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ON AN INVERSE PROBLEM FOR THE NORMAL SINGULAR INTEGRALS

1. Preliminaries

Let E be a non-empty subset of a metric space, having an accumulation point ξ_0 (in the case $E \subset R$ we admit also $\xi_0 = \infty$). Denote by K a non-negative function defined on the product $R \times E$, with values $K(t,\xi)$ 2π-periodic and even in t, non-increasing on the interval $\langle 0,\pi\rangle$, for every fixed $\xi \in E$.

D e f i n i t i o n 1. The function K, introduced above, is called a kernel if it satisfies the conditions

(1) $$\int_{-\pi}^{\pi} K(t,\xi) = 1 \qquad \text{for every } \xi \in E,$$

(2) $$\lim_{\xi \to \xi_0} K(\tau,\xi) = 0 \qquad \text{for every } \tau \in (0,\pi\rangle.$$

The kernel K is said to be normal if there are positive numbers A, δ such that for every $h \in (0,\delta)$ and for some $\xi = \xi(h) \in E$ the inequality

(3) $$h\, K(h,\xi(h)) \geq A$$

holds, and

(4) $$\lim_{h \to 0^+} \xi(h) = \xi_0 .$$

Write $L_{2\pi}$ for the class of all 2π-periodic real-valued functions of one variable, Lebesgue-integrable over $\langle -\pi,\pi\rangle$.

D e f i n i t i o n 2. A real number x is called a D-point of $f \in L_{2\pi}$ if

(5) $$\lim_{h \to 0} \frac{1}{h} \int_0^h (f(x+u) - f(x))du = 0.$$

A number x is said to be a D_s- point of the function f if

(6) $$\lim_{h \to 0^+} \frac{1}{h} \int_0^h (f(x+u) + f(x-u) - 2f(x))du = 0.$$

If x is a D- point [D_s- point] of $f \in L_{2\pi}$, then we write $x \in D(f)$ [resp. $x \in D_s(f)$].

We will consider the singular integral

$$U(x,\xi\ ;f,K) = \int_{-\pi}^{\pi} f(t)\ K(x-t,\xi)dt \quad (f \in L_{2\pi}) .$$

It is said to be normal when K denotes a normal kernel.

For arbitrary fixed $C > 0$, $\delta > 0$, $\varepsilon \in (0,1)$, $x_0 \in R$ we introduce the sets

$$Z_C = Z_C(K) = \{(x,\xi) \in R \times E : |x-x_0|K(0,\xi) \le C\} ,$$

$$Z_{C,\delta,\varepsilon} = \{(x,\xi) \in R{\times}E : (\int_{-\delta}^{\delta} |f(t+x)-f(t+x_0)|dt)^{1-\varepsilon}K(0,\xi) \le C\}.$$

The following theorem of the Romanovski type is known.

T h e o r e m. Suppose that $f \in L_{2\pi}$ and K is a kernel. If $x_0 \in D(f)$ and $C > 0$ [$x_0 \in D_s(f)$ and $C > 0$, $\delta > 0$, $\varepsilon \in (0,1)$], then

(7) $U(x,\xi;f,K) \longrightarrow f(x_0)$ as $(x,\xi) \longrightarrow (x_0,\xi_0)$ and $(x,\xi) \in Z_C$

[resp. $(x,\xi) \in Z_{C,\delta,\varepsilon}$]. In particular,

(8) $$\lim_{\xi \to \xi_0} U(x,\xi;f,K) = f(x_0) .$$

(see [4] p. 175, [3] p. 137, [1] p. 63).

Assuming that $f \in L_{2\pi}$, K is a kernel and (7) or (8) is satisfied, it is natural to ask at which points x_0 it is possible ?

In the next section some answers for this questions are given.

2. Statement of results

T h e o r e m 1. Suppose that K is a normal kernel, $f \in L_{2\pi}$, $x_0 \in R$, (8) holds and exists $\varepsilon > 0$ such that

(9) $\quad \varphi(x) = f(x)-f(x_0) \ge 0$ a.e. on $(x_0-\varepsilon, x_0+\varepsilon)$ or

(9') $\quad \varphi(x) \le 0$ a.e. on $(x_0-\varepsilon, x_0+\varepsilon)$.

Then $x_0 \in D(f)$.

T h e o r e m 2. Suppose that K is a normal kernel, $f \in L_{2\pi}$, $x_0 \in R$, (8) holds and there exists $\varepsilon > 0$ such that

(10) $\quad \psi(t) = f(x_0+t)+f(x_0-t)-2f(x_0) \ge 0$ a.e. on $(0,\varepsilon)$ or

(10') $\quad \psi(t) \le 0$ a.e. on $(0,\varepsilon)$.

Then $x_0 \in D_s(f)$.

T h e o r e m 3. Let $E = (0,1)$, $\xi_0 = 0$, and let $K(t,\xi)$ be 2π-periodic extension (in t) of the function

$$P(t,\xi) = \begin{cases} 1/(2\xi) , & t \in (-\pi,\pi) \\ 0 , & t \in \langle -\pi,-\xi\rangle \cup \langle \xi,\pi\rangle \quad (\xi \in E) . \end{cases}$$

Then, for an arbitrary $f \in L_{2\pi}$, the assumption (7) in which $C \ge 1/2$ implies $x_0 \in D(f)$.

3. Proof of Theorem 1

Suppose that (9) holds. By Definition 1 and (3) we have, for $h \in (0,\min(\delta,\varepsilon))$,

$$0 \le A \frac{1}{h} \int_0^h (f(x_0 \pm t)-f(x_0))dt \le hK(h,\xi(h)) \frac{1}{h} \int_0^h \varphi(x_0 \pm t)dt \le$$

$$\le \int_0^h \varphi(x_0 \pm t)K(t,\xi(h))dt \le \int_{-\varepsilon}^{\varepsilon} \varphi(x_0 + t)K(t,\xi(h))dt.$$

Hence

$$(11) \quad 0 \le \frac{1}{h} \int_0^h (f(x_0 \pm t)-f(x_0))dt \le A^{-1} \int_{-\varepsilon}^{\varepsilon} \varphi(x_0 + t)K(t,\xi(h))dt.$$

Combining the assumption (8) with conditions (1), (2) and monotonicity and evenness of $K(t,\xi)$ one has

$$\lim_{\xi \to \xi_0} \int_{-\varepsilon}^{\varepsilon} \varphi(x_0 + t)K(t,\xi)dt = 0 ,$$

which together with (4) yields

$$\lim_{h \to 0^+} \int_{-\varepsilon}^{\varepsilon} \varphi(x_0 + t)K(t,\xi(h))dt = 0 .$$

Thus, in view of (11), the point x_0 is a D- point of f.

Under the assumption (9') instead of (9), our thesis can be obtained paralelly.

The proof of Theorem 2 is similar to the above one (we replace the integrand $\varphi(x_0 \pm t)$ by $\psi(t)$, and the relation (6) follows).

4. Proof of Theorem 3

To establish $x_0 \in D(f)$ we observe that

$$U(x_0+\frac{\xi}{2},\ \frac{\xi}{2};\ f,K) = \frac{1}{\xi}\int_{-\xi/2}^{\xi/2} f(t+x_0+\frac{\xi}{2})dt = \frac{1}{\xi}\int_0^{\xi} f(s+x_0)ds.$$

Since $(x_0+\frac{\xi}{2},\ \frac{\xi}{2}) \in Z_{1/2}(K)$, we have

$$U(x_0+\frac{\xi}{2},\ \frac{\xi}{2};\ f,K) \longrightarrow f(x_0) \quad \text{as} \quad \xi \longrightarrow 0^+ ,$$

by assumptions of Theorem 3. Whence (5) holds for $x = x_0$ and $h \longrightarrow 0^+$. If we take $(x_0-\frac{\xi}{2},\ \frac{\xi}{2}) \in Z_{1/2}(K)$, then we obtain (5) for $x = x_0$ and $h \longrightarrow 0^-$. Therefore $x_0 \in D(f)$ when $C = 1/2$. In case $C > 1/2$, the inclusion $Z_{1/2}(K) \subset Z_C(K)$ leads to $x_0 \in D(f)$, immediately.

5. Examples

We will examine three concrete non-negative kernels.

(a) The kernel of Abel-Poisson is defined by

$$p_r(t) = \frac{1}{2\pi}\ \frac{1-r^2}{1-2r\cos t + r^2},\quad r \in E = (0,1),\ \xi_0 = 1.$$

([1] p. 53). This kernel is normal.

Indeed, for positive h,

$$hp_r(h) = \frac{1}{2\pi}\ \frac{(1-r^2)h}{(1-r)^2 + 4r\sin^2(h/2)} \ge \frac{1}{2\pi}\ \frac{1-r^2}{(1-r)^2/h + rh}.$$

Assuming that $r = r(h) = (\sqrt{h^2+4}-h)^2/4$, we have $h = (1-r)/\sqrt{r}$. Hence

$$hp_{r(h)}(h) \ge \frac{1}{2\pi}\ \frac{1-r^2}{2(1-r)\sqrt{r}} \ge \frac{1}{2\pi} \quad \text{if } h \in (0,\infty),$$

and

$$\lim_{h \to 0^+} r(h) = 1.$$

(b) The kernel of de la Vallèe Poussin is defined by

$$v_n(t) = \frac{1}{2\pi} \frac{(n!)^2}{(2n)!} (2 \cos \frac{t}{2})^{2n}, \quad n \in E = N, \ \xi_0 = \infty.$$

([1] p. 112). It is normal, too.

To prove this, we choose the positive integers

$$n = n(h) = [2/h]^2$$

corresponding to positive numbers $h \le 2$ ($[\alpha]$ denotes the integral part of α). Then, the obvious inequality

$$(\cos \frac{h}{2})^{2n} \ge (\cos \frac{h}{2})^{2(2/h)^2}$$

together with the relation

$$\lim_{\lambda \to 0} (\cos \lambda)^{2/\lambda^2} = e^{-1}$$

leads to

(12) $$\frac{1}{2e} \le (\cos \frac{h}{2})^{2n} \quad \text{for every } h \in (0, \delta_0),$$

provided that δ_0 is small enough. By Stirling's formula,

(13) $$\frac{1}{2} \sqrt{\pi [2/h]^2} \le \frac{(n!)^2 2^{2n}}{(2n)!} \quad \text{for each } h \in (0, \delta_1),$$

whenever δ_1 denotes a suitable positive number.

Applying (12) and (13), we obtain

$$h \cdot v_{n(h)}(h) \ge 0.02 \quad \text{if } h \in (0, \min(\delta_0, \delta_1)).$$

Finally, it is evident that $n(h) \longrightarrow \infty$ as $h \longrightarrow 0^+$.

(c) Consider the set $E = (0, 1/2)$ and it accumulation point $\xi_0 = 0$. Suppose that $\xi \in E$, $t_0 = e^{1-1/(2\xi)}$, and write

$$g_\xi(t) = \begin{cases} \xi/t_0 & , \quad t \in \langle 0, t_0\rangle , \\ \xi/t & , \quad t \in \langle t_0, 1\rangle , \\ 0 & , \quad t \in (1, \pi\rangle . \end{cases}$$

Denote by $K(t,\xi)$ the even 2π-periodic extension (in t) of $g_\xi(t)$ introduced just now.

The last kernel K is not normal, since $hK(h,\xi) \le \xi$ for every $h \in (0,\pi)$ and for all $\xi \in E$.

6. Remarks

Without the assumptions (9) and (9') Theorem 1 is false. Similarly, without (10) and (10') Theorem 2 is not true.

Indeed, let $f(t)$ be 2π-periodic extension of sgn t, $t \in \langle -\pi, \pi)$. Clearly, for an arbitrary kernel K,

$$\lim_{\xi \to \xi_0} \int_{-\pi}^{\pi} f(t)K(t,\xi)dt = 0 = f(0) ,$$

but $x_0 = 0 \notin D(f)$. Thus, the thesis of Theorem 1 does not hold.

As regards Theorem 2, let us put

$$f(t) = \begin{cases} \sum_{k=0}^{\infty} (-1)^k \chi_{\langle 1/2^{k+1}, 1/2^k)}(t), & t \in (0,1) , \\ 0 & , \quad t \in \langle 1, \pi\rangle \cup \{0\} , \end{cases}$$

where χ_B denotes the characteristic function of the set B. Let us extend this function on R by 2π-periodic and even way. Considering the sequence $h_n = 2^{-n}$, we observe that $x_0 = 0 \notin D_s(f)$. Let K be the kernel mentioned in Theorem 3, but with

$$E = \{ \xi \in (0,1) : \int_0^{\xi} f(t)dt = 0 \}$$

instead of $E = (0,1)$. The point $\xi_0 = 0$ is an accumulation point of so changed set E.

Obviously,

$$\int_{-\pi}^{\pi} f(t)K(t,\xi)dt = \frac{1}{2\xi} 2 \int_0^{\xi} f(t)dt = f(0) \quad \text{if} \quad \xi \in E.$$

However the thesis of Theorem 2 does not hold, because $0 \notin D_s(f)$.

7. Appendix

Similar results one can obtain in the non-periodic case (under modified assumptions on K as in [1] p. 121, 132, [2] p. 113).

Last of all we observe that every non-negative kernel χ of Fejèr's type ([1] p. 121) satisfies conditions (3) and (4). Indeed, there is $t_0 > 0$ such that $\chi(t_0) > 0$, because $\int_{-\infty}^{\infty} \chi(t)dt > 0$ and $\chi \geq 0$. Taking $\xi_0 = \infty$ and $\xi(h) = t_0/h$ we get

$$h(\xi(h)\cdot\chi(\xi(h)\cdot h)) = t_0\, \chi(t_0) > 0 .$$

Therefore (4) holds with $\xi_0 = \infty$, and (3) holds with $A = t_0 \cdot \chi(t_0)$.

For example, the kernels of Cauchy-Poisson and Weierstrass ([1] p. 125-126) satisfy (3), (4), thus they are normal.

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