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K-THEORY FOR IM-BUNDLES

Introduction

By an im-bundle we mean any quasi-bundle (i.e. any singular vector bundle) which can be represented as an image of an endomorphism of a locally trivial bundle. Some properties of the category ImVB of im-bundles are presented in [4] and [5].

Any im-bundle ξ over a space X determines a decomposition of this base space into the sets $X_1, X_2, \ldots$, $(X_i \subseteq X)$ over which the dimensions of fibres are constant. By a filtered space denoted $\mathcal{X}$ we mean any space X together with a fixed decomposition of such kind. Observe that in a natural way we can define the semiring ImVB($\mathcal{X}$) (with respect to Whitney sum and tensor product) of isomorphism classes of im-bundles which give the same decomposition of X. Then the K-functor may be applied. Let us denote K(ImVB($\mathcal{X}$)) by $K_{im}(\mathcal{X})$. The main theorem of this paper says that

$$K_{im}(\mathcal{X}) = \oplus K(X_i).$$

Let us start from the following definitions and notation :

1. The category of filtered spaces

Let X be a topological space and $r : X \longrightarrow \{1,2,3,\ldots,n\}$

be a surjective function called filtration such that for each $1 \le k \le n$ the set $r^{-1}(\{1,\dots,k\})$ is closed in X. By a filtered space denoted $\mathcal{X}^n$ or $\mathcal{X}$ we mean any such pair (X,r).

If $\mathcal{X}^n = (X,r)$ and $\mathcal{Y}^{n'} = (Y,r')$ are two filtered spaces then we say that a continuous map $\alpha : X \longrightarrow Y$ is compatible with this filtration if there exists a numerical function $\bar{\alpha}$ which makes the following diagram commute :

$$\begin{array}{ccc} X & \xrightarrow{\quad\alpha\quad} & Y \\ r \downarrow & & \downarrow r' \\ \{1,2,\dots,n\} & \xrightarrow{\quad\bar{\alpha}\quad} & \{1,2,\dots,n\} \end{array}$$

Note that if $\bar{\alpha}$ exists then it is uniquely determined by α. The category with filtered spaces as objects and maps, which are compatible with filtrations, as morphisms we denoted by FTop.

For any filtered space $\mathcal{X}$ and for any continuous map α : $X \longrightarrow Y$ there exists a natural pull-back of the filtration $r : Y \longrightarrow \{1,2,\dots,n\}$ onto X. The structure of this filtered space $\mathcal{X} = (X,r,\alpha)$ we call induced from $\mathcal{Y}$ under α.

It is suitable to define a homotopy in this category as any map $H : \mathcal{X} \times I \longrightarrow \mathcal{Y}$ where the filtration of $\mathcal{X} \times I$ is induced from $\mathcal{X}$ under the projection $X \times I \longrightarrow X$ onto the first factor.

2. The category of im-bundles and the K_{im}-functor

A real (or complex) quasi-bundle called shortly q-bundle is a triple $\xi = (E,p,X)$ where $p : E \longrightarrow X$ is a continuous map such that any fibre $p^{-1}(x) = \xi_x$, $x \in X$, is an n_x-dimensional vector space over the field F of real (or complex) numbers. A morphism $\psi : \xi \longrightarrow \xi'$ of q-bundles is a pair of maps (u, f) such that the diagram :

$$\begin{array}{ccc} E & \xrightarrow{\ u\ } & E' \\ p \downarrow & & \downarrow p' \\ X & \xrightarrow{\ f\ } & X' \end{array}$$

commutes and the restriction of u to any fibre is linear.

We consider only q-bundles with a bounded dimension of fibres (i.e. $n_x = \dim_F \xi_x \le m < \infty$) and such that the sets

$$X_{\underline{n}}(\xi) := \{x \in X : n_x \le n\}$$

$n = 0, 1, \ldots, m$ are subcomplexes of a finite CW-complex X. Any q-bundle ξ is called an im-bundle if and only if the condition $x \in X_k = X_{\underline{k}} \setminus X_{\underline{k-1}}$ implies that there exist an open neighbourhood U of x and k linearly independent sections $(s_1, s_2, \ldots, s_k : U \longrightarrow p^{-1}(U))$ of $\xi_{|U}$.

With our assumption on $(X, X_{\underline{k}})$ each im-bundle $\xi = (E, p, X)$ is an image of an endomorphism of a trivial bundle (see Th. 1 in [4]) and hence may be represented by a map $\alpha : X \longrightarrow \mathrm{End}(F^n)$ or more general by a map

$$\alpha : X \longrightarrow \mathrm{End}(F^{\infty}) = \varinjlim_{k} \mathrm{End}(F^{n+k})$$

where $\mathrm{End}(F^n) \overset{i}{\subset} \mathrm{End}(F^{n+1}) \subset \ldots \subset \mathrm{End}(F^{n+k})$ is given by $i(\varphi) = (\varphi \oplus 0)$. By definition $\mathrm{im}\alpha$ denotes the im-bundle contained in the trivial bundle $\theta^n = (X \times F^n, \pi_1, X)$ with fibres $\mathrm{im}\alpha(x)$. Observe that any im-bundle may be represented as a matrix (with function coefficients) of the endomorphism

$\alpha(x)$ in the canonical base of F^n. The set of isomorphism classes of im-bundles over X we denote by ImVB(X). It is a semiring with respect to Whitney sum and tensor product (see [4]). By an im-bundle ξ we mean both : an im-bundle and its isomorphism class.

Let $\mathcal{E}nd(F^\infty)$ denote the filtered space $(End(F^\infty), rank+1)$. Let $\mathcal{X}$ be any filtered space and let $\alpha : \mathcal{X} \longrightarrow \mathcal{E}nd(F^\infty)$ be a map in the category FTop. Observe that $\bar{\alpha}(r(x)) = rk(\alpha(x))+1$ for $x \in X$.

We know (see [4]) that the isomorphism class of an im-bundle determines the homotopy class $[\alpha] \in [\mathcal{X}, \mathcal{E}nd(F^\infty)]$ and vice versa ; any homotopy class in $[\mathcal{X}, \mathcal{E}nd(F^\infty)]$ determines an im-bundle up to isomorphism. Let ImVB($\mathcal{X}$) denote the subsemiring in ImVB(X) which consists of classes of im-bundles for which a representation may be chosen as a map $\alpha : \mathcal{X} \longrightarrow \mathcal{E}nd(F^\infty)$ in the category of filtered spaces. It is clear that

$$ImVB(\mathcal{X}) = [\mathcal{X}, \mathcal{E}nd(F^\infty)] \ .$$

The main purpose of this paper is to investigate the ring completion K(ImVB($\mathcal{X}$)) which we denote shorter by $K_{im}(\mathcal{X})$. Observe that as in the classical case K_{im} is a contravariant functor from the category of filtered spaces to the category of rings.

3. Piecewise trivial im-bundles

Let $\mathcal{X}^n = (X,r)$ be a filtered space and $h : \{1,2,\dots,n\} \longrightarrow \mathbb{Z}_{\geq 0}$ ($\mathbb{Z}_{\geq 0} = \{0\} \cup \mathbb{N}$) be an nondecreasing function ; let h denote also the n-tuple $(h(1),h(2),\dots,h(n)) \in \mathbb{Z}^n$.

Then we define $rk(h) : \mathcal{X} \longrightarrow \mathcal{E}nd(F^{\infty})$ by the formula :

$$rk(h)(x) = [\varphi_0(x)I_{k_0}]\oplus [\varphi_1(x)I_{k_1}]\oplus \ldots\oplus [\varphi_n(x)I_{k_n}]\oplus 0\oplus 0\oplus\ldots$$

where for each $i = 1,2,\ldots,n$:

- $I_{k_i} : F^{k_i}\longrightarrow F^{k_i}$ is the identity ;
- $k_{(i+1)} = h(i+1)-k_i$, $k_1 = h(1)$;
- $\varphi_i : X \longrightarrow F$ is any function such that $\varphi_i^{-1}(0) = X_{\underline{i}}$.

Observe that the homotopy class $[rk(h)] \in [\mathcal{X},\mathcal{E}nd(F^{\infty})]$ is the same for any (n+1)-tuple $(\varphi_0,\varphi_1,\ldots,\varphi_n)$ of functions such that $\varphi_i^{-1}(0) = X_{\underline{i}}$: if $(\psi_0,\psi_1,\ldots,\psi_n)$ is the any other such (n+1)-tuple then the matrices

$$\left[\begin{array}{ccc|ccc} (1-2t)\varphi_0 I_{k_0} & & 0 & t\,\psi_0\, I_{k_0} & & 0 \\ & \ddots & & & \ddots & \\ 0 & & (1-2t)\varphi_n I_{k_n} & 0 & & t\,\psi_n\, I_{k_n} \end{array}\right] \quad t\in[0,\tfrac{1}{2}]$$

$$\left[\begin{array}{ccc|ccc} (2t-1)\varphi_0 I_{k_0} & & 0 & (1-t)\psi_0\, I_{k_0} & & 0 \\ & \ddots & & & \ddots & \\ 0 & & (2t-1)\varphi_n I_{k_n} & 0 & & (1-t)\psi_n\, I_{k_n} \end{array}\right] \quad t\in[\tfrac{1}{2},1]$$

define the desired homotopy (in category FTop). Therefore $im(rk(h)) \in ImVB(\mathcal{X})$ depends only on h.

D e f i n i t i o n. The im-bundle im(rk(h)) will be called piecewise trivial and denoted by θ^h.

Equivalently the piecewise trivial im-bundle θ^h may be defined as an im-bundle contained in the trivial bundle $\theta^{h(n)} = \mathrm{span}\{e_1, e_2, \dots, e_{h(n)}\}$ over X, with fibres $\theta^h|_X = \mathrm{span}\{e_1, e_2, \dots, e_{h(r(X))}\}$ where $\{e_i : X \longrightarrow X \times F^{h(n)}\}$ denotes a basis of the space of sections of $\theta^{h(n)}$.

Let $\mathrm{im}\alpha \in \mathrm{ImVB}(\mathcal{X})$ be any im-bundle. Then we define an im-bundle $\mathrm{rk}(\mathrm{im}\alpha) \in \mathrm{ImVB}(\mathcal{X})$ as $\theta^{\bar{\alpha}}$ and the n-tuple $(\bar{\alpha}(1), \bar{\alpha}(2), \dots, \bar{\alpha}(n)) \in \mathbb{Z}^n$ by $\mathrm{rk}(\mathrm{im}\alpha)$ or simply $\mathrm{rk}(\alpha)$. Observe that if $n = 1$ and $\xi^k = \mathrm{im}\alpha$ is a k-dimensional locally trivial bundle over X, then the bundle $\mathrm{rk}(\xi^k) = \mathrm{rk}(\mathrm{im}\alpha)$ is the k-dimensional trivial bundle θ^k over X, denoted by k as the element of the ring K(X).

4. The reduced $\tilde{K}_{im}$-functor

L e m m a 1. For any function $h : \{1,2,\dots,n\} \longrightarrow \mathbb{Z}$ there exist two nondecreasing functions $h_1, h_2 : \{1,2,\dots,n\} \longrightarrow \mathbb{Z}_{\geq 0}$ such that $h = h_1 - h_2$.

P r o o f. The functions $h_1(k) = c + \sum_{i=1}^{k} \max(0, h(i)-h(i-1))$ and $h_2(k) = c - h(k) + \sum_{i=1}^{k} \max(0, h(i)-h(i-1))$, where $h(0) = 0$ and $c = \max(h(0), h(1), \dots, h(n))$, have the required properties. This proves Lemma 1.

Let us define $\mathrm{rk} : K_{im}(\mathcal{X}) \longrightarrow \mathbb{Z}^n$ as $\mathrm{rk}([\xi_1 - \xi_2]) = \mathrm{rk}\xi_1 - \mathrm{rk}\xi_2 \in \mathbb{Z}^n$. It is clear that rk is the well defined homomorphism of rings. From Lemma 1 we have immediately :

P r o p o s i t i o n 1. The ring homomorphism rk : $K_{im}(\mathcal{X}) \longrightarrow \mathbb{Z}^n$ is an epimorphism.

P r o o f. For any $h \in \mathbb{Z}^n$ there exist nondecreasing $h_1, h_2 \in (\mathbb{Z}_{\geq 0})^n$ such that $rk([\theta^{h_1} - \theta^{h_2}]) = h_1 - h_2 = h$. This proves Proposition 1.

By the proof of Lemma 1 it is also clear that $\psi : \mathbb{Z}^n \longrightarrow K_{im}(\mathcal{X}^n)$ given by the formula $\psi(h) = \psi(h_1 - h_2) = im(rk(h_1)) - im(rk(h_2)) \in K_{im}(\mathcal{X}^n)$ correctly defines a homomorphism of rings.

P r o p o s i t i o n 2. The ring homomorphism $\psi : \mathbb{Z}^n \longrightarrow K_{im}(\mathcal{X})$ is an monomorphism and $rk \circ \psi = id_{\mathbb{Z}}n$.

P r o o f. Let $h \in \mathbb{Z}^n$ and h_1, $h_2 \in (\mathbb{Z}_{\geq 0})^n$ be the two functions given in the proof of Lemma 1. We have

$$rk \cdot \psi(h) = rk(im(rk(h_1)) - im(rk(h_2))) = h_1 - h_2 = h.$$

This proves Proposition 2.

D e f i n i t i o n. The reduced K_{im}-functor, denoted $\tilde{K}_{im}$ is equal to $\ker(rk : K_{im}(\ ) \longrightarrow \mathbb{Z}^{n(\ )})$.

From the above considerations, the functor K_{im} splits :

$$K_{im}(\mathcal{X}^n) = \tilde{K}_{im}(\mathcal{X}^n) \oplus \mathbb{Z}^n.$$

The goal of this paragraph is to describe the ring $K_{im}(\mathcal{X})$. In the classical theory the ring $\tilde{K}(X)$ may be identified with the stable equivalence classes of bundles. For $\tilde{K}_{im}(\mathcal{X})$ the stable equivalence must be generalized. It turns out that it is sufficient to replace the trivial bundle θ^k by the piecewise trivial $\theta^{h(\ )}$ in the definition of stable equivalent bundles. First we prove the following

T h e o r e m 1. Let $\xi \in \mathrm{ImVB}(\mathfrak{X}^n)$ be an im-bundle such that there exists a diagonal representation $\alpha : \mathfrak{X}^n \longrightarrow \mathcal{E}\mathrm{nd}(F^m)$ of ξ (i.e. for each $x \in X$ $\alpha(x)$ is a diagonal matrix). Then ξ is a piecewise trivial im-bundle.

P r o o f. Let $\alpha = \mathrm{diag}(\alpha_1,\dots,\alpha_m)$, $\mathrm{im}\alpha = \xi$ be a diagonal representation of im-bundle ξ. It is sufficient to prove that there exists a homotopy (in the category FTop) between α and the map $\beta = \mathrm{diag}(\beta_1,\dots,\beta_m) : \mathfrak{X}^n \longrightarrow \mathcal{E}\mathrm{nd}(F^\infty)$ such that

- for each $k = 1,\dots,m$, β_k is a real nonnegative function $\beta_k : X \longrightarrow \mathbb{R}_{\geq 0}$ and $\beta_k^{-1}(0) = X_{\underline{i}}$ for some $i = 0,1,\dots,\bar{\alpha}(n)$.
- for each point $x \in X$ the relation $\mathrm{sgn}\beta_1(x) \geq \dots \geq \mathrm{sgn}\beta_m(x)$ holds.

It is clear that any im-bundle $\mathrm{im}(\mathrm{diag}(\beta_1,\dots,\beta_m))$ is a piecewise trivial one. Let us define β_k to be equal to the k-th symmetric function in the indeterminates $|\alpha_k|,\dots,|\alpha_m|$ (i.e. $\beta_k(x) := \sum_{i(k')} |\alpha_{i_1}||\alpha_{i_2}|\dots|\alpha_{i_{k'}}|$, where $i(k') = (i_1 < \dots < i_{k'}) \in \mathbb{Z}^{k'}$). Observe that the relation $\beta_k(x) = 0$ implies $\beta_j(x) = 0$ for $j \geq k$ and moreover $\beta_k^{-1}(0) = X_{\underline{i}}$.

We start an induction with respect to m. The first step is clear : $\mathrm{im}\alpha_1 = \mathrm{im}|\alpha_1|$ for $\alpha_1 : \mathfrak{X} \longrightarrow F$.

It is enough to prove that $\mathrm{im}(\mathrm{diag}(\gamma,\beta_1,\dots,\beta_m))$ is isomorphic to $\mathrm{im}(\mathrm{diag}(\gamma+\beta_1,\gamma\beta_1+\beta_2,\gamma\beta_2+\beta_3,\dots,\gamma\beta_{m-1}+\beta_m,\beta_m)$ where $\gamma = |\alpha_{m+1}|$.

Consider the relation $A\cdot\mathrm{diag}(\gamma,\beta_1,\dots,\beta_m) = C$ where :

$$
A = \left[\begin{array}{c|c|c|c|c|c|c|c}
1 & 1 & 0 & \cdots \quad \cdots & 0 & 0 & 0 & 0 \\
\hline
-\beta_1 & \gamma & 1 & 0.. \quad \cdots & 0 & 0 & 0 & 0 \\
\hline
\beta_1\beta_2 & -\gamma\beta_2 & \gamma\beta_1 & 1.. \quad 0 & 0 & 0 & 0 & 0 \\
\hline
-\beta_1\beta_2\beta_3 & \gamma\beta_2\beta_3 & -\gamma\beta_1\beta_3 & \gamma\beta_1\beta_2 & 0 & 0 & 0 & 0 \\
\hline
 & & & & & & & 0 \\
\hline
 & & & & & & & 1 \\
\hline
\pm \prod_{i=1}^{m} \beta_i & \mp\gamma \prod_{i\neq 1} \beta_i & \pm\gamma \prod_{i\neq 2} \beta_i & \cdots & .. & .. & .. & \gamma \prod_{i=1}^{m-1} \beta_i
\end{array}\right]
$$

$$
B = \left[\begin{array}{c|c|c|c|c|c|c|c}
1 & 0 & 0 & \cdots \quad \cdots & 0 & 0 & 0 & 0 \\
\hline
1 & 1 & 0 & \cdots \quad \cdots & 0 & 0 & 0 & 0 \\
\hline
0 & 1 & 1 & 0 \quad \cdots & 0 & 0 & 0 & 0 \\
\hline
\vdots & 0 & 1 & 1 & & & & \\
\hline
0 & 0 & 0 & & 1 & 1 & 0 & 0 \\
\hline
0 & 0 & 0 & & 0 & 1 & 1 & 0 \\
\hline
0 & 0 & 0 & & 0 & 0 & 1 & 1
\end{array}\right]
$$

$$C = \begin{bmatrix} \gamma+\beta_1 & \beta_1 & 0 & 0 & \dots\ \dots & 0 & 0 & 0 \\ 0 & \gamma\beta_1+\beta_2 & \beta_2 & 0 & \dots\ \dots & 0 & 0 & 0 \\ 0 & 0 & \gamma\beta_1\beta_2+\beta_3 & \beta_3 & 0\ \dots & 0 & 0 & 0 \\ & & & & & & & \\ 0 & 0 & 0 & & \dots & \ddots & \beta_{m-1} & \\ 0 & 0 & 0 & 0 & \dots\dots & 0 & \ddots & \beta_m \\ 0 & 0 & 0 & 0 & \dots\dots & 0 & 0 & \gamma\prod_{i=1}^{m-1}\beta_i+\beta_m \end{bmatrix}$$

The methods described in [4] give us isomorphism : $\mathrm{im}(\mathrm{diag}(\gamma,\beta_1,\beta_2,\dots,\beta_m)) \cong \mathrm{im}C \cong \mathrm{im}(\mathrm{diag}(C_{11},C_{22},\dots,C_{mm}))$.

The relation $\{ \beta_i(x) = 0 \Rightarrow (\beta_{i+j}(x) = 0 \text{ for } j \geq 0),\ x \in B\}$, implies that the following homotopy is of constant rank : $\mathrm{diag}[C_{11},C_{22},(1-t)C_{33}+ t(\gamma\beta_1+ \beta_2),\dots,(1-t)C_{mm}+ t(\gamma\beta_{m-1}+ \beta_m)$.

The above homotopy determines the required isomorphism of im bundles. This proves Theorem 1.

From Theorem 1 we have immediately the following characterisation of the piecewise trivial im-bundles :

P r o p o s i t i o n 3. The im-bundle $\xi = (E,p,X) \in \mathrm{ImVB}(\mathfrak{X}^n)$ is a piecewise trivial im-bundle if and only if there exists an embedding of ξ into the trivial bundle $\theta^m = (X \times F^m,\pi_1,X)$ of dimension $m = \max_{x\in X} \{\dim E_x\}$ and sections $s_1,\dots,s_n : X \longrightarrow E$ of $\xi \leq \theta^m$ such that $\xi = \mathrm{span}\{s_1,\dots,s_n\}$ and $\langle s_i,s_j\rangle = 0$ for any pair (i,j), $i\neq j$. (Here $\langle .,.\rangle$ denotes the standard product in θ^m).

P r o o f. The existence of such sections means that there exists the following diagonal representation of $\xi \subseteq \theta^m$: $\alpha = \mathrm{diag}(|s_1|, \dots, |s_n|)$. For the inverse observe that any ξ with diagonal representation $\mathrm{im}\alpha = \xi$, admits such sections defined by the columns of α.

Now we are able to prove the following.

T h e o r e m 2. Let $\xi \in \mathrm{ImVB}(\mathcal{X}^n)$ be any im-bundle. Then there exists an im-bundle $\eta \in \mathrm{ImVB}(\mathcal{X}^n)$, such that $\xi \oplus \eta \in \mathrm{ImVB}(\mathcal{X}^n)$ is a piecewise trivial im-bundle.

P r o o f. Let $\xi = \bigcup \xi_k = \bigcup (E^k, p, V^k)$ be the decomposition (described in [4]) of ξ into locally trivial bundles ξ^k ($k = 0, 1, \dots, m$) over V^k , where each V^k is an open neighbourhood of X_k. Let us consider ξ^k over X_k. There exists a locally trivial bundle (in particular an im-bundle) $\mathrm{im}\gamma_k$, where γ_k: $X_k \longrightarrow \mathrm{End}(F^{n_k})$ such that $(\mathrm{im}\gamma_k \oplus \xi^k)$ is a trivial bundle. Each im-bundle $\mathrm{im}\gamma_k$ (defined over X_k) may be extended to the whole space X by :

$$\mathrm{im}\bar{\gamma}_k := \begin{cases} 0 & \text{over } \underline{X_{k-1}} \\ \mathrm{im}\gamma_k & \text{over } X_k \\ \theta^{n_k} & \text{over } X_{\overline{k+1}} = X \setminus \underline{X_k} . \end{cases}$$

Note that $(\mathrm{im}\bar{\gamma}_1 \oplus \mathrm{im}\bar{\gamma}_2 \oplus \dots \oplus \mathrm{im}\bar{\gamma}_n) \oplus \xi$ is an im-bundle which is trivial over each set X_k . Therefore we may assume that im-bundle ξ is trivial over the set X_k of the smallest non-zero dimension.

Let $s_1 \dots s_k$ be k orthogonal to one another non-zero sections of ξ^k over $V^k \supseteq X_k$ extended continuously by 0 onto the whole base space X. Let η_k denotes an im-bundle which is trivial, k-dimensional with (orthogonal to one another) independent sections $t_1,\dots,t_k$: $X_{\overline{k+1}} \longrightarrow E(\eta_k)$ over $X_{\overline{k+1}} =$ $= X \setminus X_{\underline{k}}$ and zero dimensional over $X_{\underline{k}}$. The sections $s_i + t_i$ of the im-bundle $\xi \oplus \eta_k$ spann over $X_{\bar{k}}$ the k-dimensional trivial bundle, whose orthogonal complement in $(\xi\oplus \eta_k)_{|X_{\bar{k}}}$ is an im-bundle denoted by ξ'. Observe that $\langle s_i+ t_i, s_j+ t_j\rangle = 0$ for $i \neq j$. The fibres dimension of ξ' is non-zero only over $X_{\overline{k+1}}$, so the induction argument may be used. In no more than m steps we obtain the im-bundle $\eta=\eta_{k_1} \oplus \eta_{k_2} \oplus \dots \oplus \eta_{k_m} \in ImVB(\mathcal{X})$ such that $\xi \oplus \eta$ admits sections desribed in Proposition 3. This proves Theorem 2.

From Theorem 2 we have :

P r o p o s i t i o n 4. The map $\psi : ImVB(\mathcal{X}) \longrightarrow \tilde{K}_{im}(\mathcal{X})$ defined by $\psi(\xi) = \xi- rk(\xi)$ is surjective. There is one to one correspondence between $\tilde{K}_{im}(\mathcal{X})$ and the set $IMVB(\mathcal{X})/_\sim$ where $\xi_1\sim \xi_2$ if and only if there exist piecewise trivial im-bundles θ_1 and θ_2 such that $\xi_1\oplus \theta_1 \cong \xi_2\oplus \theta_2$.

P r o o f. First statement is obvious because for any $[\xi_1- \xi_2] \in \tilde{K}_{im}(\mathcal{X})$ there exists (by Theorem 2) $\eta_2\in ImVB(\mathcal{X})$ such that $\xi_2\oplus \eta_2$ is piecewise trivial and therefore $\xi_1- \xi_2 =$ $\psi(\xi_1\oplus \eta_2)$. For the second observe that $\tilde{\psi} : ImVB(\mathcal{X})/_\sim \longrightarrow \tilde{K}_{im}(\mathcal{X})$ is correctly defined by $\tilde{\psi}([\xi])= [\xi-rk(\xi)]$ and it is bijective. In fact $\tilde{\psi}([\xi_1]) = \tilde{\psi}([\xi_2])$ implies that $[\xi_1- rk(\xi_1)] = [\xi_2-$ $- rk(\xi_2)]$ and $[\xi_1\oplus rk(\xi_2)] = \xi_2\oplus rk(\xi_1)]$ so $\xi_1\sim \xi_2$. This proves Proposition 4.

5. Main Theorem

Let $\mathcal{X}^n = (X,r)$ be a filtered space and let $A = X_{\underline{k}} = r^{-1}(1,2,\dots,k)$ for fixed $1 \le k \le n$. It is clear that $\mathcal{A}^k = (A, r_{|A})$ is also a filtered space. Let us define the filtration on X/A :

$$\mathcal{X}/\mathcal{A} := (X/A, r')$$

where

$$\begin{cases} r(x) - k + 1 & \text{for } x \in X\setminus A \\ 1 & \text{for } [x] = [A] . \end{cases}$$

Let us consider two maps of filtered spaces $\mathcal{A} \xrightarrow{i} \mathcal{X} \xrightarrow{j} \mathcal{X}/\mathcal{A}$. Just as in classical K-theory we obtain two ring homomorphisms

$$\tilde{K}_{im}(\mathcal{X}/\mathcal{A}) \xrightarrow{j^*} \tilde{K}_{im}(\mathcal{X}) \xrightarrow{i^*} \tilde{K}_{im}(\mathcal{A}) .$$

On the other hand there exist two natural homomorphisms which we call collaps (c) and blowup (b) :

$$\tilde{K}_{im}(\mathcal{A}) \xrightarrow{b} \tilde{K}_{im}(\mathcal{X}) \xrightarrow{c} \tilde{K}_{im}(\mathcal{X}/\mathcal{A}) .$$

The blowup homomorphism $b : \tilde{K}_{im}(\mathcal{A}) \longrightarrow \tilde{K}_{im}(\mathcal{X})$ we define as follows :

$$b([\xi]) := [\mathrm{im}(b'(\alpha))]$$

$$b'(\alpha)(x) := \left[\begin{array}{c|c} (1-\varphi_A(x))\tilde{\alpha}(x) & \varphi_A(x)\ I_n \\ \hline \varphi_A(x)\ I_n & 0 \end{array}\right]$$

where :

- $x \in \mathcal{A}$, $\xi = \mathrm{im}\alpha$, $\alpha : \mathcal{A} \longrightarrow \mathcal{E}nd(F^n) \subset \mathcal{E}nd(F^{2n}) \subset \mathcal{E}nd(F^\infty)$;
- $\tilde{\alpha} : X \longrightarrow \mathrm{End}(F^n)$, $\tilde{\alpha}$ is continuous extension of α ;

- $\varphi_A : X \longrightarrow [0,1]$ any function such that $\varphi_A^{-1}(0) = A$.

However it is clear that the above definition is independent of choices of φ_A and α (in the "Ftop homotopy class" $[\alpha]$) and therefore it correctly defines the homomorphism ; it seems to be unnecessarily complicated. To explain the sense of blowup observe that $b([\xi])$ is simply determined by the im-bundle over X which is ξ over A and trivial of high dimension over $X\setminus A$.

The collaps homomorphism $c : \tilde{K}_{im}(\mathcal{X}) \longrightarrow \tilde{K}_{im}(\mathcal{X}/\mathcal{A})$ is defined as follows : $c([\xi]) = [im(c'(\alpha))]$ where

$$c'(\alpha)([x]) = \begin{cases} \varphi_A(x)\alpha(x) & [x] \neq [A] \\ 0 & [x] = [A] \end{cases}$$

and $\varphi_A : X \longrightarrow [0,1]$ is any function such that $\varphi_A^{-1}(0) = A$.

With help of the above definition we are able to formulate

T h e o r e m 3. The sequence $0 \longrightarrow \tilde{K}_{im}(\mathcal{X}/\mathcal{A}) \xrightarrow{j^*} \tilde{K}_{im}(\mathcal{X}) \xrightarrow{i^*} \tilde{K}_{im}(\mathcal{A}) \longrightarrow 0$ is exact.

Moreover : (1) $i^* \circ b = id_{\tilde{K}_{im}(\mathcal{A})}$

(2) $c \circ j^* = id_{\tilde{K}_{im}(\mathcal{X}/\mathcal{A})}$.

P r o o f. Observe that :

(1) implies that i^* is epimorphic ,

(2) implies that j^* is monomorphic.

Therefore it is enough to prove that $imj^* = \ker i^*$ and the relations 1), 2).

L e m m a 2. Let $\xi, \xi_0 \in \mathrm{ImVB}(\mathcal{X})$ be two im-bundles such that $\xi = \mathrm{im}\alpha$ and $\xi_0 = \mathrm{im}(\varphi_A \alpha)$. Then the relation $\xi - \mathrm{rk}(\xi) = \xi_0 - \mathrm{rk}(\xi_0)$ holds if and only if $\xi_{|A} = \mathrm{rk}(\xi_{|A})$.

P r o o f of the Lemma 2. If we assume that $\xi - \mathrm{rk}(\xi) = \xi_0 - \mathrm{rk}(\xi_0)$ then we have $(\xi - \mathrm{rk}(\xi))_{|A} = (\xi_0 - \mathrm{rk}(\xi_0))_{|A}$, but $(\xi_0 - \mathrm{rk}(\xi_0))_{|A} = 0$, hence we obtain $\xi_{|A} = \mathrm{rk}(\xi)_{|A}$.

For inverse observe that the condition $\xi_{|A} = \mathrm{rk}(\xi)_{|A}$ implies that we may choose the base of θ^n, an embedding of ξ into θ^m and a representation $\alpha : \mathcal{X} \longrightarrow \mathcal{E}nd(F^m)$ of the im-bundle ξ, such that $\alpha_{|A}$ is a diagonal matrix. Therefore, we may assume that the rank of the matrix $[\alpha(x), \beta(x)] \in M(m \times 2m)$ - where β is a diagonal representation of the im-bundle $\mathrm{rk}(\xi)$ - is equal to the rank of $\alpha(x) \in \mathcal{E}nd(F^m)$ for any $x \in X$. Let us consider the map given by the matrix :

$$H_t(x) = \left[\begin{array}{c|c} \varphi_A(x)\alpha(x) & 0 \\ \hline t\ \alpha(x) & \beta \end{array}\right].$$

Observe that $\mathrm{rk}\, H_t(x) = \begin{cases} 2\ \mathrm{rk}\alpha(x) & \text{dla } x \in X \setminus A \\ \mathrm{rk}\alpha(x) & \text{dla } x \in A \end{cases}$,

so it is independent of the parameter $t \in I$. Now it is clear ([4]) that :

$$\operatorname{im}\left[\begin{array}{c:c}\varphi_A(\ )\alpha(\ ) & 0\\ \hline 0 & \beta\end{array}\right] \cong \operatorname{im}\left[\begin{array}{c:c}\varphi_A(\ )\alpha(\ ) & 0\\ \hline \alpha(\ ) & \beta\end{array}\right] \cong$$

$$\operatorname{im}\left[\begin{array}{c:c}I-\varphi_A(\ )\ I & 0\\ \hline 0 & I\end{array}\right] \cdot \left[\begin{array}{c:c}\varphi_A(\ )\alpha(\ ) & 0\\ \hline \alpha(x) & \beta(x)\end{array}\right] \cong$$

$$\operatorname{im}\left[\begin{array}{c:c}0 & -\varphi_A(\ )\beta(\ )\\ \hline \alpha(\ ) & \beta(\ )\end{array}\right] \cong \operatorname{im}\left[\begin{array}{c:c}0 & -\varphi_A(\ )\beta(\ )\\ \hline \alpha(\ ) & t\ \beta(\ )\end{array}\right] \cong$$

$$\operatorname{im}\left[\begin{array}{c:c}\alpha(x) & 0\\ \hline 0 & \varphi_A(\ )\beta(\ )\end{array}\right].$$

This proves Lemma 2.

Now we are able to prove Theorem 3.

For (1) let $[\xi - rk(\xi)] \in \tilde{K}_{im}(\mathcal{A})$, $\operatorname{im}\alpha = \xi$. By the definition $\operatorname{im}[b'(\alpha)]_{|A} = [\xi]$, so $(i^* \circ b) = id_{\tilde{K}_{im}(\mathcal{A})}$ because i^* is just the restriction. For (2) let $[\xi - rk(\xi)] \in \tilde{K}_{im}(\mathcal{X}/\mathcal{A})$, $\operatorname{im}\alpha = \xi$. By definition $(c \circ j^*)[\xi - rk(\xi)] = [\xi_0 - rk(\xi_0)]$, and we obtain the relation (2) immediately from Lemma 2 when we put $\mathcal{X} := \mathcal{X}/\mathcal{A}$ and $\mathcal{A} := [A]$.

For $\ker i^* = \operatorname{im} j^*$ let $\xi \in ImVB(\mathcal{X})$, $[\xi] \in \tilde{K}_{im}(\mathcal{X})$. From Lemma 2 we have $i^*([\xi]) = 0$ if and only if $[\xi] = [\xi_0] \in \operatorname{im} j^*$.

This finshes the proof of Theorem 3.

C o r o l l a r y 1. Let $\mathcal{X}^2 = (X,r)$, $r^{-1}(1) = x_0 \in X$, $r^{-1}(2) = X \setminus X_0$, be a filtered space. Then there exists an isomorphism $i^* : \tilde{K}_{im}(\mathcal{X}) \longrightarrow \tilde{K}(X\setminus x_0)$ defined by the relation $i^*([\xi]) = [\xi_{|X\setminus x_0}]$. If $[im\alpha] \in \tilde{K}(X \setminus x_0)$ then we have the relation $(i^*)^{-1}([im\alpha]) = [im\varphi_0(\ )\alpha(\ )] \in \tilde{K}_{im}(\mathcal{X})$ (where $\varphi_0 : \mathcal{X} \longrightarrow \mathcal{R} = (\mathbb{R}, |sgn|)$ is any real function such that $\varphi_0^{-1}(0) = = x_0$).

C o r o l l a r y 2. There is an isomorphism $K_{im}(\mathcal{X}^n) = = \oplus K(X_i)$.

P r o o f. From Theorem 3 (where we put $\mathcal{A} := \mathcal{X}_{\underline{1}}, \mathcal{X}_{\underline{2}}, \dots, \mathcal{X}_{\underline{n}}$) we obtain $\tilde{K}_{im}(\mathcal{X}^n) = \bigoplus_{i=1}^{n} \tilde{K}(X_i)$. From Corollary 1 we have :

$$K_{im}(\mathcal{X}^n) = \tilde{K}_{im}(\mathcal{X}^n) \oplus \mathbb{Z}^n = \left(\bigoplus_{i=1}^{n} \tilde{K}(X_i)\right) \oplus \mathbb{Z}^n = \bigoplus_{i=1}^{n} (\tilde{K}(X_i) \oplus \mathbb{Z}) = \bigoplus_{i=1}^{n} K(X_i).$$

C o r o l l a r y 3. Let X be a finite CW-complex, let $K_{im}(X)$ denote the ring completion of ImVB(X) and let $\mathbb{Z}^X$ be the set of all functions $f : X \longrightarrow \mathbb{Z}$. Then the homomorphism $rk : K_{im}(X) \longrightarrow \mathbb{Z}^X$ defined by the relation $rk[\xi_1 - \xi_2] = = \dim\xi_{1|x} - \dim\xi_{2|x}$ is monomorphic.

P r o o f. Let $rk[\xi_1 - \xi_2] = 0 \in \mathbb{Z}^X$. We may assume that there exists a filtration $r : X \longrightarrow \{1,2,\dots,n\}$ (determined by any triangulation of X which is compatible with the CW-structures given by the im-bundles ξ_i) such that :

- $\xi_1, \xi_2 \in ImVB(\mathcal{X})$, $rk(\xi_1) = rk(\xi_2)$;
- each X_i is contracible.

Then we have $K_{im}(\mathcal{X}^n) = \tilde{K}_{im}(\mathcal{X}^n) \oplus \mathbb{Z}^n = 0 \oplus \mathbb{Z}^n$.

Because $[\xi_1 - \xi_2] = 0$ as an element of $\tilde{K}_{im}(\mathcal{X}^n)$, so there exists $\eta \in ImVB(X)$ such that $\xi_1 \oplus \eta = \xi_2 \oplus \eta$ therefore $[\xi_1 - \xi_2]=0$ as an element of $K_{im}(X)$.

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