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THE FUNCTIONAL LAW OF THE ITERATED LOGARITHM FOR WEAKLY MULTIPLICATIVE SYSTEMS

In this note the functional law of the iterated logarithm for a uniformly bounded p-weakly multiplicative system (abbr. p-WMS system), $1 \le p < 2$, will be proved. A sequence (X_i) of random variables is called a p-WSM system, if

(1) $$C_p = \left[1 + \sum |E(X_{i_1} X_{i_2} \dots X_{i_n})|^p \right]^{1/p} < \infty , \quad 1 \le p < 2 ,$$

where the sum is taken over all combinations of indices. The law of the iterated logarithm for such systems were investigated by Móricz [4], Rèvèsz [5,6], Jakubowski [3]. In Berkes' paper [2] the functional law of the iterated logarithm was proved for a uniformly bounded 1-WMS system.

Let $S_n = X_1 + \dots + X_n$. We define random functions φ_n, ψ_n on $[0,1]$, $n = 1,2,\dots$, as follows :

$$\varphi_n(t) = n^{-1/2} S_{[nt]} , \quad \psi_n(t) = (2n \ln \ln n)^{-1/2} \chi_n(t) ,$$

where $\chi_n(t)$, $0 \le t \le 1$, is a random function, linear in each interval $[\frac{k-1}{n}, \frac{k}{n}]$, $1 \le k \le n$, such that $\chi_n(\frac{k}{n}) = S_n$, $0 \le k \le n$. We recall that a sequence of random variables $X_1, X_2, \dots$ is said to obey the functional (Strassen-type) law of the iterated logarithm if the sequence ψ_n is equicontinuous with probability 1, and the set of its norm-limit points (in the norm C[0,1]) coincides with the set : $K = \{x : x$ is absolutely continuous in $[0,1]$, $x(0) = 0$, and $\int_0^1 (x'(t))^2 dt \le 1\}$ ([1]). In this note Berkes' result is generalized. The following theorem is proved.

T h e o r e m. Let (X_i) be a uniformly bounded p-WMS system, $1 \le p \le 2$, such that (X_i^2-1) is a 1-WMS system. Then (X_i) obeys the functional law of the iterated logarithm.

Berkes in his paper [1] imposed some condition on dependent random variables to obey the functional law of the iterated logarithm. Namely, he obtained the following result.

For every $t = (t_1,\dots,t_r)$ and $t' = (t'_1,\dots,t'_r)$ with

(2) $$0 \le t_1 < t'_1 \le t_2 < t'_2 < \dots \le t_r < t'_r \le 1,$$

let $t = \left[\sum_{i=1}^{r} t_i^2 \right]^{1/2}$, $t_0 = \min(t'_1-t_1,\dots,t'_r-t_r)$ and let $f_n^{(\underline{t},\underline{t}')}$ be the characteristic function of a random vector

$$\left[\frac{\varphi_n(t'_1)-\varphi_n(t_1)}{(t'_1-t_1)^{1/2}}, \dots, \frac{\varphi_n(t'_r)-\varphi_n(t_r)}{(t'_r-t_r)^{1/2}} \right].$$

T h e o r e m A. [1]. Let $X_1, X_2, \dots$ be a sequence of uniformly bounded random variables such that

(3) for every $\underline{t}$, $\underline{t}'$ satisfying (2), for each $n \in N$ and for each $\underline{s} \in R^r$ such that $s \le B_1(nt_0)^{\alpha}$ there holds

$$| f_n^{(\underline{t},\underline{t}')}(s_1,s_2,\dots,s_r) - \exp(-s^2/2) | < B_2(nt_0)^{-\beta}$$

for some $\alpha > 0$, $\beta > 0$, where constants B_1, B_2 depend only on r and the sequence (X_i). Then (X_i) obeys the functional law of the iterated logarithm.

P r o o f of Theorem. It is enough to check that assumptions of Theorem implay assumptions of Theorem A. It will be shown that for uniformly bounded (by $K \geq 1$) p-WMS system the condition (3) is fulfilled with an arbitrary $\alpha \in (0,1/6)$, $\beta = (1/2)-3\alpha$, $B_1 = \min\left\{ \frac{1}{2K^3} , \frac{2^{1/2}}{\max(1,K^2-1)} \right\}$ and a constant B_2 which is described at the end of the proof. The left side of the second inequality of the condition (3) is less or equal to 2, and also $B_3x^{-\beta} > 2$ for $0 < x \leq 1$, $\beta > 0$, provided $B_3 > 2$. Therefore (3) is fulfilled if $nt_0 \leq 1$ and $B_2 > 2$. So we may and do assume that $nt_0 > 1$. It is well known that

(4) $\exp(ix) = (1+ix)\exp\left[-(1/2)x^2+ r(x)\right]$, $|r(x)|\leq|x|^3$ for $|x|<1$.

Let $u_k = \dfrac{s_k}{(n(t'_k - t_k))^{1/2}}$. Then, by (4), we have

(5)
$$f_n^{(\underline{t},\underline{t}')}(s_1,s_2,\ldots,s_r) =$$
$$= E\left[\left[\prod_{k=1}^{r} \prod_{l=[nt_k]+1}^{[nt'_k]} (1+iu_kX_l)\right] \exp\left[-\sum_{k=1}^{r}(1/2)u_k^2N_k + \sum_{k=1}^{r} M_k\right]\right] ,$$

where $N_k = \prod_{l=[nt_k]+1}^{[nt'_k]} X_l^2$, $M_k = \prod_{l=[nt_k]+1}^{[nt'_k]} r(u_kX_l)$.

Since $|e^{-x}- ae^{-y}| \leq |1-a|+ |a|\cdot|e^{-x}- e^{-y}|$ for $x \geq 0$, $a,y\in \mathbb{C}$ and $|1+ib| \leq e^{b^2/2}$ for $b \in R$, we obtain, by (5) :

$$|f_n^{(\underline{t},\underline{t}')}(s_1,s_2,\ldots,s_r) - \exp(-s^2/2)| \leq$$

$$\le \left| E\left(\prod_{k=1}^{r} \prod_{l=[nt_k]+1}^{[nt'_k]} (1+iu_k X_l) \right) - 1 \right| +$$

$$+ E\left| \exp\left(-\sum_{k=1}^{r} (1/2)u_k^2 N_k - s^2/2 \right) - \exp\left(\sum_{k=1}^{r} M_k \right) \right| = I_1 + I_2.$$

To estimate I_1 it will be convenient to introduce one more notation. For $l = 1,2, \ldots, [nt'_r]$ let $v_l = u_k$ if $[nt_k] < l \le [nt'_k]$, and $v_l = 0$ otherwise. Then

$I_1 = | \sum_A v_{l_1} v_{l_2} \ldots v_{l_m} E(X_{l_1} X_{l_2} \ldots X_{l_m})|$, where A is a set of combinations of indices l_i such that $1 \le l_1 < l_2 < \ldots < l_m \le [nt'_r]$, $m = 1, \ldots, [nt'_r]$. Now we consider separately two cases $p = 1$ and $1 < p < 2$. We start from the case $p = 1$. Taking into consideration that $0 < \alpha < 1/6$, $nt_0 > 1$ we have $|u_k| \le \dfrac{s}{(nt_0)^{1/2}}$

and

(6) $$\frac{s}{(nt_0)^{1/2}} \le B_1 \le \frac{1}{2}.$$

Hence $|v_{l_1} v_{l_2} \ldots v_{l_m}| \le \dfrac{s}{(nt_0)^{1/2}}$ and $I_1 \le C_1 \dfrac{s}{(nt_0)^{1/2}}$,

where C_1 is the constant from (1).

Now let $1 < p < 2$ and $q = p/(p-1)$. Since

(7) $$[nt'_k] - [nt_k] \le n(t'_k - t_k) + 1$$

we have $\displaystyle\sum_{l=1}^{[nt'_r]} v_l^q \le \frac{2s^q}{(nt_0)^{q/2-1}}$, and

$$\sum_{1 \le l_1 < \ldots < l_m \le [nt'_r]} (v_{l_1} v_{l_2} \ldots v_{l_m})^q \le \frac{2s^{mq}}{(nt_0)^{mq/2-1}}.$$

By this estimations we obtain

$$I_1 \le \Big[\sum_A (v_{l_1} v_{l_2} \dots v_{l_m})^q \Big]^{1/q} \Big[\sum_A |E(X_{l_1} X_{l_2} \dots X_{l_m})|^p \Big]^{1/p} \le$$

$$\le C_p \Big[\sum_{l=1}^{[nt'_r]} v_l^q + \sum_{1 \le l_1 < l_2 \le [nt'_r]} (v_{l_1} v_{l_2})^q + \dots + v_1 v_2 \dots v_{[nt'_r]} \Big]^{1/q} \le$$

$$\le C_p \Big[2nt_0 \sum_{m=1}^{\infty} \Big[\frac{s^q}{(nt_0)^{q/2}} \Big]^m \Big]^{1/q} \le C_p\, 4^{1/q} \frac{s}{(nt_0)^{1/2-1/q}} ,$$

for, by (6), $0 \le \Big[1 - \frac{s^q}{(nt_0)^{1/q}} \Big]^{-1} \le 2$.

To estimate I_2 we use the well known inequality

(8) $$|e^u - 1| \le 3u \quad \text{for } |u| \le 1.$$

Hence we obtain

$$I_2 \le E\Big|\exp\Big[-\sum_{k=1}^{r}(1/2)u_k^2 N_k - s^2/2\Big] - 1\Big| + 3E\Big|\sum_{k=1}^{r} M_k\Big| = I_3 + 3E\Big|\sum_{k=1}^{r} M_k\Big|,$$

provided $\big|\sum_{k=1}^{r} M_k\big| \le 1$. From (7) and from definition of B_1 we have

$$\Big|\sum_{k=1}^{r} M_k\Big| \le \sum_{k=1}^{r} \sum_{l=[nt_k]+1}^{[nt'_k]} |u_k X_l|^3 \le \frac{2K^3 s^3}{(nt_0)^{1/2}} \le 1.$$

On account that $|e^x - 1| \le \sum_{k=1}^{\infty} |x|^k/k!$, we get $I_3 \le \sum_{k=1}^{\infty} E|Z|^k/k!$, where

$$Z = (1/2)\sum_{k=1}^{r} (u_k^2 N_k - s_k^2) = (1/2)\sum_{k=1}^{r} \frac{s_k^2}{n(t'_k - t_k)} \Big[\sum_{l=[nt_k]+1}^{[nt'_k]} (X_l^2 - 1) + \alpha_k \Big],$$

with $\alpha_k = n(t'_k - t_k) - [nt'_k] + [nt_k]$, so $|\alpha_k| \le 1$. The sequence

(Y_n) : $Y_0 = 1$, $Y_n = X_n^2 - 1$, $n \geq 1$, is, by assumption, the 1-WMS system. Using the estimations for moments of (Y_n) (cf. [3], Theorem 6) we obtain that $E|Z|^m \leq LK_1^m D_m \left[\frac{s^4}{2nt_0} \right]^{m/2}$, where $K_1 = \max(1, K^2 - 1)$, L is the constant C_1 from (1) for the sequence (Y_n), $n = 0,1,\ldots$, D_m is a constant from Khinthine inequality. Taking the best constants $D_{2m} = (2m-1)!!$ ([7]) and using the inequalities

$$\frac{t^{2j+1}}{(2j+1)!} \leq \frac{2^{1/2}}{2}\left[\frac{t^{2j}}{(2j)!} + \frac{t^{2j+2}}{(2j+2)!}\right], \quad \frac{D_{2m}}{(2m-1)!} \leq \frac{1}{m!}, \text{ and } \frac{s^4K_1^2}{2nt_0} \leq 1$$

(which is true by assumption (3) and by definition of B_1), we get by (8),

$$I_3 \leq E|Z| + (1+2^{1/2}) \sum_{k=1}^{\infty} E|Z|^{2k}/(2k)! \leq (4+3\cdot 2^{1/2})LK_1 \frac{s^2}{(nt_0)^{1/2}} .$$

Let $B_4 = 6K^3 + (4+3\cdot 2^{1/2})LK_1$, and $B_2 = B_4 + \max(C_1, 4^{1/q} C_p)$.

Taking all estimations together we have

$$I_1 + I_2 < \frac{C_1 + B_4}{(nt_0)^{1/2}} \max(s, s^2, s^3) \leq B_2 (nt_0)^{3\alpha - 1/2} \quad \text{for } p = 1,$$

and

$$I_1 + I_2 < \frac{B_4}{(nt_0)^{1/2}} \max(s^2, s^3) + 4^{1/q} C_p \frac{1}{(nt_0)^{q/2-1/2}} < B_2 (nt_0)^{3\alpha - 1/2}$$

for $p > 1$. Hence taking arbitrary positive $\alpha < 1/6$ and $\beta = 3\alpha + 1/2$ we see that the condition (3) is fulfilled. The Theorem is proved.

REFERENCES

[1] I. B e r k e s : The functional law of the iterated logarithm for dependent random variables, Z. Wahrscheinlichkeitstheorie ver. Geb. 26 (1973), 245-258.

[2] I. B e r k e s : On Strassen's version of the loglog law for multiplicative systems, Studia Sci. Math. Hung. 8 (1973), 425-431.

[3] J. J a k u b o w s k i : Structural theorems for multiplicative systems of functions, Acta Math. Acad. Sci. Hungar. 46 (1985), 111-126.

[4] F. M ó r i c z : The law of the iterated logarithm for weakly multiplicative systems, Analysis. Math. 2 (1976), 211-229.

[5] P. R è v è s z : A law of the iterated logarithm for weakly multiplicative systems and its applications, Acta Math. Acad. Sci. Hungar. 25 (1974), 425-433.

[6] P. R è v è s z : The law of the iterated logarithm for multiplicative systems, Indiana Univ. Math. J.21 (1972), 557-564.

[7] S. B. S t e č k i n : On the best lacunary systems of functions (in Russian), Izv. Acad. Nauk SSSR, ser. Mat. 25 (1961), 357-366.

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