquad |\langle G,w\rangle| = |\langle F,S^{-1}w\rangle| \le C\|S^{-1}w\| \le \frac{C}{\alpha}\|w\| \qquad \forall\ w \in D.$$

The domain D of functional G is dense in Y. For $v_0 \in \overline{D}^{\perp}$ we can find a sequence $v_n \in X$ such that $v_n \longrightarrow v_0$ in Y. If there exists $v_k \in \overline{D}^{\perp}$, $\|v_k\| > 0$ then we have $0 = [v_k,Sv] = A(v_k,v)$ for all $v \in X$, also for $v = v_k$. Thus $0 = A(v_k,v_k) \ge \alpha\|v_k\|^2 > 0$ which is a contradiction. Therefore all $v_n \in \overline{D}$ and we have $0 = [v_n,v_0] \longrightarrow [v_0,v_0] = \|v_0\|^2$ which implies $v_0 = 0$.

Hence the functional G has only one extension $\overline{G}$ on Y. Once again using the Riesz theorem, we conclude that there exists exactly one element $u \in Y$ such that $\langle\overline{G},w\rangle = [w,u]$ for all $w \in Y$. This element u is a solution of (3.5) because and for all $v \in X$ and $w = Sv$

$$A(u,v) = [u,Sv] = [u,w] = \langle\overline{G},w\rangle = \langle F,S^{-1}w\rangle = \langle F,v\rangle$$

is fulfilled. ∎

Existence and uniquness of the solution of GVP follows from this lemma.

T h e o r e m 2. The generalized variational problem (GVP) has exactly one solution in the space $\overline{V}$.

P r o o f. It is easy to verify that for $X = V(\Delta)$, $Y = \overline{V}$, $F = \overline{f}$ and $A : Y \times X \longrightarrow \mathbb{R}$, $A(u,v) := \langle \overline{P}_v, u\rangle$ the assumptions of Theorem 2 are fulfilled.
(cf lemmas 4,5 and the equality for $v \in V$ $A(v,v) = e(v,v) = \|v\|_V^2$). ∎

4. Variational principle

For a, a^*, f, g we define a bilinear form $\hat{a} : \hat{V} \times \hat{V} \longrightarrow \mathbb{R}$ and a linear functional $\hat{f} : \hat{V} \longrightarrow \mathbb{R}$ by

$$\hat{a}(\hat{u},\hat{v}) := a(u_1,v_2) + a^*(u_2,v_1) \ ,$$

$$\langle\hat{f},\hat{v}\rangle := \langle f,v_2\rangle + \langle g,v_1\rangle \ ,$$

where $\hat{u} := (u_1,u_2)$ and $\hat{v} := (v_1,v_2)$ are elements of the space $\hat{V} := V \times V$. Since the forms a, a^* are adjoint (cf (1.5)) the form $\hat{a}$ is selfadjoint. This fact permits us to associete the VP and VP^* problems with a certain functional whose variation is vanishing on solutions of these adjoint problems.

T h e o r e m 3. Let $\hat{u} = (u_1,u_2) \in \hat{V}$. Then $\hat{u}$ is a solution of the problem

Find $\hat{u} = (u_1,u_2) \in \hat{V}$ such that :

(4.9a) $\quad a(u_1,v) = \langle f,v\rangle$

(4.9b) $\quad a^*(u_2,v) = \langle g,v\rangle \qquad \forall\, v \in V$

iff $\hat{u}$ is a critical point of the functional $X : V \times V \longrightarrow \mathbb{R}$

(4.10) $X(u_1, u_2) := a(u_1, u_2) - \langle f, u_2 \rangle - \langle g, u_1 \rangle$.

P r o o f. Since $\hat{a}(\hat{u}, \hat{u}) = 2a(u_1, u_2)$ for all $\hat{u} = (u_1, u_2) \in \hat{V}$ it is possible to write the functional X in the following form

$$X(\hat{u}) = \frac{1}{2}\, \hat{a}(\hat{u}, \hat{u}) - \langle \hat{f}, \hat{u} \rangle \ .$$

Then $\langle X'(\hat{u}), \hat{v} \rangle := \lim_{t \to 0} \frac{1}{t} [\, X(\hat{u}+t\hat{v}) - X(\hat{u})] = \hat{a}(\hat{u}, \hat{v}) - \langle \hat{f}, \hat{v} \rangle$

$\forall\ \hat{v} \in \hat{V}$ and $X'(\hat{u}) = 0$ iff $\hat{a}(\hat{u}, \hat{v}) = \langle \hat{f}, \hat{v} \rangle$ for all $\hat{v} \in \hat{V}$. Finally we remark that $\hat{u} = (u_1, u_2) \in \hat{V}$ is a solution of the system (4.9) iff it fulfils the last equality. ∎

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