DEMONSTRATIO MATHEMATICA
Vol. XXII No 3 1989

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GENERALIZATION OF THE PASCAL THEOREM TO THE N-DIMENSIONAL PROJECTIVE SPACE

1. Introduction

In nearly every book concerned with projective geometry (e.g. [4], page 194) it is possible to find the Pascal theorem about the hexagon inscribed into the conical curve. This theorem is often used in the proofs of other theorems and in the constructions linked with conical curves on a projective plane. It turns out that important generalizations of these theorems exist with regard for the space dimension as for the type of the skew polygon inscribed in the normal curve. In the present work methods of synthetic geometry are being applied.

Notations :

c^n - a normal curve of a projective space $\mathbb{P}^n$,

$[A_kA_l]$ - a line determined by the points A_k and A_l ,

$[A_1A_2...A_k]$-a hyperplane determined by the points $A_1, A_2, ... A_k$,

$\mathcal{B}^k$ - a k-dimensional hyperplane ,

$[k_1k_2...k_s]$-a hyperplane determined by the lines $k_1, k_2, ..., k_s$,

$[\mathcal{B}^kA]$ - a hyperplane determined by a k-dimensional hyperplane and point A ,

$\mathcal{B}^k(\alpha_1,\alpha_2,\dots)$ - a pencil of k+1 - dimensional hyperplanes with a k-dimensional $\mathcal{B}^k$ hyperedge ,

$\mathcal{B}^k(\alpha_1,\alpha_2,\dots) \barwedge p(A_1,A_2,\dots)$ - a pencil of hyperplanes α_1, $\alpha_2,\dots$ with a hyperedge $\mathcal{B}^k$ which is perspective to a series of points $p(A_1,A_2,\dots)$,

$\mathcal{B}^k_1(\alpha'_1,\alpha'_2,\dots) \barwedge \mathcal{B}^k_2(\alpha''_1,\alpha''_2,\dots)$ - a pencil $\mathcal{B}^k_1$ of k+1 - dimensional hyperplanes $\alpha'_1,\alpha'_2,\dots$ is projective to a pencil $\mathcal{B}^k_2$ of k+1 - dimensional hyperplanes $\alpha''_1,\alpha''_2,\dots$,

$S_{k,l}$ - the Serge manifold which has k and l as dimensional generating lines ,

PC - the Pascal system.

2. The Pascal theorem about the n+5 angle inscribed into a normal curve

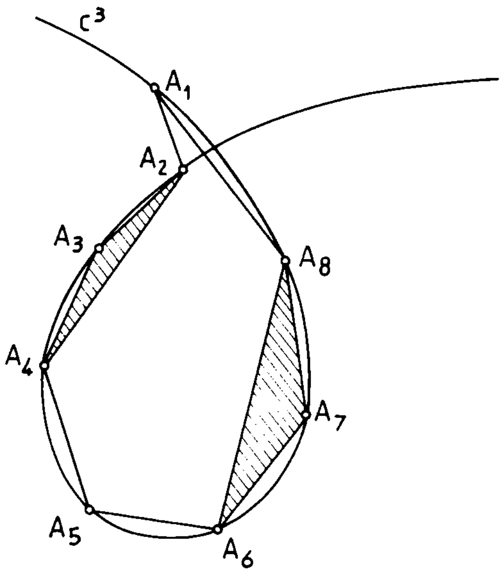
We designate a normal curve in the projective space $\mathbb{P}^n$ as algebraic curve of the order n of this space which is not included in the space whose dimension is n-1 ([3], page 5).

D e f i n i t i o n 2.1.

1. On the normal curve c^n we choose n+5 of different points $A_0,A_1,\dots,A_{n+4}$ (where $n = 2k+1$, $k \in N$). The cycle of points $(A_0,A_1,\dots,A_{n+4})$ together with the lines $[A_0A_1]$, $[A_1A_2]$ ..., $[A_{n+4}A_0]$ will be designated as n+5-angle inscribed into the normal curve c^n. We denominate the points A_i as vertexes and the lines $[A_iA_{i+1}]$ as sides of the n+5-angle ($i = 0,1,\dots,n+4$, $A_{n+5} = A_0$).

2. We denominate as opposite sides the pairs of sides ($[A_i A_{i+1}], [A_{\frac{2i+n+5}{2}} \ A_{\frac{2i+n+7}{2}}]$) ; by which indexes greater than n+4 are reduced modulo n+5.

3. We designate as opposite faces two $\frac{n+1}{2}$ dimensional hyperplanes from which one is determined by $\frac{n+1}{2}$ of the next sides and the second by $\frac{n+1}{2}$ of the sides opposite to them.



Figure

For n = 4k - 1 (k ∈ N) the following theorem is true.

T h e o r e m 2.1. The intersection lines of $\frac{n+5}{2}$ pairs of opposite faces of the n+5-angle inscribed into the normal curve c^n are included in one hypersurface of the second order.

P r o o f. We write down pairs of opposite faces of the n+5-angle inscribed into the normal curve c^n using the appropriate diagram :

$$
\left.\begin{array}{llll|llll}
A_0 & A_1 & \ldots & A_{\frac{n+3}{2}} & A_{n+5}A_{n+7} & \ldots & A_{n+4} & \\
\hline
[A_0 & A_1 & \ldots & A_{\frac{n+1}{2}}] & [A_{\frac{n+5}{2}}A_{\frac{n+7}{2}} & \ldots & A_{n+3}] & \ldots l_0 \\
[A_1 & A_2 & \ldots & A_{\frac{n+3}{2}}] & [A_{\frac{n+7}{2}}A_{\frac{n+9}{2}} & \ldots & A_{n+4}] & \ldots l_1 \\
[A_2 & A_3 & \ldots & A_{\frac{n+5}{2}}] & [A_{\frac{n+9}{2}}A_{\frac{n+1}{2}} & \ldots & A_0\ ] & \ldots l_2 \\
\ldots & \ldots & \ldots & \ldots & & & & \\
[A_{\frac{n+3}{2}} & A_{\frac{n+5}{2}} & \ldots & A_{n+1}] & [A_{n+4}A_1 & \ldots & A_{\frac{n-1}{2}}] & \ldots l_{\frac{n+3}{2}}
\end{array}\right\} \subset H^2
$$

In order to show that the lines $l_0, l_1, \ldots, l_{\frac{n+3}{2}}$ are included in one hypersurface of the second order, it is sufficient to prove that the Serge manifold $S_{\frac{n-1}{2}}$, to which belong those lines, exists. In this aim we have to show three $\frac{n-1}{2}$ - dimensional hyperplanes which are reciprocally skew and intersect the lines l_0, $l_1, \ldots, l_{\frac{n+3}{2}}$ (compare [2], page 230).

It is known that the points of the normal curve c^n are projected from two different $\frac{n-1}{2}$ - dimensional hyperplanes $\frac{n+1}{2}$ - dimensional hyperplanes ([1], page 323). Let us consider these two projective pencile with hyperedges $\mathcal{B}_1^{\frac{n-1}{2}} = [A_3A_4 \ldots A_{\frac{n+5}{2}}]$ and $\mathcal{B}_2^{\frac{n-1}{2}} = [A_{\frac{n+9}{2}}A_{\frac{n+11}{2}} \ldots A_{\frac{n+4}{2}}]$. Let α_{ij} denote a $\frac{n+1}{2}$ - dimensional hyperplane which belongs to the pencil $\mathcal{B}_1^{\frac{n-1}{2}}$ and goes through point A_j. Then we get the following projection :

$$(1) \quad \mathcal{B}_1^{\frac{n-1}{2}}(\alpha_{10},\alpha_{11},\alpha_{12},\alpha_{1\frac{n+7}{2}},\dots) \;\bar{\wedge}\; \mathcal{B}_2^{\frac{n-1}{2}}(\alpha_{20},\alpha_{21},\alpha_{22},\alpha_{2\frac{n+7}{2}},\dots)$$

We cut a pencil $\mathcal{B}_1^{\frac{n-1}{2}}$ with the line $p_1 = [A_0A_1]$, and the pencil $\mathcal{B}_2^{\frac{n-1}{2}}$ with the line $p_2 = [A_1A_2]$. We obtain series of points which are perspective to the pencils of the hyperplanes :

$$(2) \quad \begin{cases} p_1(A_0,A_1,A_2,A_{\frac{n+7}{2}},\dots) \;\bar{\bar{\wedge}}\; \mathcal{B}_1^{\frac{n-1}{2}}(\alpha_{10},\alpha_{11},\alpha_{12},\alpha_{1\frac{n+7}{2}}\dots), \\ p_2(A_0'',A_1,A_2,A''_{\frac{n+7}{2}},\dots) \;\bar{\bar{\wedge}}\; \mathcal{B}_2^{\frac{n-1}{2}}(\alpha_{20},\alpha_{21},\alpha_{22},\alpha_{2\frac{n+7}{2}}\dots). \end{cases}$$

From the equivalence (1) and (2) results the following projection of series :

$$p_1(A_0,A_1,A_2',A'_{\frac{n+7}{2}},\dots) \;\bar{\wedge}\; p_2(A_0'',A_1,A_2,A''_{\frac{n+7}{2}},\dots).$$

These series have got a common point A_1, hence they are perspective. Both series are included in the plane $\alpha_0=[A_0A_1A_2]$. Let us notice that the centre of perspectivity of the series (p_1) and (p_2) is the point $\{L_2\} = [A_0A_0''] \cap [A_2A_2']$, where $\{L_2\} = l_2 \cap \alpha_0$. As a matter of fact ,

$$[A_0A_0''] = \alpha_0 \cap [A_{\frac{n+9}{2}}A_{\frac{n+11}{2}}\dots A_{n+4}A_0] ,$$

$$[A_2A'_2] = \alpha_0 \cap [A_3A_4 \ \dots \ A_{\frac{n+5}{2}}A_2] \ ,$$

$$[A_0A''_0] \cap [A_2A'_2] = \alpha_0 \cap [A_2A_3A_4 \dots A_{\frac{n+5}{2}}] \cap [A_{\frac{n+9}{2}}A_{\frac{n+11}{2}} \dots A_{n+4}A_0] =$$

$$= \alpha_0 \cap l_2 = \{L_2\}.$$

Hence $L_2 \in [A'_{\frac{n+7}{2}}A''_{\frac{n+7}{2}}]$. Let us also remark that

$$A'_{\frac{n+7}{2}} \in [A_3A_4 \dots A_{\frac{n+7}{2}}] \cap [A_{\frac{n+11}{2}}A_{\frac{n+13}{2}} \dots A_0A_1] = l_3 \ ,$$

$$A''_{\frac{n+7}{2}} \in [A_1A_2 \dots A_{\frac{n+3}{2}}] \cap [A_{\frac{n+7}{2}}A_{\frac{n+9}{2}} \dots A_{n+4}] = l_1 \ .$$

Therefore we have shown the line $k_0 = [A'_{\frac{n+7}{2}}A''_{\frac{n+7}{2}}]$ included in α_0 which intersects the lines l_1, l_2, l_3.
After that it is possible to show the line $k_1 \subset [A_1A_2A_3]$ which intersects the lines l_2, l_3, l_4 , if we take $[A_5A_6 \dots A_{\frac{n+9}{2}}]$ and $[A_{\frac{n+11}{2}}A_{\frac{n+13}{2}} \dots A_0]$ as hyperedges of the considered pencils of the hyperplanes. In the same way it is possible to show the line $k_{\frac{n-3}{2}} \subset [A_{\frac{n-3}{2}}A_{\frac{n-1}{2}}A_{\frac{n+1}{2}}]$ intersecting the lines $l_{\frac{n-1}{2}}$, $l_{\frac{n+1}{2}}$, $l_{\frac{n+3}{2}}$ and the line $k_{\frac{n-1}{2}} \subset [A_{\frac{n-1}{2}}A_{\frac{n+1}{2}}A_{\frac{n+3}{2}}]$ intersecting the lines $l_{\frac{n+1}{2}}$, $l_{\frac{n+3}{2}}$, l_0. At the end one can show the line $k_{\frac{3n-1}{4}} \subset [A_{\frac{3n-1}{4}}A_{\frac{3n+3}{4}}A_{\frac{3n+7}{4}}]$ intersecting the lines $l_{\frac{3n+3}{4}}, l_{\frac{3n+7}{4}}$, $l_{\frac{3n+11}{4}}$ (we reduce the indexes greater than $\frac{n-3}{2}$ modulo $\frac{n+5}{2}$).

The $\frac{n-1}{2}$ - dimensional hyperplane $\mathcal{B}_0$ determined by $\frac{n+1}{4}$ of skew lines k_0, k_3, ..., $k_{\frac{3n-9}{4}}$ intersecting the lines l_0, l_1, ..., $l_{\frac{n+3}{2}}$. The second and third hyperplane looked for are correspondingly equal to $\mathcal{B}_1=[k_1k_4\ldots k_{\frac{3n-5}{4}}]$, $\mathcal{B}_2=[k_2k_5\ldots k_{\frac{3n-1}{4}}]$. The hyperplanes $\mathcal{B}_0,\mathcal{B}_1,\mathcal{B}_2$ are skew pairs. Actually, if two from them $\mathcal{B}_0$ and $\mathcal{B}_1$ intersect, this would determine a n-1 - dimensional hyperplane $\mathcal{B}$ of at least, which would include the lines $l_0,l_1,\ldots,l_{\frac{n+3}{2}}$ in spite of the fact that the lines in question undo the n-dimensional space. Therefore the lines $l_0,l_1,\ldots,l_{\frac{n+3}{2}}$ are included in the manifold Serge $S_{\frac{n-1}{2},1}$.

If $n = 4k-1$ $(k \in N)$, it means a second order surface H^2 exists and that it contains $S_{\frac{n-1}{2},1}$ (compare [2], page 230).

In the case where $n = 2s+3$ $(s \in N)$ on the basis of the proof of Theorem 2.1., it is possible to formulate the two following theorems :

T h e o r e m 2.2. The intersection lines of $\frac{n+5}{2}$ pairs of opposite faces of the n+5 - angle inscribed into the normal curve c^n belong to one manifold Serge $S_{\frac{n-1}{2},1}$.

T h e o r e m 2.3. If the points A_0, $A_1,\ldots,A_{n+4}$ are different and belong to the normal curve c^n $(n = 2k+1,\ k\in N)$ and $[A_iA_{i+1}] \cap [A_{i+3}A_{i+4}\ldots A_{2i+n+5}] = \{C_i\}$,

$$[A_{i+2}A_{i+3}\ldots A_{\frac{2i+n+3}{2}}] \cap [A_{\frac{2i+n+7}{2}}A_{\frac{2i+n+9}{2}}\ldots A_i] = l_i ,$$

$$[A_{\frac{2i+n+5}{2}} A_{\frac{2i+n+7}{2}} \dots A_{i+n+3}] \cap [A_{i+1} A_{i+2}] = \{D_i\} ,$$

It results that the lines $[C_i D_i]$ and l_i intersect.

3. The Pascal theorem about the 2n+2 - angle inscribed into a normal curve

The second of the generalizations concerns the 2n+2 skew angle inscribed into the normal curve c^n. Let us assume the following definitions :

D e f i n i t i o n 3.1. We will consider the set of n-2 - dimensional n+1 hyperplanes out of which every two ones have a common hyperplane which is at the utmost n-4 - dimensional and 2n+2 of reciprocally skew lines which intersect all these hyperplanes. We refer to this set as "Pascal system" and designate it by the symbol PC.

D e f i n i t i o n 3.2.

1. On the normal curve c^n we choose 2n+2 different points $(A_0, A_1, \dots, A_{2n+1})$ $(n \in N)$. We consider the cycle of points $(A_0, A_1, \dots, A_{2n+1})$ together with the lines $[A_0 A_1]$, $[A_1 A_2], \dots, [A_{2n+1} A_0]$ and we call it 2n+2 - angle inscribed in c^n. We call the points A_i vertexes and the lines $[A_i A_{i+1}]$ sides of the 2n+2 angle $(i = 0, 1, \dots, 2n+1,\ A_{2n+2} = A_0)$.

2. We denominate as opposite sides the pairs of sides $([A_i A_{i+1}], [A_{i+n} A_{i+n+1}])$ where the indexes greater than 2n+1 are reduced modulo 2n+2.

3. We denominate as opposite faces of the 2n+2 angle inscribed into c^n two n-1 - dimensional hyperplanes from which one is determined by n-1 consecutive sides and the

second by n-1 of opposite sides.
We will prove the next theorem.

T h e o r e m 3.1. The n-2 - dimensional hyperplanes of intersection of n+1 pairs of opposite sides of the 2n+2 - angle inscribed in the normal curve c^n belong to the Pascal system PC.

P r o o f. We write down the pairs of opposite faces of the 2n+2 skew angle inscribed in the normal curve c^n using a diagram similar to the Pascal diagram applied in the proof of the Theorem 2.1.

$$
\begin{array}{l|l}
A_0 \quad A_1 \quad \ldots \quad A_n & A_{n+1} A_{n+2} \quad \ldots \quad A_{2n+1} \\
\hline
[A_0 \quad A_1 \quad \ldots \quad A_{n-1}] & [A_{n+1} A_{n+2} \quad \ldots \quad A_{2n}] \ \ldots \ \gamma_0^{n-2} \\
[A_1 \quad A_2 \quad \ldots \quad A_n] & [A_{n+2} A_{n+3} \quad \ldots \quad A_{2n+1}] \ \ldots \ \gamma_1^{n-2} \\
\ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
[A_n \quad A_{n+1} \ \ldots \ A_{2n-1}] & [A_{2n+1} A_0 \quad \ldots \quad A_n] \ \ldots \ \gamma_n^{n-2}
\end{array}
\left.\vphantom{\begin{array}{l}a\\a\\a\\a\\a\end{array}}\right\} \subset PC
$$

In order to show that the n-2 - dimensional $\gamma_0^{n-2}, \gamma_1^{n-2}, \ldots, \gamma_n^{n-2}$ hyperplanes belong to PC we have to prove the existence of the lines $k_0, k_1, \ldots, k_{2n+1}$ with pairs of skew lines intersecting these hyperplanes. It is known that the points of the normal curve are projected from two different n-2 - dimensional hyperplanes n-1 secant as two projective pencils of hyperplanes n-1 - dimensional ([1], page 323). Let us consider these projective pencils with hyperedges $\mathcal{B}_1^{n-2}$ =

$[A_3\ A_4 \ldots A_{n+1}]$. Let α_{1j} denote the hyperplane n-1 -dimensional belonging to the pencil $(\mathcal{B}_1^{n-2})$ and going through point A_j. Then we have got the following projection :

$$(3)\quad \mathcal{B}_1^{n-2}(\alpha_{10},\alpha_{11},\alpha_{12},\alpha_{1\ n+2},\ldots)\,\bar{\wedge}\,\mathcal{B}_2^{n-2}(\alpha_{20},\alpha_{21},\alpha_{22},\alpha_{2\ n+2},\ldots).$$

Let us intersect the pencil $(\mathcal{B}_1^{n-2})$ with the line $p_1 = [A_0A_1]$ and the pencil $(\mathcal{B}_2^{n-2})$ with the line $p_2 = [A_1A_2]$. We will obtain series of points which are perspective to the pencils of the hyperplanes :

$$(4)\quad \begin{cases} p_1(A_0,A_1,A'_2,A'_{n+2},\ldots)\ \bar{\wedge}\ \mathcal{B}_1^{n-2}(\alpha_{10},\alpha_{11},\alpha_{12},\alpha_{1\ n+2},\ldots)\ , \\ p_2(A''_0,A_1,A_2,A''_{n+2},\ldots)\ \bar{\wedge}\ \mathcal{B}_2^{n-2}(\alpha_{20},\alpha_{21},\alpha_{22},\alpha_{2\ n+2},\ldots). \end{cases}$$

From the corresondence (3) and (4) results the projection :

$$p_1(A_0,A_1,A'_2,A'_{n+2},\ldots)\ \bar{\wedge}\ p_2(A''_0,A_1,A_2,A''_{n+2},\ldots)\ .$$

These series have a common point A_1, they are therefore perspective. Both series are included in the plane $\alpha_0 = [A_0A_1A_2]$. The following relations are resulting from this :

$$[A_0A''_0] \cap [A_2A''_2] \cap [A'_{n+2}A''_{n+2}] = \{S\}\ ,$$

$$[A_0A''_0] = [A_{n+3}A_{n+4}\ldots A_{2n+1}A_0] \cap \alpha_0\ ,$$

$$[A_2A'_2] = [A_3A_4\ldots A_{n+1}] \cap \alpha_0\ ,$$

$$\{S\} = [A_0A''_0] \cap [A_2A'_2] = \gamma_2^{n-2} \cap \alpha_0\ ,$$

$$A'_{n+2} \in [A_3A_4\ldots A_{n+2}] \cap [A_{n+4}A_{n+5}\ldots A_{2n+1}A_0A_1] = \gamma_3^{n-2}\ ,$$

$$A''_{n+2} \in [A_1A_2\ldots A_n] \cap [A_{n+2}A_{n+3}\ldots A_{2n+1}] = \gamma_1^{n-2}.$$

As a consequence the line $k_0 = [A_{n+2}A_{n+2}] \cap \alpha_0$, to which belongs the point S, intersects γ_1^{n-2}, γ_2^{n-2}, γ_3^{n-2}. These three hyperplanes are intersections of the second, third and fourth pair of hyperplanes which appear in the Pascal diagram.

Because α_0 is also included in one of the hyperplanes of every other pair of hyperplanes from the Pascal diagram, therefore intersects the remaining hyperplanes γ_i^{n-2} for i = 0,4,5,...,n. Changing only the symbols in the above mentioned reasoning it is possible to show the lines $k_1 \subset [A_1A_2A_3]$, $k_2 \subset [A_2A_3A_4]$,...,$k_{2n+1} \subset [A_{2n+1}A_0A_1]$ which intersect each of the hyperplanes γ_i^{n-2} (i = 0, 1,..., n). From the property of the normal curve it results that the lines which were pointed out are skew. Hence the hyperplanes γ_0^{n-2}, γ_1^{n-2}, ,...,γ_n^{n-2} belong to the system PC.

Let us consider that, in a three-dimensional projective space, Theorems 2.1. and 3.1. concern a skew octagon inscribed into the normal curve c^3. In this space the manifold Serge $S_{1,1}$ is equivalent to the second order surface ([2], page 230) and the Pascal system is equivalent to the set of four skew lines of one family of lines od a determined second order surface and to the set of eight other lines from the second family of lines of this surface. Theorems 2.1. and 3.1. can be formulated under one theorem : the intersection lines of four pairs of opposite faces of the octagon inscribed into the normal curve c^3 belong to one family of a second order surface H^2.

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Received December 9, 1987.