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ON SOME RETROSPECTIVE AND PROSPECTIVE EQUATIONS FOR A VECTOR-VALUED QUASI-DIFFUSION PROCESS

Let $\langle\Omega, F, P\rangle$ be a probability space, $T_0 = [t_0, T) \subset \mathbb{R}$, and let $X_t : \Omega \times T_0 \longrightarrow \mathbb{R}^m$ be a m-dimensional stochastic process. Further, let $t_0 \le t_1 < \ldots < t_n < t < \tau < T$, $\vec{t}_n = \langle t_1, \ldots, t_n\rangle$, $\vec{x}_n = \langle x_1, \ldots, x_n\rangle \in \mathbb{R}^{nm}$, where $x_k = \langle x_k^1, \ldots, x_k^m\rangle \in \mathbb{R}^m$, for $k = 1, \ldots, n$, $x = \langle x^1, \ldots, x^m\rangle \in \mathbb{R}^m$, and $A, B \in \mathcal{B}^m$, where $\mathcal{B}^m$ is a σ-algebra of Borel sets in $\mathbb{R}^m$.

In papers [1], [2], [3] there are given definitions of a quasi-diffusion process and there are investigated properties of this process in the one-dimensional case. Since quasi-diffusion processes are useful for description and stochastic investigation of some system of storage, it is of great importance to find their properties for the m-dimensional case with m > 1. In our paper, we are going to prove the Kolmogorov-type retrospective and prospective theorem for a non-Markov vector-valued quasi-diffusion process X_t.

We are calling the m-dimensional process X_t with values in $\mathbb{R}^m$ a quasi-diffusion process if for $n \ge 1$ and for arbitrarily chosen $\varepsilon > 0$, $\vec{t}_m$, t, τ, $\vec{x}_n$, x the following conditions are fulfilled ([2]) :

(1) $$\lim_{\tau \to t} \frac{1}{\tau - t} P_{\vec{t}_n, \vec{x}_n}(t,x;\tau,V'_{\varepsilon}(x)) = 0 ,$$

(2) $$\lim_{\tau \to t} \frac{1}{\tau - t} \int_{V_{\varepsilon}(x)} (y-x)\, P_{\vec{t}_n, \vec{x}_n}(t,x;\tau,dy) = A_{\vec{t}_n, \vec{x}_n}(t,x) ,$$

(3) $$\lim_{\tau \to t} \frac{1}{\tau - t} \int_{V_{\varepsilon}(x)} (y-x,z)^2 P_{\vec{t}_n, \vec{x}_n}(t,x;\tau,dy) = (B_{\vec{t}_n, \vec{x}_n}(t,x)z,z)$$

for $z = \langle z^1,\dots,z^m\rangle \in \mathbb{R}^m$, where $P_{\vec{t}_n, \vec{x}_n}(t,x;\tau,A)$ is the probability of the event that the quasi-diffusion process is in the state belonging to the Borel set A at the moment τ, while it was in $x_1,\dots,x_n,x$ states at earlier moments $t_1,\dots, t_n,t$. It means that in general we have

$$P_{\vec{t}_n, X_{t_1},\dots,X_{t_n}}(t,X_t;\tau,A) = P(X_\tau \in A \mid X_{t_1},\dots,X_{t_n},X_t).$$

Moreover,

$$V_{\varepsilon}(x) = \{ y \in \mathbb{R}^m : |y-x| < \varepsilon \},\quad V'_{\varepsilon}(x) = \mathbb{R}^m - V_{\varepsilon}(x) ,$$

$$A_{\vec{t}_n, \vec{x}_n}(t,x) = \langle A^1_{\vec{t}_n, \vec{x}_n}(t,x),\ \dots,\ A^m_{\vec{t}_n, \vec{x}_n}(t,x) \rangle ,$$

where

$$A^k_{\vec{t}_n, \vec{x}_n}(t,x) = \lim_{\tau \to t} \frac{1}{\tau - t} \int_{V_{\varepsilon}(x)} (y^k - x^k)\, P_{\vec{t}_n, \vec{x}_n}(t,x;\tau,dy) ,$$

for $k = 1,\dots,m$, and

$$B_{\vec{t}_n, \vec{x}_n}(t,x) = [\, b^{i,j}_{\vec{t}_n, \vec{x}_n}(t,x) \,] ,$$

$i,j = 1,\dots,m$, is a positively defined matrix for which

$$b^{i,j}_{\vec{t}_n,\vec{x}_n}(t,x) = \lim_{\tau \to t} \frac{1}{\tau - t} \int_{V_\varepsilon(x)} (y^i - x^i)(y^j - x^j) P_{\vec{t}_n,\vec{x}_n}(t,x;\tau,dy),$$

for $i,j = 1,\dots,m$.

We denote as $(B_{\vec{t}_n,\vec{x}_n}(t,x)z,z)$ and $(y-x,z)$ the scalar products. For example

$$(B_{\vec{t}_n,\vec{x}_n}(t,x)z,z) = \sum_{i=1}^{m} \sum_{j=1}^{m} b^{i,j}_{\vec{t}_n,\vec{x}_n}(t,x)\ z^i\ z^j .$$

We will use the following notation for differential operators

$$(A_{\vec{t}_n,\vec{x}_n}(t,x),\nabla) = \sum_{i=1}^{m} A^i_{\vec{t}_n,\vec{x}_n}(t,x)\ \frac{\partial}{\partial x^i},$$

$$(B_{\vec{t}_n,\vec{x}_n}(t,x)\nabla,\nabla) = \sum_{i=1}^{m} \sum_{j=1}^{m} b^{i,j}_{\vec{t}_n,\vec{x}_n}(t,x)\ \frac{\partial^2}{\partial x^i\ \partial x^j}.$$

Let $g : \mathbb{R}^m \longrightarrow \mathbb{R}$ denote a function integrable in $\mathbb{R}^m$ with respect to probability measures $P_{\vec{t}_n,\vec{x}_n}(t,x;\tau,dy)$, for all $\vec{t}_n$, t, τ, $\vec{x}_n$, x, which means that for

$$u_{\vec{t}_n,\vec{x}_n}(t,x;\tau) := \int_{\mathbb{R}^m} g(y)\ P_{\vec{t}_n,\vec{x}_n}(t,x;\tau,dy)$$

we have

$$\left|u_{\vec{t}_n,\vec{x}_n}(t,x)\right| \le K < +\infty .$$

Using now the known equality

(4) $$P_{\vec{t}_{n-1},\vec{x}_{n-1}}(t_n,x_n;\tau,A) =$$

$$= \int_{\mathbb{R}^m} P_{\vec{t}_n,\vec{x}_n}(t,x;\tau,A)\ P_{\vec{t}_{n-1},\vec{x}_{n-1}}(t_n,x_n;t,dx)$$

we easily show that

(5)
$$u_{\vec{t}_{n-1},\vec{x}_{n-1}}(t_n,x_n;\tau) =$$
$$= \int_{\mathbb{R}^m} u_{\vec{t}_n,\vec{x}_n}(t,y;\tau)\; P_{\vec{t}_{n-1},\vec{x}_{n-1}}(t_n,x_n;t,dy)\ .$$

Conditions (1)-(3) can be written in the following form :

$$(1^*)\qquad P_{\vec{t}_n,\vec{x}_n}(t,x;\tau,V'_\varepsilon(x)) = o(\tau-t)\ ,$$

$$(2^*)\quad \int_{V_\varepsilon(x)} (y-x)\; P_{\vec{t}_n,\vec{x}_n}(t,x;\tau,dy) = A_{\vec{t}_n,\vec{x}_n}(t,x)\;(\tau-t) + o_\varepsilon(\tau-t),$$

$$(3^*)\quad \int_{V_\varepsilon(x)} (y-x,z)^2 P_{\vec{t}_n,\vec{x}_n}(t,x;\tau,dy) = (B_{\vec{t}_n,\vec{x}_n}(t,x)z,z)(\tau-t) + o_\varepsilon(\tau-t),$$

where

$$\int_{V_\varepsilon(x)} (y-x)\; P_{\vec{t}_n,\vec{x}_n}(t,x;\tau,dy) =$$

$$< \int_{V_\varepsilon(x)} (y^1-x^1)\; P_{\vec{t}_n,\vec{x}_n}(t,x;\tau,dy),\dots,\int_{V_\varepsilon(x)} (y^m-x^m)\; P_{\vec{t}_n,\vec{x}_n}(t,x;\tau,dy >,$$

$$\int_{V_\varepsilon(x)} (y-x,z)^2 P_{\vec{t}_n,\vec{x}_n}(t,x;\tau,dy) =$$
$$= \sum_{i=1}^{m} \sum_{j=1}^{m} \left[\int_{\varepsilon(x)} (y^j-x^j)(y^i-x^i)\; z^j z^i\; P_{\vec{t}_n,\vec{x}_n}(t,x;\tau,dy \right] .$$

T h e o r e m 1. If conditions (1)-(3) are fulfilled and the bounded, continuous function $g : \mathbb{R}^m \longrightarrow \mathbb{R}$ is such that the function $u_{\vec{t}_{n-1},\vec{x}_{n-1}}(t_n,x_n;\tau)$ posesses continuous derivatives

$$\frac{\partial u_{\vec{t}_n,\vec{x}_n}(t,x;\tau)}{\partial x^i}\ ,\ \frac{\partial^2 u_{\vec{t}_n,\vec{x}_n}(t,x;\tau)}{\partial x^i\,\partial x^j}\ ,\ i,j = 1,\dots,m\ ,$$

then, the function $u_{\vec{t}_{n-1},\vec{x}_{n-1}}(t_n,x_n;\tau)$ posesses the derivative

$$\frac{\partial u_{\vec{t}_{n-1},\vec{x}_{n-1}}(t_n,x_n;\tau)}{\partial t_n}\ ,$$

the following equation is fulfilled

$$\frac{\partial u_{\vec{t}_{n-1},\vec{x}_{n-1}}(t_n,x_n;\tau)}{\partial t_n} =$$

$$= (A_{\vec{t}_{n-1},\vec{x}_{n-1}}(t_n,x_n),\nabla)\ u_{\vec{t}_{n-1},\vec{x}_{n-1}}(t_n,x_n;\tau) +$$

$$+ \frac{1}{2}\,(B_{\vec{t}_{n-1},\vec{x}_{n-1}}(t_n,x_n)\nabla,\nabla)\ u_{\vec{t}_{n-1},\vec{x}_{n-1}}(t_n,x_n;\tau)\ ,$$

and the following condition holds

$$\lim_{t_n \to \tau} u_{\vec{t}_{n-1},\vec{x}_{n-1}}(t_n,x_n;\tau) = g(x_n)\ .$$

P r o o f. We have $t_0 \le t_1 < \dots < t_{n-1} < t_n < t < \tau$. Expanding the function $u_{\vec{t}_n,\vec{x}_n}(t,y;\tau)$ into the corresponding Taylor power series in the neighbourhood $V_\varepsilon(x_n)$ of the point $x_n \in \mathbb{R}^m$ we have

$$(6)\quad u_{\vec{t}_n,\vec{x}_n}(t_n,y;\tau) - u_{\vec{t}_n,\vec{x}_n}(t,x_n;\tau) = (y-x,\nabla)u_{\vec{t}_n,\vec{x}_n}(t,x_n;\tau) +$$

$$+ \frac{1}{2}\,(y-x,\nabla)^2\, u_{\vec{t}_n,\vec{x}_n}(t,x_n;\tau) + r_{\vec{t}_n,\vec{x}_n}(t,\tau;x_n;y)\ ,$$

where $y \in V_\varepsilon(x_n) = \{\ z \in \mathbb{R}^m\ ;\ |z-x_n| < \varepsilon\ \}$, and

(7) $$\left|r_{\vec{t}_n,\vec{x}_n}(t_n,\tau;x_n;y)\right| \le \alpha_\varepsilon\cdot|y-x_n|^2,$$

$$\alpha_\varepsilon = \sup_{\substack{s\in(t_n,\tau)\\ y\in V_\varepsilon(x_n)\\ 1\le i,j\le m}} \left| \frac{\partial^2 u_{t_n,x_n}(s,y;\tau)}{\partial y^i\,\partial y^j} - \frac{\partial^2 u_{t_n,x_n}(s,x_n;\tau)}{\partial x_n^i\,\partial x_n^j} \right| ,$$

and

(8) $$\lim_{\varepsilon\to 0} \alpha_\varepsilon = 0 .$$

Moreover, form (5) and (1)-(3) we have

$$u_{\vec{t}_n,\vec{x}_n}(t_n,x_n;\tau) - u_{\vec{t}_n,\vec{x}_n}(t,x_n;\tau) =$$

$$= \int_{\mathbb{R}^m} [u_{\vec{t}_n,\vec{x}_n}(t,y;\tau) - u_{\vec{t}_n,\vec{x}_n}(t,x_n;\tau)]\, P_{\vec{t}_{n-1},\vec{x}_{n-1}}(t_n,x_n;t,dy) =$$

$$= \int_{V_\varepsilon(x_n)} [u_{\vec{t}_n,\vec{x}_n}(t,y;\tau) - u_{\vec{t}_n,\vec{x}_n}(t,x_n;\tau)] P_{\vec{t}_{n-1},\vec{x}_{n-1}}(t_n,x_n;t,dy) +$$

$$+ o_\varepsilon(t-t_n).$$

Using (6) we have

(9) $$u_{\vec{t}_{n-1},\vec{x}_{n-1}}(t,x_n;\tau) - u_{\vec{t}_n,\vec{x}_n}(t,x_n;\tau) =$$

$$= r^*_{\vec{t}_n,\vec{x}_n}(t,\tau;x_n) + o_\varepsilon(t-t_n) +$$

$$\int_{V_\varepsilon(x_n)} \left[(y-x_n,\nabla)u_{\vec{t}_n,\vec{x}_n}(t,x_n;\tau) + \frac{1}{2}(y-x_n,\nabla)^2 u_{\vec{t}_n,\vec{x}_n}(t,x_n;\tau)\right]\cdot$$

$$\cdot P_{\vec{t}_{n-1},\vec{x}_{n-1}}(t_n,x_n;t,dy) ,$$

where

$$o_\varepsilon(t-t_n)=\int_{V'_\varepsilon(x_n)}\left[u_{t_n,\vec{x}_n}(t,y;\tau)-u_{t_n,\vec{x}_n}(t,x_n;\tau)\right]P_{\vec{t}_{n-1},\vec{x}_{n-1}}(t_n,x_n;t,dy)$$

$$r^*_{\vec{t}_n,\vec{x}_n}(t,\tau;x_n) = \int_{V_\varepsilon(x_n)} r_{\vec{t}_n,\vec{x}_n}(t,\tau;x_n,y)P_{\vec{t}_{n-1},\vec{x}_{n-1}}(t_n,x_n;t,dy) \quad .$$

From (7) we have the inequality

$$|r^*_{\vec{t}_n,\vec{x}_n}(t,\tau;x_n)| \le \alpha_\varepsilon \int_{V_\varepsilon(x_n)} |y-x|^2 P_{\vec{t}_{n-1},\vec{x}_{n-1}}(t_n,x_n;t,dy) \ ,$$

and consequently using (3) and (8) we have

$$\left|\lim_{t\to t_n}\frac{1}{t-t_n} r^*_{\vec{t}_n,\vec{x}_n}(t,\tau;x_n)\right| \le \alpha_\varepsilon\left(B_{\vec{t}_n,\vec{x}_n}(t,x_n)\frac{y-x_n}{|y-x_n|} \ , \ \frac{y-x_n}{|y-x_n|}\right)$$

and this quantity tends to 0 if only $\varepsilon \longrightarrow 0$. Since ε is an arbitrarily chosen number and $\varepsilon > 0$, we may write

$$r^*_{\vec{t}_n,\vec{x}_n}(t,\tau;x_n) = o_\varepsilon(t-t_n) \ . \tag{10}$$

Finally, from (9), (10), (2), (3), ((2*) and (3*)) we get

$$\frac{u_{\vec{t}_{n-1},\vec{x}_{n-1}}(t_n,x_n;\tau) - u_{\vec{t}_n,\vec{x}_n}(t,x_n;\tau)}{t - t_n} =$$

$$= (A_{\vec{t}_{n-1},\vec{x}_{n-1}}(t_n,x_n),\nabla)\ u_{\vec{t}_n,\vec{x}_n}(t,x_n;\tau) +$$

$$+ \frac{1}{2}\,(B_{\vec{t}_{n-1},\vec{x}_{n-1}}(t_n,x_n)\nabla,\nabla)\ u_{\vec{t}_n,\vec{x}_n}(t,x_n;\tau) + \frac{o_\varepsilon(t-t_n)}{t-t_n} \ .$$

Since partial derivatives of the function $u_{\vec{t}_n,\vec{x}_n}(t,x_n;\tau)$ are continuous, then taking the limit with $t \longrightarrow t_n$ we have

$$- \frac{\partial}{\partial t_n} u_{\vec{t}_{n-1},\vec{x}_{n-1}}(t_n,x_n;\tau) = (A_{\vec{t}_{n-1},\vec{x}_{n-1}}(t_n,x_n),\nabla) \cdot$$

$$\cdot\, u_{\vec{t}_{n-1},\vec{x}_{n-1}}(t_n,x_n;\tau) +$$

$$+ \frac{1}{2} (B_{\vec{t}_{n-1},\vec{x}_{n-1}}(t_n,x_n)\nabla,\nabla)\, u_{\vec{t}_{n-1},\vec{x}_{n-1}}(t_n,x_n;\tau) ,$$

or in the equivalent form

$$-\frac{\partial u_{\vec{t}_{n-1},\vec{x}_{n-1}}(t_n,x_n;\tau)}{\partial t_n} = \sum_{i=1}^{m} A^{i}_{\vec{t}_{n-1},\vec{x}_{n-1}}(t_n,x_n)\frac{\partial u_{\vec{t}_{n-1},\vec{x}_{n-1}}(t_n,x_n;\tau)}{\partial x_n^i}$$

$$+ \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} b^{i,j}_{\vec{t}_{n-1},\vec{x}_{n-1}}(t_n,x_n) \frac{\partial^2 u_{\vec{t}_{n-1},\vec{x}_{n-1}}(t_n,x_n;\tau)}{\partial x_n^i\, \partial x_n^j} .$$

The second part of our Theorem follows easily from the equality

$$u_{\vec{t}_{n-1},\vec{x}_{n-1}}(t_n,x_n;\tau) = \int_{\mathbb{R}^m} g(y)\, P_{\vec{t}_{n-1},\vec{x}_{n-1}}(t_n,x_n;\tau,dy) =$$

$$= g(x_n) + \int_{\mathbb{R}^m} (g(y)-g(x_n))\, P_{\vec{t}_{n-1},\vec{x}_{n-1}}(t_n,x_n;\tau,dy) =$$

$$= g(x_n) + \int_{V_\varepsilon(x_n)} (g(y)-g(x_n))\, P_{\vec{t}_{n-1},\vec{x}_{n-1}}(t_n,x_n;\tau,dy) + o_\varepsilon(\tau-t_n).$$

From our assumption the function $g : \mathbb{R}^m \longrightarrow \mathbb{R}$ is bounded and continuous. Therefore, taking into account that ε is arbirarily chosen we have

$$\lim_{t_n \longrightarrow \tau} u_{\vec{t}_{n-1},\vec{x}_{n-1}}(t_n,x_n;\tau) = g(x_n) .$$

This completes the proof of our Theorem.

Let us assume now that the measures $P_{\vec{t}_n,\vec{x}_n}(t,x;\tau,dy)$ are absolute continuous with respect to the Lebesgue measure in $\mathbb{R}^m$. It means that there exist functions $f_{\vec{t}_n,\vec{x}_n}(t,x;\tau,y)$ such that for each set $B \in \mathcal{B}^m$ we have

$$(11) \qquad P_{\vec{t}_n,\vec{x}_n}(t,x;\tau,B) = \int_B f_{\vec{t}_n,\vec{x}_n}(t,x;\tau,y)\, dy.$$

T h e o r e m 2. If conditions (1)-(3) are fulfilled, $A_{\vec{t}_n,\vec{x}_n}(t,x)$ and $B_{\vec{t}_n,\vec{x}_n}(t,x)$ are continuous and such that for each function $f_{\vec{t}_{n-1},\vec{x}_{n-1}}(t_n,x_n;t,x)$ there exist continuous derivatives

$$\frac{\partial}{\partial t} f_{\vec{t}_{n-1},\vec{x}_{n-1}}(t_n,x_n;t,x), \quad \frac{\partial}{\partial x^i}\left[A^i_{\vec{t}_n,\vec{x}_n}(t,x)\, f_{\vec{t}_{n-1},\vec{x}_{n-1}}(t_n,x_n;t,x)\right],$$

$$\frac{\partial^2}{\partial x^i\, \partial x^j}\left[b^{i,j}_{\vec{t}_n,\vec{x}_n}(t,x)\, f_{\vec{t}_{n-1},\vec{x}_{n-1}}(t_n,x_n;t,x)\right],$$

for $i,j = 1,\dots,m$, then for $x \in \mathbb{R}^m$ and $t > t_n$ we have

$$\frac{\partial}{\partial t} f_{\vec{t}_{n-1},\vec{x}_{n-1}}(t_n,x_n;t,x) +$$

$$+ \sum_{i=1}^{m} \frac{\partial}{\partial x^i}\left[A^i_{\vec{t}_n,\vec{x}_n}(t,x)\, f_{\vec{t}_{n-1},\vec{x}_{n-1}}(t_n,x_n;t,x)\right] =$$

$$= \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} \frac{\partial^2}{\partial x^i\, \partial x^j}\left[b^{i,j}_{\vec{t}_n,\vec{x}_n}(t,x)\, f_{\vec{t}_{n-1},\vec{x}_{n-1}}(t_n,x_n;t,x)\right].$$

P r o o f. Let $t_0 \le t_1 < \dots < t_{n-1} < t_n < s < t < \tau$ and let $\varphi : \mathbb{R}^m \longrightarrow \mathbb{R}$, $\varphi \in \mathcal{C}^2(\mathbb{R}^m)$ be a function such that $\varphi(x) = 0$ for $x \notin \Gamma$, where Γ is a compact set in $\mathbb{R}^m$.

If $y \in V_\varepsilon(x) \subset \Gamma$, then from the Taylor expansion we have

$$\varphi(y) - \varphi(x) = (y-x,\nabla)\ \varphi(x) + \frac{1}{2}\ (y-x,\nabla)^2\varphi(x) + r(x,y)\ ,$$

where

$$|r(x,y)| \le \alpha_\varepsilon |y-x|^2\ ,\quad \alpha_\varepsilon = \sup_{\substack{y\in V_\varepsilon(x)\\ 1\le i,j\le m}} \left| \frac{\partial^2\varphi(x)}{\partial x^i\ \partial x^j} - \frac{\partial^2\varphi(y)}{\partial y^i\ \partial y^j} \right| ,$$

and $\lim\limits_{\varepsilon\to 0} \alpha_\varepsilon = 0$.

Therefore for the arbitrarily chosen $\varepsilon > 0$ we have

$$\frac{1}{\tau-s}\left[\ \int_{\mathbb{R}^m} \varphi(y)\ P_{\vec{t}_n,\vec{x}_n}(s,x;\tau,dy) - \varphi(x)\ \right] =$$

$$= \frac{1}{\tau-s}\left\{\ \int_{V_\varepsilon(x)} [\varphi(y)-\varphi(x)]\ P_{\vec{t}_n,\vec{x}_n}(s,x;\tau,dy)\right\} + \frac{o_\varepsilon(\tau-s)}{\tau-s} =$$

$$= \frac{1}{\tau-s}\left\{\ \int_{V_\varepsilon(x)} [(y-x,\nabla)\varphi(x)]\ P_{\vec{t}_n,\vec{x}_n}(s,x;\tau,dy) + \right.$$

$$+ \frac{1}{2}\int_{V_\varepsilon(x)} [(y-x,\nabla)^2\varphi(x)]\ P_{\vec{t}_n,\vec{x}_n}(s,x;\tau,dy) +$$

$$\left. + \int_{V_\varepsilon(x)} r(x,y)\ P_{\vec{t}_n,\vec{x}_n}(s,x;\tau,dy)\right\} + \frac{o_\varepsilon(\tau-s)}{\tau-s}\ .$$

Therefore,

$$\frac{1}{\tau-s}\left[\ \int_{\mathbb{R}^m} \varphi(y)\ P_{\vec{t}_n,\vec{x}_n}(s,x;\tau,dy) - \varphi(x)\ \right] = \tag{12}$$

$$= (A_{\vec{t}_n,\vec{x}_n}((s,x),\nabla)\varphi(x) + \frac{1}{2}\ (B_{\vec{t}_n,\vec{x}_n}(s,x)\nabla,\nabla)\varphi(x) + \frac{o_\varepsilon(\tau-s)}{\tau-s}\ ,$$

since

$$\varphi(x)\ P_{\vec{t}_n,\vec{x}_n}(s,x;\tau,V_\varepsilon(x)) = o_\varepsilon(\tau-s)\ ,$$

$$\int_{V'_\varepsilon(x)} \varphi(y)\ P_{\vec{t}_n,\vec{x}_n}(s,x;\tau,dy) = o_\varepsilon(\tau-s)\ ,$$

and

$$\left|\int_{V_\varepsilon(x)} r(x,y)\ P_{\vec{t}_n,\vec{x}_n}(s,x;\tau,dy)\right| \le \alpha_\varepsilon \int_{V_\varepsilon(x)} |y-x|^2 P_{\vec{t}_n,\vec{x}_n}(s,x;\tau,dy)$$

$$= \alpha_\varepsilon \left[(B_{\vec{t}_n,\vec{x}_n}(s,x)\ \frac{y-x}{|y-x|},\ \frac{y-x}{|y-x|})(\tau-s) + o_\varepsilon(\tau-s)\right] = o_\varepsilon(\tau-s)$$

since ε is arbitrarily chosen and $\alpha_\varepsilon \longrightarrow 0$ as $\varepsilon \longrightarrow 0$.

Moreover, it is easy to see that $M = \sup_{x\in\mathbb{R}^m} |\varphi(x)| < \infty$ which means that

$$\text{(13)}\qquad \left|\int_{\mathbb{R}^m} \varphi(x)\ P_{\vec{t}_{n-1},\vec{x}_{n-1}}(s,x;\tau,dy) - \varphi(x)\right| \le 2\ M < \infty\ .$$

On the other hand, if we use (4) and (13), then we may write

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^m} \varphi(x)\ P_{\vec{t}_{n-1},\vec{x}_{n-1}}(t_n,x_n;t,x) =$$

$$= \lim_{\substack{s\nearrow t\\ \tau\searrow t}} \frac{1}{\tau-s}\int_{\mathbb{R}^m} \varphi(x)\left[P_{\vec{t}_{n-1},\vec{x}_{n-1}}(t_n,x_n;\tau,dx) - P_{\vec{t}_{n-1},\vec{x}_{n-1}}(t_n,x_n;s,dx)\right] =$$

$$= \lim_{\substack{s\nearrow t\\ \tau\searrow t}} \frac{1}{\tau-s}\int_{\mathbb{R}^m}\left[\int_{\mathbb{R}^m} \varphi(y)P_{\vec{t}_n,\vec{x}_n}(s,x;\tau,dy) - \varphi(x)\right]P_{\vec{t}_{n-1},\vec{x}_{n-1}}(t_n,x_n;s,dx) =$$

$$= \int_{\mathbb{R}^m} \lim_{\substack{s\nearrow t\\ \tau\searrow t}}\left\{\frac{1}{\tau-s}\left[\int_{\mathbb{R}^m} \varphi(y)P_{\vec{t}_n,\vec{x}_n}(s,x;\tau,dy) - \varphi(x)\right]P_{\vec{t}_{n-1},\vec{x}_{n-1}}(t_n,x_n;s,dx)\right\}.$$

This gives together with (11) and (12)

$$\frac{\partial}{\partial t} \int_{\mathbb{R}^m} \varphi(x)\, P_{\vec{t}_{n-1}, \vec{x}_{n-1}}(t_n, x_n; t, x) =$$

$$= \int_{\mathbb{R}^m} \varphi(x) \frac{\partial}{\partial t} f_{\vec{t}_{n-1}, \vec{x}_{n-1}}(t_n, x_n; t, x)\, dx =$$

$$= \int_{\mathbb{R}^m} \lim_{\substack{s \nearrow t \\ \tau \searrow t}} f_{\vec{t}_{n-1}, \vec{x}_{n-1}}(t_n, x_n; s, x) \Big[(A_{\vec{t}_n, \vec{x}_n}(s, x), \nabla)\, \varphi(x) +$$

$$+ \frac{1}{2} (B_{\vec{t}_n, \vec{x}_n}(s, x)\nabla, \nabla)\, \varphi(x) + \frac{o_\varepsilon(\tau - s)}{\tau - s} \Big]\, dx =$$

$$= \int_{\mathbb{R}^m} \Big[(A_{\vec{t}_n, \vec{x}_n}(t, x), \nabla)\, \varphi(x) \Big] f_{\vec{t}_{n-1}, \vec{x}_{n-1}}(t_n, x_n; t, x)\, dx +$$

$$+ \frac{1}{2} \int_{\mathbb{R}^m} \Big[B_{\vec{t}_n, \vec{x}_n}(t, x)\nabla, \nabla)\, \varphi(x) \Big] f_{\vec{t}_{n-1}, \vec{x}_{n-1}}(t_n, x_n; t, x)\, dx =$$

$$= \sum_{i=1}^{m} \int_{\mathbb{R}^m} \frac{\partial \varphi(x)}{\partial x} A^i_{\vec{t}_n, \vec{x}_n}(t, x)\, f_{\vec{t}_{n-1}, \vec{x}_{n-1}}(t_n, x_n; t, x)\, dx +$$

$$+ \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} \int_{\mathbb{R}^m} \frac{\partial^2 \varphi(x)}{\partial x^i\, \partial x^j} b^{i,j}_{\vec{t}_n, \vec{x}_n}(t, x) f_{\vec{t}_{n-1}, \vec{x}_{n-1}}(t_n, x_n; t, x)\, dx .$$

If we integrate by parts the above integral, then we have

$$\int_{\mathbb{R}^m} \frac{\partial \varphi(x)}{\partial x^i} A^i_{\vec{t}_n, \vec{x}_n}(t, x)\, f_{\vec{t}_{n-1}, \vec{x}_{n-1}}(t_n, x_n; t, x)\, dx =$$

$$= - \int_{\mathbb{R}^m} \varphi(x) \frac{\partial}{\partial x^i} \Big[A^i_{\vec{t}_n, \vec{x}_n}(t, x)\, f_{\vec{t}_{n-1}, \vec{x}_{n-1}}(t_n, x_n; t, x) \Big]\, dx ,$$

and

$$\int_{\mathbb{R}^m} \frac{\partial^2 \varphi(x)}{\partial x^i\ \partial x^j}\, b^{i,j}_{\vec{t}_n,\vec{x}_n}(t,x) f_{\vec{t}_{n-1},\vec{x}_{n-1}}(t_n,x_n;t,x)\ dx =$$

$$= \int_{\mathbb{R}^m} \varphi(x)\ \frac{\partial^2}{\partial x^i\ \partial x^j} \left[b^{i,j}_{\vec{t}_n,\vec{x}_n}(t,x) f_{\vec{t}_{n-1},\vec{x}_{n-1}}(t_n,x_n;t,x) \right] dx\ ,$$

for $i,j = 1,\dots,m$. Finally, we have

$$\int_{\mathbb{R}^m} \varphi(x)\ \frac{\partial}{\partial t}\ f_{\vec{t}_{n-1},\vec{x}_{n-1}}(t_n,x_n;t,x)\ dx =$$

$$= - \sum_{i=1}^{m} \int_{\mathbb{R}^m} \varphi(x)\ \frac{\partial}{\partial x^i} \left[A^i_{\vec{t}_n,\vec{x}_n}(t,x)\ f_{\vec{t}_{n-1},\vec{x}_{n-1}}(t_n,x_n;t,x) \right] dx\ +$$

$$+ \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} \int_{\mathbb{R}^m} \varphi(x) \frac{\partial^2}{\partial x^i\ \partial x^j} \left[b^{i,j}_{\vec{t}_n,\vec{x}_n}(t,x) f_{\vec{t}_{n-1},\vec{x}_{n-1}}(t_n,x_n;t,x) \right] dx,$$

and since the function φ is arbitrarily chosen it ends the proof of our Theorem 2.

In the case of $m = 1$ Theorem 2 (the prospective theorem) is proved in [1], [2] and [3]. Theorem 1 is a generalization of the retrospective Kolmogorov equation for the case of non-Markov quasi-difussion processes.

REFERENCES

[1] A. P l u c i ń s k a : Some transformations of multi-dimensional quasi-diffusion process, Rev. Roum. Math. Pure. and Appl. 23, 8 (1978).

[2] A. P l u c i ń s k a : Remarks on prospective equations, Teor. Wierojatn. i Prim. 25, 2 (1980).

[3] A. K o s y k : Some properties of diffusion processes in the wide sense, Demonstratio Math. 4 (1974), 495-505.

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