DEMONSTRATIO MATHEMATICA
Vol. XXII No 3 1989

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VARIATIONAL FORMULATION FOR A HYPERBOLIC PROBLEM

The aim of this work is to give a variational formulation of a second Fourier problem for a linear hyperbolic equation. We prove the existence and uniqueness of a generalized solution of this problem.

1. Symbols and conventions

Throughout this paper we use the Einstein summation convention writing $k_{ij}x_ix_j$, $i,j=1,..,N$ instead of $\sum_{i,j=1}^{N} k_{ij}x_ix_j$.

Besides commonly used spaces we use the following ones :

$H^{1,2}(G)$ - where G is an open and bounded set in $\mathbb{R}^N$, is the normed space of functions $u \in L^2(G)$ such that

$$\partial_i u,\ \partial_t u,\ \partial^2_{it} u,\ \partial^2_t u \ \in\ L^2(G),\ i=1,..,N,$$

($\partial_i u$ denotes $\frac{\partial u}{\partial x_i}$, $(x,t)=(x_1,..x_N,t) \in \mathbb{R}^{N+1}$) with the norm

$$\|u\|^2_{H^{1,2}} = (\|u\|^2_{H^1} + \|\partial^2_t u\|^2_{L^2} + \sum_{i=1}^{N} \|\partial^2_{it} u\|^2_{L^2}) ;$$

moreover, it is a Hilbert space;

$C^{k,1}(G)$, $k = 0,1$, is the normed space of real functions from $C^k(G)$ such that their derivatives of order k are Lipschitz functions.

Following notations from [2], we denote by $\mathcal{N}^{k,1}(\mathbb{R}^n)$ the class of sets $A \subset \mathbb{R}^n$ with boundaries of class $C^{k,1}$.

2. Variational formulation

Let $G \subset \mathbb{R}^N$ be an open and bounded set of class $\mathcal{N}^{0,1}$; denote by $(n_1,..,n_N)$ the outward normal vector to ∂G of length 1.

Denote by $\Gamma = \partial G \times (0,T)$, $G_0 = G \times \{0\}$, $G_s = G \times \{s\}$, $\Gamma_S = \partial G \times (0,s)$, $Q_s = G \times (0,s)$, $s > 0$, $T > 0$, $s \le T$.

Define the following differential problem :

P r o b l e m I Find $u \in H^2(Q_T)$ such that

(1) $$\Box u := \partial_t^2 u - \partial_i(k_{ij}\partial_j u) + p\partial_t u + qu = f \text{ in } L^2(Q_T),$$

(2) $$tr_{|G_0} u = \varphi_0 \quad \text{in } H^1(G),$$

(3) $$tr_{|G_0} \partial_t u = \varphi_1 \quad \text{in } L^2(G),$$

(4) $$tr_{|\Gamma}\, k_{ij}\partial_j u \circ n_i = 0 \quad \text{in } L^2(\Gamma).$$

We assume that

(5) $$\begin{cases} p\in L^\infty(G),\ p\ge 0 \text{ a.e. in } G,\ q\in L^\infty(G),\ q\ge 0 \text{ a.e. in } G, \\ f\in L^2(Q_T),\ \varphi_0\in H^1(G),\ \varphi_1\in L^2(G), \\ k_{ij}\in L^\infty(G),\ k_{ij}=k_{ji} \text{ a.e. in } G,\ i,j = 1,..,N,\ N \ge 2; \\ \text{there exists } \alpha>0 \text{ such that} \\ k_{ij}z_i z_j \ge \alpha\circ(z_1^2 +..+z_N^2),\ z_i\in \mathbb{R},\ i,j = 1,..,N. \end{cases}$$

Consider besides the following variational problem:

P r o b l e m 2 Find $u \in H^{1,2}(Q_T)$ such that

(6) $$(Lu,v) = F(v) \quad \forall\, v\in H^{1,2}(Q_T),$$

where $Lu\colon H^{1,2}(Q_T) \longrightarrow \mathbb{R}$ is a linear functional given by :

$$(7)\qquad (Lu,v) = \int_0^T -(\partial_t u, \partial_t^2 v)_{L^2(Q_s)}\, ds + (\partial_t u, \partial_t v)_{L^2(Q_T)} +$$

$$+ \int_0^T (k_{ij}\partial_j u, \partial_{it}^2 v)_{L^2(Q_s)}\, ds + 2T(k_{ij} tr|_{G_0} \partial_j u, tr|_{G_0} \partial_i v)_{L^2(G)} +$$

$$+ \int_0^T (q\circ u, \partial_t v)_{L^2(Q_s)}\, ds + T(q\circ tr|_{G_0} u, tr|_{G_0} v)_{L^2(G)} +$$

$$+ 0,5T(tr|_{G_0} u, tr|_{G_0} v)_{L^2(G)} + \int_0^T (p\circ\partial_t u, \partial_t v)_{L^2(Q_s)}\, ds$$

and $F : H^{1,2}(Q_T) \longrightarrow \mathbb{R}$ is a linear functional of the form

$$(8)\qquad F(v) = \int_0^T (f, \partial_t v)_{L^2(Q_s)}\, ds + T(\varphi_1, tr|_{G_0} \partial_t v)_{L^2(G)} +$$

$$2T(k_{ij}\partial_j \varphi_0, tr|_{G_0} \partial_i v)_{L^2(G)} + T(q\circ\varphi_0, tr|_{G_0} v)_{L^2(G)} +$$

$$+0.5T(\varphi_0. tr|_{G_0} v)_{L^2(G)}.$$

It is easy to verify, that under assumptions (5), problem 2 makes sense. One might have noticed that, in a formal way, we can get problem 2 from problem 1 by taking the scalar product in $L^2(Q_s)$ of the latter and $\partial_t v$ and next integrating over $[0,T]$ in ds. This leads us directly to the question of the relation between the problems 1, 2.

Assume that

$$(9)\qquad k_{ij} \in C^{0,1}(G).$$

T h e o r e m 1. If we assume (5) and (9) then problem 1 has a solution $u \in H^2(Q_T)$ iff it is a solution of problem 2.

Begin with a proof of a lemma stating the relationship of problem 1 to the following intermediate problem.

P r o b l e m 3 Find $u \in H^2(Q_T)$ such that

$$(Lu,v) = F(v) \quad \forall\, v \in H^{1,2}(Q_T), \tag{10}$$

where

$$\begin{aligned}(Lu,v) = {} & -(\partial_t u, \partial_t^2 v)_{L^2(Q_T)} + (\partial_t u, \partial_t v)_{L^2(G_T)} + \\ & + (k_{ij}\partial_j u, \partial_{it}^2 v)_{L^2(Q_T)} + 2(k_{ij}\, tr|_{G_0}\partial_j u, tr|_{G_0}\partial_i v)_{L^2(G)} + \\ & + (q\circ u, \partial_t v)_{L^2(Q_T)} + (q\circ tr|_{G_0} u, tr|_{G_0} v)_{L^2(G)} + \\ & + 0{,}5(tr|_{G_0} u, tr|_{G_0} v)_{L^2(G)} + (p\circ\partial_t u, \partial_t v)_{L^2(Q_T)},\end{aligned} \tag{11}$$

$$\begin{aligned}F(v) = {} & (f, \partial_t v)_{L^2(Q_T)} + (\varphi_1, tr|_{G_0}\partial_t v)_{L^2(Q_s)} + \\ & + 2(k_{ij}\partial_j\varphi_0, tr|_{G_0}\partial_i v)_{L^2(G)} + (q\circ\varphi_0, tr|_{G_0} v)_{L^2(G)} + \\ & + 0{,}5(\varphi_0 . tr|_{G_0} v)_{L^2(G)} .\end{aligned} \tag{12}$$

L e m m a 1. Under the assumptions of Theorem 1, distribution $u \in H^2(Q_T)$ is a solution of problem 1 iff it is a solution of problem 3.

P r o o f. In order to show the implication "$\Rightarrow$", take $v \in C^\infty(\bar{Q}_T)$, multiply equation (1) by $\partial_t v$ and integrate over Q_T. Taking into account the boundary conditions (2) - (4) and in3egrating by parts we get equation (10) for $v \in C^\infty(\bar{Q}_T)$; as such functions are dense in $H^{1,2}(Q_T)$, from the continuity of all the expressions in the norm of $H^{1,2}(Q_T)$, we get the desired implication.

Now we shall show the implication "⇐". As $u \in H^2(Q_T)$ and $k_{ij} \in C^{0,1}(G)$ for $i,j=1,\dots,N$, $\Box u \in L^2(Q_T)$ and, if we take in equation (10) $v \in C^\infty(\bar{Q}_T)$ such that $v(x,t) = \int_0^t \varphi(s,x)ds$, $\varphi \in C_0^\infty(Q_T)$, we easily get the fulfilment of equation (1) in the distributional sense; as $\Box u$ and f belong to $L^2(Q_T)$, u satisfies (1) in $L^2(Q_T)$.

Now we shall show the fulfilment of the boundary conditions (2) - (4). Take $v \in C^\infty(\bar{Q}_T)$ such that $\partial_t v = 0$ in the neigbourhood of Γ. As u fulfils (1), we get

$$(tr|_{G_0}\partial_t u, tr|_{G_0}\partial_t v)_{L^2(G)} + 2(k_{ij}tr|_{G_0}\partial_j u, tr|_{G_0}\partial_i v)_{L^2(G)} +$$

$$+ (q\circ tr|_{G_0}u, tr|_{G_0}v)_{L^2(G)} + 0.5(tr|_{G_0}u, tr|_{G_0}v)_{L^2(G)} =$$

$$= (\varphi_1, tr|_{G_0}\partial_t v)_{L^2(G)} + 2(k_{ij}\partial_j\varphi_0, tr|_{G_0}\partial_i v)_{L^2(G)} +$$

$$+ (q\circ\varphi_0, tr|_{G_0}v)_{L^2(G)} + 0,5T(\varphi_0 . tr|_{G_0}v)_{L^2(G)} .$$

If now we take v such that $v(x,0) = 0$, then from the arbitrariness of $\partial_t v(x,0)$ we get condition (3). To prove (2) take $v(x,t) = tr|_{G_0}u - \varphi_0$, $v \in H^{1,2}(Q_T)$, $\partial_t v = 0$.

Taking into account that u satisfies (1) and (3), from equation (10) we get the equation

$$(k_{ij}tr|_{G_0}\partial_j v, tr|_{G_0}\partial_i v)_{L^2(G)} + ((q+0.5)tr|_{G_0}v, tr|_{G_0}v)_{L^2(G)} = 0$$

which, together with the positive definiteness of k_{ij} and the nonnegativeness of q, gives the condition $tr_{|G_0} v = v = 0$ in $H^1(G)$ being equivalent to (2).

As we have proved that u satisfies (1) - (3), from (10) and by the assumption that $u \in H^2(Q_T)$ we conclude that u satisfies (4). This completes the proof of Lemma 1.

In the proof of Theorem 1 we shall need two easy facts.

F a c t 1. If $f \in L^1(0,T)$, then

$$\int_0^T \Big(\int_0^s f(t)\, dt \Big)\, ds = \int_0^T (T-t)\circ f(t)\, dt.$$

F a c t 2. For $u,v \in H^{1,2}(Q_T)$ the mapping $s \to tr_{|G_s} \partial_i u$ is a continuous function from $(0,T)$ to $L^2(G)$ and

$$(\partial_i u, \partial_j v)_{L^2(Q_T)} = \int_0^T (tr_{|G_s} \partial_i u, tr_{|G_s} \partial_j v)_{L^2(G)}\, ds,$$

$$\int_0^T \| tr_{|G_s} \partial_i u \|^2_{L^2(G)}\, ds = \| \partial_i u \|_{L^2(Q_T)}.$$

P r o o f of Theorem 1. Assume that $u \in H^2(Q_T)$ is a solution of problem 2. Integrating by parts the first and third members of (Lu,v) in (7) and applying facts 1, 2 we get the following equation for u:

$$((T-t)\square u, \partial_t v)_{L^2(Q_T)} + T(tr_{|G_0} \partial_t u, tr_{|G_0} \partial_t v)_{L^2(G)} +$$

$$+ \int_0^T \int_{\Gamma_s} tr_{|\Gamma_s} (k_{ij} \partial_j u)\circ tr_{|\Gamma_s} (\partial_t v)\circ n_i\, dm ds +$$

$$+ T(q\circ tr_{|G_0} u, tr_{|G_0} v)_{L^2(G)} + 2T(k_{ij} tr_{|G_0} \partial_j u, tr_{|G_0} \partial_i v)_{L^2(G)} +$$

$$+ 0,5T(tr_{|G_0}u, tr_{|G_0}v)_{L^2(G)} = T(q\circ\varphi_0, tr_{|G_0}v)_{L^2(G)} +$$

$$+ ((T-t)f, \partial_t v)_{L^2(Q_T)} + T(\varphi_1, tr_{|G_0}\partial_t v)_{L^2(G)} +$$

$$+ 2T(k_{ij}\partial_j\varphi_0, tr_{|G_0}\partial_i v)_{L^2(G)} + 0,5T(\varphi_0, tr_{|G}v)_{L^2(G)},$$

$$\forall\ v\in H^{1,2}(Q_T).$$

Using fact 1 we can write the following equation

$$(13)\qquad \int_0^T\int_{\Gamma_s} tr_{|\Gamma_s}(k_{ij}\partial_j u)\circ tr_{|\Gamma_s}\partial_t v\circ n_i\, dm\, ds =$$

$$= \int_\Gamma (T-t)tr_{|\Gamma}(k_{ij}\partial_j u)\circ tr_{|\Gamma}\partial_t v\circ n_i\, dm.$$

Reasoning simmilarly as in the proof of Lemma 1 we obtain

$$(T-t)\Box u = (T-t)f \text{ in } L^2(Q_T),$$

$$(T-t)tr_{|\Gamma}(k_{ij}\partial_j u)n_i = 0 \text{ in } L^2(\Gamma),$$

and conditions (2), (3). As $(T-t) \in L^\infty(Q_T)$, $T-t > 0$ in Q_T, thus from (13) we get the fulfilment of (1) and (4). This proves that u is a solution of problem 1.

To prove the reverse implication we shall use Lemma 1, stating that $u \in H^2(Q_T)$ is a solution of problem 1 iff it is a solution of problem 3. Notice that, if $s \in (0,T)$, then $u \in H^2(Q_T)$ is also a solution of problem 3 in Q_S. If we integrate equation (10) written for Q_S in parameter s over (0,T), then, from fact 2, we see that $u \in H^2(Q_T)$ satisfies (1).

3. Exístence anduniquennes of a generalized solution

Define a mapping $[.,.] : H^{1,2}(Q_T) \times H^{1,2}(Q_T) \longrightarrow \mathbb{R}$ with a formula :

$$[u,v] = (u,v)_{L^2(Q_T)} + \sum_{i=1}^{N} (\partial_i u, \partial_i v)_{L^2(Q_T)} +$$

$$+(tr_{|G_0} \partial_t u, tr_{|G_0} \partial_t v)_{L^2(G)} + \sum_{i=1}^{N} (tr_{|G_0} \partial_i u, tr_{|G_0} \partial_i v)_{L^2(G)} +$$

$$+ (\partial_t u, \partial_t v)_{L^2(Q_T)} + (tr_{|G_0} u, tr_{|G_0} v)_{L^2(G)} .$$

We also define $\|u\|_V = \sqrt{[u,u]}$, $\|.\|_V : H^{1,2}(Q_T) \longrightarrow \mathbb{R}$. It is obvious that $\|.\|_V$ is a norm in $H^{1,2}(Q_T)$, but this space is not complete in this norm.

D e f i n i t i o n 1. Space V is a completion of $H^{1,2}(Q_T)$ in the norm $\| \; \|_V$.

As $\| \; \|_V$ is a norm in $H^{1,2}(Q_T)$ and $[.,.]$ has the properties of a scalar product, V is a Hilbert space with this product. Now we shall give another useful characterization of this space.

Let $I : H^{1,2}(Q_T) \longrightarrow H^1(Q_T) \times H^1(G) \times L^2(G)$ be defined by the formula

$$(14) \qquad I(u) = (I_H u, I_G u, I_t u) = (u, tr_{|G_0} u, tr_{|G_0} \partial_t u).$$

As $H^{1,2}(Q_T)$ is dense in V and in the norm of the cartesian product $\|Iu\| = \|u\|_V$, we can extend I to an isometry of V into a closed subspace of the cartesian product $H^1(Q_T) \times H^1(G) \times L^2(G)$. This isometry will be also denoted by $I = (I_H, I_G, I_t)$.

Now we shall state a general version of a theorem on the existence of a generalized solution of a variational problem. It is one of the possible formulation of the Lax-Milgram theorem. A simmilar theorem can be found in [1].

Let $(V,[.,.],\| \|_V)$ and $(W,(.,.),\| \|)$ be Hilbert spaces, suppose V is the completion of W in the norm $\| \|_V$, this norm is defined on elements of W and there exists a constant c >0 such that $\|w\|_V \le c \circ \|w\|$ $\forall w \in W$. Let $F : V \longrightarrow \mathbb{R}$ be linear and continuous ($F \in L(V,\mathbb{R})$) and $a : W \times W \longrightarrow \mathbb{R}$ a bilinear form satisfying conditions

(15) $\exists C>0: \forall u,v \in W \quad |a(u,v)| \le C \circ \|u\|_V \circ \|v\|,$

(16) $\exists \alpha>0: \forall w \in W \quad a(w,w) \ge \alpha \circ \|w\|_V^2.$

For a variational problem

(17) find $u \in W$ such that $a(u,w) = F(w)$ $\forall w \in W$

we can define an extension to V and a generalized solution. Its definition will be given in the proof of the following theorem:

T h e o r e m 2. Under the above assumptions problem (17) has exactly one generalized solution.

P r o o f. Define by $A_w : W \longrightarrow \mathbb{R}$, $w \in W$, a functional given by the formula $A_w(u) = a(u,w)$. From the assumptions (15), (16), the bilinear form a is is a linear functional continuous in the norm $\| \|_V$. From the Riesz theorem we conclude that there exsists precisely one $w_0 \in V$ such that $A_w(u) = [u,w_0]$ $\forall u \in W$. Let us denote by $S : W \longrightarrow V$ the mapping assigning the element w_0 to w. Thus we can write $A_w(u) = [u,Sw]$. From the properties (15), (16) we conclude that $\forall\ w \in W \quad \alpha \circ \|w\|_V^2 \le a(w,w) = [w,Sw] \le \|w\|_V \circ \|Sw\|_V$ and thus $\|w\|_V \le \frac{1}{\alpha} \circ \|Sw\|_V \quad \forall\ w \in W$. We can now write

$$|F(w)| \le C \circ \|w\|_V \le \frac{C}{\alpha} \circ \|Sw\|_V .$$

Let us denote by M the functional given by the formula $M(S(w)) = F(w)$, $M : S(W) \longrightarrow \mathbb{R}$. It is a linear and continuous functional on S(W).

Notice that S(W) is dense in V. For suppose the contrary : there exists an element $v_0 \in V$, $v_0 \in \overline{S(W)}^{\perp}$ (the space orthogonal to $\overline{S(W)}$), $\|v_0\|_V > 0$. Then we can find a sequence $w_n \in W$, $w_n \longrightarrow v_0$ in V. There are two possibilities :

(a) there exists n_0 such that $w_{n_0} \in \overline{S(W)}^{\perp}$, $\|w_{n_0}\|_V > 0$;

(b) for every $n \in \mathbb{N}$, $w_n \in \overline{S(W)}$.

In the case (a) we can write

$$0 = [w_{n_0}, Sw] = A_w(w_{n_0}) = a(w_{n_0}, w) \quad \forall\ w \in W,$$

and taking $w = w_{n_0}$, we get

$$0 = a(w_{n_0}, w_{n_0}) \ge \alpha \circ \|w_{n_0}\|_V^2, \text{ thus } \|w_{n_0}\|_V = 0,$$

which is a contradiction.

In the case (b) we have

$$0 = [v_0, w_n] \underset{n\to\infty}{\longrightarrow} [v_0, v_0] = \|v_0\|_V^2 ,$$

thus $\|v_0\|_V = 0$, which is a contradiction.

From the density of S(W) in V it follows that there is only one extension of $M : S(W) \longrightarrow \mathbb{R}$ to V. Denoting this extension by G we notice that $M \in L(V,\mathbb{R})$.

Once again, using Riesz theorem, we conclude that there exsists a unique $\tilde{u} \in V$ such that $G(v) = [\tilde{u}, v] \quad \forall\, v \in V$.

This $\tilde{u}$ is the unique generalized solution in this sense that, if $v \in S(W)$ and $v = S(w)$, $w \in W$, then

$$G(v) = F(w) = [\tilde{u}, S(w)] \quad \forall\, w \in W, \tag{18}$$

moreover, if $\tilde{u} \in W$, then

$$G(v) = F(w) = [\tilde{u}, S(w)] = a(\tilde{u}, w) \quad \forall\, w \in W.$$

This completes the proof of Theorem 2.

Now we shall apply Theorem 2 to problem 2. Take $W = H^{1,2}(Q_T)$, V as in Definition 1; it is obvious that the assumptions of Theorem 2 are satisfied. To check the fulfilment of the other assumptions we use integration by parts, fact 2 and conditions (5), and we get

$$(Lu, u) \geq 0{,}5(\partial_t u, \partial_t u)_{L^2(Q_T)} + 0{,}5T(tr|_{G_0} u, tr|_{G_0} u)_{L^2(G)} +$$

$$+ 0{,}5\alpha \sum_{i=1}^{N} (\partial_i u, \partial_i u)_{L^2(Q_T)} + 0{,}5T(tr|_{G_0} \partial_t u, tr|_{G_0} \partial_t u)_{L^2(G)} +$$

$$+ 1{,}5T\alpha \sum_{i=1}^{N} (tr|_{G_0} \partial_i u, tr|_{G_0} \partial_i u)_{L^2(G)}.$$

In order to prove the positive definiteness of L we notice that there exists $C > 0$ such that for $v \in V$ we have:

$$\|v\|_V^2 < C^2\Big(\sum_{i=1}^{N} \|\partial_i I_H v\|^2_{L^2(Q_T)} + \|\partial_t I_H v\|^2_{L^2(Q_T)} +$$

$$+ \|I_G v\|^2_{L^2(G)} + \|I_t v\|^2_{L^2(G)} \Big).$$

To prove this we shall use the representation of V as a closed subspace of the cartesian product of $H^1(Q_T) \times H^1(G) \times L^2(G)$. Take $u \in C^\infty(\bar{Q}_T)$, then

$$|u(x,t)|^2 \le 2[|u(x,0)|^2 + (\int_0^t |\partial_t u(x,s)| ds)^2] \le$$

$$\le 2[|u(x,0)|^2 + (\int_0^T |\partial_t u(x,s) \circ 1| ds^2)] \le 2[|u(x,0)|^2 +$$

$$+ T^2 \int_0^T |\partial_t u(x,s)|^2 ds],$$

and thus $\|u\|^2_{L^2(Q_T)} \le 2T\|tr_{|G_0} u\|^2_{L^2(G)} + 2T^3\|\partial_t u\|^2_{L^2(Q_T)}$.

As the trace operator $tr_{|G_0}: H^1(Q_T) \to L^2(G)$ is continuous, the above estimate is true for $u \in H^1(Q_T)$ ($C^\infty(Q_T)$ is dense in $H^1(Q_T)$), it is also true for $I_H v \in H^1(Q_T)$.

As obviously $|(Lu,v)| \le C\|u\|_V \circ \|v\|_{H^{1,2}(Q_T)}$ and $|F(v)| \le$ $\le C_1\|v\|_V$ all these facts prove the fulfilment of the assumptions of Theorem 2.

C o r r o l a r y Problem 2 has a unique generalized solution in the space V.

Now we shall give another characterization of the generalized solution of Problem 2.

T h e o r e m 3. An element $\tilde{u} \in V$ is a generalized solution of Problem 2 iff for every $v \in H^{1,2}(Q_T)$ we have $(LI\tilde{u},v) = F(v)$, where

$$(LI\tilde{u},v) = \int_0^T -(\partial_t I_H \tilde{u}, \partial_t^2 v)_{L^2(Q_S)} ds + (\partial_t I_H \tilde{u}, \partial_t v)_{L^2(Q_T)} +$$

$$+\int_0^T (k_{ij}\partial_j \dot{I}_H\tilde{u}, \partial_{it}^2 v)_{L^2(Q_s)} ds + 2T(k_{ij} tr|_{G_0} \cdot \partial_j I_G u, \tilde{tr}|_{G_0} \partial_i v)_{L^2(G)} +$$

$$+\int_0^T (q\circ I_H\tilde{u}, \partial_t v)_{L^2(Q_s)} ds + T(q\circ I_G\tilde{u}, tr|_{G_0} v)_{L^2(G)} +$$

$$+ 0.5T(I_G\tilde{u}, tr|_{G_0} v)_{L^2(G)} + \int_0^T (p\circ\partial_t I_H\tilde{u}, \partial_t v)_{L^2(Q_s)} ds.$$

Moreover, $\tilde{u}$ has the following properties:

operator $\Box$ is defined on $I_H\tilde{u}$ and $\Box I_H\tilde{u} \in L^2(Q_T)$,

$\Box I_H\tilde{u} = f$ in $L^2(Q_T)$, $I_G\tilde{u} = \varphi_0$ in $H^1(G)$.

P r o o f. The first part of the thesis follows obviously from Theorem 1. If we look closer at the conditions satisfied by the generalized solution we shall notice that in our case (for the form (Lu, v)) we may write

$$\forall\ w \in H^{1,2}(Q_T) \quad (LIw, v) = (Lw, v) \quad \text{for } v \in H^{1,2}(Q_T),$$

moreover, from the continuity of operator I we get the identity of the two following problems :

find $w \in H^{1,2}(Q_T)$ such that $(LIw, v) = F(v)$ $\forall\ v \in H^{1,2}(Q_T)$

and

find $w \in H^{1,2}(Q_T)$ such that $(Lw, v) = F(v)$ $\forall\ v \in H^{1,2}(Q_T)$.

As operator I defined on V is linear and continuous, thus (18) may be written as follows:

$\tilde{u} \in V$ is a generalized solution of problem 2 iff

(19) $$(LI\tilde{u}, v) = F(v) \quad \forall\ v \in H^{1,2}(Q_T).$$

Now we move to the proof of the second part of the thesis. Take $v \in C^{\infty}(\bar{Q}_T)$ given by the formula

$$v(x,t) = \int_0^t v_t(x,s)ds, \quad v_t \in C_0^{\infty}(Q_T).$$

For such v, from (19) we get

$$\langle (T-t)\Box I_H\tilde{u}, v_t \rangle = ((T-t)\circ f, v_t)_{L^2(Q_T)}.$$

On the left side of this equation we have a distributional derivative which is regular, as $(T-t)\circ f \in L^2(Q_T)$. Since $T-t > 0$ for $t \in (0,T)$, we have $I_H\tilde{u} = f$ in $L^2(Q_T)$.

The only thing we have to prove is $I_G\tilde{u} = \varphi_0$ in $H^1(Q_T)$. Take another $v \in H^{1,2}(Q_T)$ given by the formula $v = v_{G_0}(x)$, $v_{G_0} \in H^1(Q_T)$. For such v we have $tr_{|G_0}\partial_t v = \partial_t v = \partial_t^2 v = \partial_{it}^2 v = 0$ and our equation reduces to

$$2T(k_{ij}\partial_j(I_G\tilde{u}-\varphi_0), \partial_i v_{G_0})_{L^2(G)} + T((q+0,5)\circ(I_G\tilde{u}-\varphi_0), \partial_t v)_{L^2(G)} = 0.$$

From the arbitrariness of v_{G_0} and assumptions (5) on k_{ij} and q we get the desired equation, which ends the proof of Theorem 3.

There are some final remarks we want to make :

1) our generalized solution is too weak to give a meaning to the initial condition $tr_{|G_0}\partial_t u = \varphi_1$ in $L^2(G)$ other than the one given by (18), but such a situation is not uncommon with such solutions;

2) if we had a theorem giving sufficient conditions for the generalized solution to be regular in the sense of being

an element of $H^{1,2}(Q_T)$, then Theorems 1 and 3 would give us the equivalence of this solution and the common generalized solution of problem 1.

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Received January 7, 1988.