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ON THREE DIMENSIONAL PSEUDO-UMBILICAL ISOTROPIC SUBMANIFOLDS

1. Introduction

A submanifold M in a Riemannian manifold N is said to be isotropic (or λ-isotropic) if for each point p in M and each unit vector t tangent to M at p, the length λ of the normal curvature vector h(t,t) depends only on p, not on t at p. In particular, when λ is also independent of the point p in M, then M is said to be constant isotropic. It is known (see [2]) that M is isotropic at p if and only if the second fundamental form h satisfies

$$\langle h(x,x),h(x,y)\rangle = 0 \tag{1.1}$$

for any orthonormal vectors x and y of the tangent space $T_p(M)$, where $\langle,\rangle$ is the scalar product on N.

Let H be the mean curvature vector of M in N. M is said to be pseudo-umbilical if there exists a function ρ on M such that $\langle h(x,y),H\rangle = \rho\langle x,y\rangle$ for all vectors x and y tangent to M. Recently, B. Y. Chen and P. Verheyen [1] proved that an isotropic surface in a Riemannian manifold must be pseudo-umbilical.

In this paper, we study 3-dimensional isotropic submanifolds. We shall prove the following theorems.

Theorem 1. Let M be a 3-dimensional λ-isotropic submanifold in a space form N. If M is pseudo-umbilical, then the mean curvature H of M satisfies $\|H\| \le \lambda$, the equality holds if and only if M is totally umbilical.

Theorem 2. Let M be a 3-dimensional λ-isotropic submanifold in a space form N. If M is pseudo-umbilical, then M has constant mean curvature if and only if M is constant isotropic.

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2. Preliminaries

Let M be an n-dimensional submanifold in an m-dimensional Riemannian manifold N. We choose a local field of orthogonal frames $(e_1,\dots,e_n;\ e_{n+1},\dots,e_m)$ in N such that, restricted to M. We denote by $(\theta^1,\dots,\theta^m)$ the field of dual frames. The structure equations of N are given by

(2.1) $$d\theta^A = -\sum \theta^A_B \wedge \theta^B\ ,\quad \theta^A_B + \theta^B_A = 0\ ,$$

(2.2) $$d\theta^A_B = -\sum \theta^A_C\wedge \theta^C_B + \Phi^A_B,\ \Phi^A_B = \frac{1}{2}\sum K^A_{BCD}\theta^C\wedge \theta^D,\ K^A_{BCD} + K^A_{BDC} = 0,$$

$A,B,C,D,\dots = 1,\dots,m$. Restricting these forms on M, we have $\theta^r = 0$, $r,s,t,\dots = n+1,\dots,m$. Since

(2.3) $$0 = d\theta^r = -\sum \theta^r_i\wedge \theta^i\ ,\quad i,j,k,\dots = 1,\dots,n.$$

Cartan's lemma implies

(2.4) $$\theta^r_i = \sum h^r_{ij}\theta^j , \quad h^r_{ij} = h^r_{ji} .$$

If we denote by ∇ and $\tilde{\nabla}$ the covariant derivatives of M and N, espectively, then for any two vector fields x,y tangent to M and any vector field ξ normal to M, we have

(2.5) $$\tilde{\nabla}_x y = \nabla_x y + h(x,y) ,$$

(2.6) $$\tilde{\nabla}_x \xi = -A_\xi x + D_x \xi ,$$

where $-A_\xi x$ and $D_x\xi$ denote the tangential and normal components of $\tilde{\nabla}_x\xi$ respectively. A_ξ is called the Weingarten map in the direction of ξ and D gives a connection in the normal bundle. We have

(2.7) $$< A_\xi x, y > = < h(x,y), \xi >.$$

We define the covariant derivative $\bar{\nabla}h$ of h by

(2.8) $$(\bar{\nabla}_x h)(y,z) = D_x h(y,z) - h(\nabla_x y, z) - h(y, \nabla_x z) ,$$

for any vector fields x,y,z tangent to M, then if N is a space form, the equation of Codazzi becomes

$$(\bar{\nabla}_x h)(y,z) = (\bar{\nabla}_y h)(x,z).$$

For later use, we recall the following lemma due to B. O'Neill [2].

L e m m a A. If a submanifold M in a Riemannian manifold N is λ-isotropic, then for any orthogonal vectors x,y,z in $T_p(M)$,

(2.10) (i) $$< h(x,x), h(y,y) > + 2\|h(x,y)\|^2 = \lambda^2 ,$$

(2.11) (ii) $< h(x,x),h(y,z) > + 2 < h(x,y),h(x,z) > = 0.$

We define the discriminant Δ of the second fundamental form h, a real-valued function on planes (through p) in $T_p(M)$, such that if x and y span π, then

(2.12) $$\Delta(\pi) = \Delta_{xy} = \left[< h(x,x),h(y,y) > - \|h(x,y)\|^2\right] / \|x \wedge y\|^2.$$

3. Weingarten map

Let M be an n-dimensional λ-isotropic submanifold in an m-dimensional Riemannian manifold N and $(e_1,\ldots,e_n)$ be an orthogonal frame tangent to M. Then $H = \frac{1}{n}\sum_{i=1}^{n} h(e_i,e_i)$. Since M is λ- isotropic, we may use (1.1) and Lemma A to compute A_H as follows.

(3.1) $$<A_He_j,e_j> = <H,h(e_j,e_j)> = \frac{1}{n}\sum_{i=1}^{n}<h(e_i,e_i),h(e_j,e_j)> =$$
$$= \lambda^2 - \frac{2}{n}\sum_{\substack{i=1\\ i\neq j}}^{n}\|h(e_i,e_j)\|^2 ,$$

(3.2) $$<A_He_j,e_k> = <H,h(e_j,e_k)> = \frac{1}{n}\sum_{i=1}^{n}<h(e_i,e_i),h(e_j,e_k)> =$$
$$= -\frac{2}{n}\sum_{i=1}^{n}<h(e_i,e_j),h(e_i,e_k)>, \quad i,j,k = 1,\ldots,n , \quad j \neq k.$$

The following lemma is similar to a lemma obtained by B. Y. Chen and P. Verheyen [1].

L e m m a 1. Let M be a 3-dimensional λ-isotropic submanifold in an m-dimensional Riemannian manifold N. If M is pseudo-umbilical, then with respect to a suitable orthonormal frame $(e_1, e_2, e_3; e_4, \dots, e_m)$, where e_1, e_2, e_3 are tangent to M and $e_4, \dots, e_m$ are normal to M, we have

$$(3.3)\quad A_4 = \begin{bmatrix} 0 & b & 0 \\ b & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_5 = \begin{bmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ b & 0 & 0 \end{bmatrix}, \quad A_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & b \\ 0 & b & 0 \end{bmatrix}$$

$$A_7 = \begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & \frac{c^2 - b^2}{c} \end{bmatrix}, \quad A_8 = \begin{bmatrix} b & 0 & 0 \\ 0 & -b & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_9 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{3}\,ab}{c} \end{bmatrix},$$

$$A_{10} = \dots = A_m = 0 ,$$

for some functions a, b, c on M, where $A_r = A_{e_r}$.

P r o o f. Let M be a 3-dimensional λ-isotropic submanifold in an m-dimensional Riemannian manifold N. If M is pseudo-umbilical, then from (3.1) and (3.2), we have

$$(3.4)\qquad \|h(e_i, e_j)\| = b, \quad \text{for any } i \neq j ,$$

where b is a function on M, and

$$(3.5)\qquad \langle h(e_i, e_i), h(e_j, e_k)\rangle = 0 ,$$

$$(3.6)\qquad \langle h(e_i, e_j), h(e_i, e_k)\rangle = 0 , \quad \text{for any } i \neq j \neq k \neq i.$$

Consider (3.4), if $b \equiv 0$, by (2.10), we have

$$(3.7)\qquad h(e_1, e_1) = h(e_2, e_2) = h(e_3, e_3).$$

Thus we may choose e_7 in the direction of $h(e_1, e_1)$ and obtain (3.3) with $b \equiv 0$ immediately. From now we assume that $b \not\equiv 0$. Then by (3.4)-(3.6), we may choose e_4, e_5, e_6 as follows.

(3.8) $h(e_1,e_2) = be_4$, $h(e_1,e_3) = be_5$, $h(e_2,e_3) = be_6$.

Since $h(e_1,e_1) + h(e_2,e_3)$ and $h(e_1,e_1) - h(e_2,e_2)$ are perpendicular to each other and do not vanish, we may put, with the help of (3.5),

(3.9) $$h(e_1,e_1) + h(e_2,e_2) = 2ce_7, \quad c > 0 ,$$

(3.10) $$h(e_1,e_1) - h(e_2,e_2) = 2qe_8, \quad q > 0.$$

Then by (2.10) and (3.4), we have $c^2 = \lambda^2 - b^2$ and $q^2 = b^2$, i.e., $c = \sqrt{a^2 - b^2}$ and $q = b$. Consequently, we have

(3.11) $$h(e_1,e_1) = ce_7 + be_8 ,$$

(3.12) $$h(e_2,e_2) = ce_7 - be_8 .$$

Let $a = \|H\|$ and $M_1 = \{p \in M \mid a(p) = 0\}$. On M_1, we have

(3.13) $$\lambda^2 = \|h(e_3,e_3)\|^2 = \|-h(e_1,e_1) - h(e_2,e_2)\|^2 = 4c^2.$$

Then since $c^2 = \lambda^2 - b^2$, we have $3\lambda^2 - 4b^2 = 0$ and

(3.14) $$h(e_3,e_3) = -2ce_7 = \frac{c^2 - b^2}{c} e_7 .$$

Combining these with (3.8), (3.11) and (3.12), we obtain (3.3) with $a^2 = (3\lambda^2 - 4b^2)/3 = 0$ on M_1 .

On $M \setminus M_1$, we may still have (3.8), (3.11) and (3.12), but we claim that $h(e_i,e_i)$, $i= 1,2,3$, are linear independent. Suppose that it is not true and let

(3.15) $$h(e_3,e_3) = fh(e_1,e_1) + gh(e_2,e_2) ,$$

for some functions f and g on $M \setminus M_1$.

Compute $\langle h(e_3,e_3),h(e_i,e_i)\rangle$, i = 1,2,3, we would have, with the help of (2.10),

$$\lambda^2 f + (\lambda^2 - 2b^2)g = \lambda^2 - 2b^2 , \tag{3.16}$$

$$(\lambda^2 - 2b^2)f + \lambda^2 g = \lambda^2 - 2b^2 , \tag{3.17}$$

$$(\lambda^2 - 2b^2)(f + g) = \lambda^2 . \tag{3.18}$$

Then from (3.16)-(3.18), we would have $f = g = -1$ and $3\lambda^2 - 4b^2 = 0$. These contradict $a^2 = (3\lambda^2 - 4b^2)/3 \neq 0$. Now wemay choose e_9 in the following way

$$we_9 = h(e_3,e_3) + k[h(e_1,e_1) + h(e_2,e_2)], \quad w > 0 , \tag{3.19}$$

where k is an undetermined function. We shall choose a suitable value of k in order that e_9 is really perpendicular to e_7 and e_8. By (2.10), (3.11) and (3.12), we have

$$0 = \langle we_9, h(e_1,e_1)+h(e_2,e_2)\rangle = \tag{3.20}$$

$$= \langle h(e_3,e_3),h(e_1,e_1)+h(e_2,e_2)\rangle + k\|h(e_1,e_1)+h(e_2,e_2)\|^2 =$$

$$= 2(\lambda^2 - 2b^2) + 4k(\lambda^2 - b^2).$$

Thus $k = -(\lambda^2 - 2b^2)/2(\lambda^2 - b^2) = -(c^2 - b^2)/2c^2$. Substituting it into (3.19), we have $w^2 = b^2(3\lambda^2 - 4b^2)/(\lambda^2 - b^2) = 3a^2b^2/c^2$ and

$$h(e_3,e_3) = \frac{c^2 - b^2}{c} e_7 + \frac{\sqrt{3}\,ab}{c} e_9 , \tag{3.21}$$

since $a^2 = (3\lambda^2 - 4b^2)/3 > 0$ and $c^2 = \lambda^2 - b^2 > 0$. Combining (3.21) with (3.8), (3.11) and (3.12), we obtain (3.3) on $M \setminus M_1$.

4. Proofs of Theorems

Let M be a 3-dimensional λ-isotropic submanifold in an m-dimensional space form N. If M is pseudo-umbilical, then by Lemma 1, we have (3.8), (3.11) and (3.21), from which we may compute $(\bar{\nabla}_{e_i} h)(e_j, e_k)$, $i, j, k = 1,2,3$, on the subset of M on which $c > 0$. Since N is a space form, we may use the equation (2.9) of Codazzi and have the following three groups of equations, (4.1) - (4.10), (4.11)-(4.20) and (4.21)- (4.27).

(4.1) $$c\theta_7^4(e_2) + b\theta_8^4(e_2) - 2b\theta_1^2(e_2) = e_1(b) ,$$

(4.2) $$c\theta_7^5(e_3) + b\theta_8^5(e_3) - 2b\theta_1^3(e_3) = e_1(b) ,$$

(4.3) $$e_1(c) - b\theta_8^7(e_1) = b\theta_4^7(e_2) ,$$

(4.4) $$c\theta_7^8(e_1) - e_1(b) = b[\theta_8^4(e_2) + 2\theta_1^2(e_2)],$$

(4.5) $$c\theta_7^9(e_1) - b\theta_8^9(e_1) = b\theta_4^9(e_2) ,$$

(4.6) $$c\theta_7^5(e_3) - b\theta_8^5(e_3) = b[\theta_6^5(e_2) - \theta_2^1(e_2)] ,$$

(4.7) $$e_1\left(\frac{c^2 - b^2}{c}\right) + \frac{\sqrt{3}ab}{c}\,\theta_9^7(e_1) = b[\theta_5^7(e_3) + \frac{b}{c}\,\theta_1^3(e_3)],$$

(4.8) $$\frac{c^2 - b^2}{c}\,\theta_7^8(e_1) + \frac{\sqrt{3}ab}{c}\,\theta_9^8(e_1) = b[\theta_5^8(e_3) - \theta_3^1(e_3)],$$

(4.9) $$\frac{c^2 - b^2}{c}\,\theta_7^4(e_2) + \frac{\sqrt{3}ab}{c}\,\theta_9^4(e_2) = b[\theta_6^4(e_3) - \theta_3^1(e_3)],$$

(4.10) $$b[\theta_4^6(e_3) - \theta_1^3(e_3)] = b[\theta_5^6(e_2) - \theta_1^2(e_2)] = e_1(b);$$

$$e_2(c) + b\theta_8^7(e_2) = b\theta_4^7(e_1) \ , \tag{4.11}$$

$$c\theta_7^8(e_2) + e_2(b) = b[\theta_8^4(e_1) + 2\theta_1^2(e_1)], \tag{4.12}$$

$$c\theta_7^9(e_2) + b\theta_8^9(e_2) = b\theta_4^9(e_1) \ , \tag{4.13}$$

$$c\theta_7^6(e_3) + b\theta_8^6(e_3) = b[\theta_5^6(e_1) - \theta_1^2(e_1)] \ , \tag{4.14}$$

$$c\theta_7^4(e_1) - b\theta_8^4(e_1) - 2b\theta_2^1(e_1) = e_2(b) \ , \tag{4.15}$$

$$c\theta_7^6(e_3) - b\theta_8^6(e_3) - 2b\theta_2^3(e_3) = e_2(b) \ , \tag{4.16}$$

$$\frac{c^2 - b^2}{c}\,\theta_7^4(e_1) + \frac{\sqrt{3}\,ab}{c}\,\theta_9^4(e_1) = b[\theta_5^4(e_3) - \theta_3^2(e_3)], \tag{4.17}$$

$$e_2\left(\frac{c^2 - b^2}{c}\right) + \frac{\sqrt{3}\,ab}{c}\,\theta_9^7(e_2) = b[\theta_6^7(e_3) + \frac{b}{c}\,\theta_2^3(e_3)], \tag{4.18}$$

$$\frac{c^2 - b^2}{c}\,\theta_7^8(e_2) + \frac{\sqrt{3}\,ab}{c}\,\theta_9^8(e_2) = b[\theta_6^8(e_3) + \theta_3^2(e_3)], \tag{4.19}$$

$$b\ [\theta_4^5(e_3) - \theta_2^3(e_3)] = e_2(b) = b[\theta_6^5(e_1) - \theta_2^1(e_1)]; \tag{4.20}$$

$$e_3(c) + b\theta_8^7(e_3) = b[\theta_5^7(e_1) + \frac{b}{c}\,\theta_1^3(e_1)] \ , \tag{4.21}$$

$$c\theta_7^8(e_3) + e_3(b) = b[\theta_5^8(e_1) - \theta_3^1(e_1)] \ , \tag{4.22}$$

$$c\theta_7^5(e_1) - b\theta_8^5(e_1) = b[\theta_4^5(e_2) - \theta_2^3(e_2)] \ , \tag{4.23}$$

$$e_3(c) - b\theta_8^7(e_3) = b[\theta_6^7(e_2) + \frac{b}{c}\,\theta_2^3(e_2)] \ , \tag{4.24}$$

$$c\theta_7^8(e_3) - e_3(b) = b[\theta_6^8(e_2) + \theta_3^2(e_2)] \ , \tag{4.25}$$

$$e_3(b) = b[\theta_5^4(e_2) - b\theta_3^2(e_2) = b[\theta_6^4(e_1) - \theta_3^1(e_1)], \tag{4.26}$$

(4.27) $$c\theta_7^6(e_2) + b\theta_8^6(e_2) = b[\theta_4^6(e_1) - \theta_1^3(e_1)],$$

where $a = \|H\| = \sqrt{3\lambda^2 - 4b^2}/\sqrt{3}$ and $c = \sqrt{\lambda^2 - b^2}$ ($\neq 0$). Now we claim that $(\lambda^2 - 3b^2)$ is constant on M. First observe the first group of the equations. Comparing (4.1) with (4.4), we have

(4.28) $$\theta_7^8(e_1) = \theta_7^4(e_2) ,$$

and substituting (4.28) into (4.3), we get

(4.29) $$e_1(c) = 2b\theta_4^7(e_2).$$

From (4.6) and (4.10), we have

(4.30) $$e_1(b) = c\theta_5^7(e_3) - b\theta_5^8(e_3) ,$$

comparing it with (4.2), we get

(4.31) $$c\theta_5^7(e_3) = -b\theta_1^3(e_3) ,$$

(4.32) $$e_1(b) = -b[\theta_5^8(e_3) + \theta_1^3(e_3)].$$

Then substituting (4.28), (4.32) into (4.8) ; and (4.10) into (4.9), and then from comparing the results, we have

(4.33) $$ab\theta_8^9(e_1) = ab\theta_4^9(e_2) .$$

Substituting (4.33) into (4.5), we get

(4.34) $$ac\theta_7^9(e_1) = 2ab\theta_4^9(e_2).$$

Using (4.29), (4.31) and (4.34), (4.7) becomes

$$\frac{1}{c}e_1(c^2 - b^2) - \frac{2b}{c}\left[\frac{c^2 - b^2}{c}\theta_4^7(e_2) + \frac{\sqrt{3}ab}{c}\theta_4^9(e_2)\right] = 0 ,$$

then by (4.9) and (4.10), we obtain

(4.36) $$e_1(\lambda^2 - 2b^2) - 2be_1(b) = 0\ ,$$

i.e., $e_1(\lambda^2 - 3b^2) = 0$.

Similarly, from the second group of the equations, we have

(4.37) $$\theta_7^8(e_2) = -\theta_7^4(e_1)\ ,$$

(4.38) $$e_2(c) = 2b\theta_4^7(e_1)\ ,$$

(4.39) $$c\theta_6^7(e_3) = -b\theta_2^3(e_3)\ ,$$

(4.40) $$e_2(b) = c\theta_6^7(e_3) + b\theta_6^8(e_3) = b[\theta_6^8(e_3) - \theta_2^3(e_3)]\ ,$$

(4.41) $$ab\theta_8^9(e_2) = -ab\theta_4^9(e_1)\ ,$$

(4.42) $$ac\theta_7^9(e_2) = 2ab\theta_4^9(e_1)\ ,$$

(4.43) $$\frac{1}{c}\, e_2(c^2 - b^2) - \frac{2b}{c}\left[\frac{c^2 - b^2}{c}\,\theta_4^7(e_1) + \frac{\sqrt{3}\,ab}{c}\,\theta_4^9(e_1)\right] = 0.$$

Then by (4.17) and (4.20), (4.43) becomes

(4.44) $$e_2(\lambda^2 - 2b^2) - 2be_2(b) = 0\ ,$$

i.e., $e_2(\lambda^2 - 2b^2) = 0$.

Finally, in the third group of the equations, multiplicating (4.21) and (4.22) by c and -b, respectively, and then summing up the results, we have

(4.45) $$ce_3(c) + 2bc\theta_8^7(e_3) - be_3(b) = b[c\theta_5^7(e_1) - b\theta_5^8(e_1)].$$

Then by (4.23) and (4.26), (4.45) becomes

(4.46) $$ce_3(c) - 2be_3(b) = 2bc\theta_7^8(e_3).$$

On the other hand, from (4.21) and (4.24), we have

(4.47) $$2b\theta_8^7(e_3) = b\theta_5^7(e_1) + \frac{b^2}{c}\theta_1^3(e_1) - b\theta_6^7(e_2) - \frac{b^2}{c}\theta_2^3(e_2).$$

And from (4.22) and (4.45), we have

(4.48) $$2c\theta_7^8(e_3) = b\theta_5^8(e_1) - b\theta_3^1(e_1) + b\theta_6^8(e_2) + b\theta_3^2(e_2).$$

Combining (4.47) and (4.48), we get, with the help of (4.23), (4.26) and (4.27),

(4.49) $$4bc\theta_7^8(e_3) = [bc\theta_6^7(e_2) + b^2\theta_6^8(e_2)] - [bc\theta_5^7(e_1) - b^2\theta_5^8(e_1)] =$$
$$= e_3(b) - e_3(b) = 0.$$

Thus we obtain, from (4.46) and (4.49),

(4.50) $$e_3(\lambda^2 - 3b^2) = 0.$$

The claim is proved by (4.36), (4.44) and (4.50). Next, we use this result to prove the following.

L e m m a 2. Let M be a 3-dimensional isotropic submanifold in an m-dimensional space N. If M is pseudo-umbilical, then the descriminant Δ of h on M is constant.

P r o o f. We have proved above that $\lambda^2 - 3b^2$ is constant on M. Now we are going to prove $\Delta = \lambda^2 - 3b^2$. Let π be an arbitrary plane through p in T_pM determined by $x = \sum_{i=1}^{3} a_i e_i$ and $y = \sum_{i=1}^{3} b_i e_i$ which are two arbitrary orthonormal vectors tangent to M at p, where $\sum_{i=1}^{3} a_i^2 = 1$, $\sum_{i=1}^{3} b_i^2 = 1$ and $\sum_{i=1}^{3} a_i b_i = 0$. Then we have

$$(4.51)\quad \Delta(\pi) = \Delta_{xy} = \langle h(x,x),h(y,y)\rangle - \|h(x,y)\|^2 =$$

$$= \langle\, h(\sum_{i=1}^{3} a_i e_i, \sum_{j=1}^{3} a_j e_j),\ h(\sum_{k=1}^{3} b_k e_k, \sum_{u=1}^{3} b_u e_u)\rangle +$$

$$-\langle\, h(\sum_{i=1}^{3} a_i e_i, \sum_{j=1}^{3} b_j e_j),\ h(\sum_{k=1}^{3} a_k e_k, \sum_{u=1}^{3} b_u e_u)\rangle =$$

$$= \sum_{\substack{i,j,\\ k,u=1}}^{3} (a_i a_j b_k b_u - a_i b_j a_k b_u)\langle h(e_i,e_j),h(e_k,e_u)\rangle =$$

$$= \sum_{\substack{i\neq u\\ j\neq k}} a_i b_u (a_j b_k - b_j a_k)\langle h(e_i,e_j),h(e_k,e_u)\rangle =$$

(by (1.1) and (3.6))

$$= \sum_{i,k} a_i b_k (a_i b_k - b_i a_k)\langle h(e_i,e_i),h(e_k,e_k)\rangle +$$

$$+ \sum_{i,j} a_i b_j (a_j b_i - b_j a_i)\langle h(e_i,e_j),h(e_i,e_j)\rangle =$$

(by (3.5))

$$= \sum_{i,j} a_i b_j (a_i b_j - b_i a_j)(\lambda^2 - 2b^2) + \sum_{i,j} a_i b_j (a_j b_i - b_j a_i) b^2 =$$

(by (3.8), (3.11), (3.12) and (3.21))

$$= \sum_{i,j} (a_i^2 b_j^2 - a_i b_i a_j b_j)(\lambda^2 - 3b^2) =$$

$$= \lambda^2 - 3b^2 .$$

Since π is arbitrary, we obtain the Lemma.

P r o o f o f T h e o r e m 1. Since $\|H\|^2 = a^2 = = (3\lambda^2 - 4b^2)/3$, we have $0 \le \|H\| \le \lambda$. If $\|H\| = \lambda$, then $b = 0$. Thus by (3.3), M is totally umbilical. The converse is also true.

P r o o f o f T h e o r e m 2. Since by Lemma 2, we have $\Delta = \lambda^2 - 3b^2 =$ const. on M, it implies $9\|H\|^2 - 5\lambda^2 =$ const. Thus, $\|H\| =$ const. if and only if $\lambda =$ const.

5. Application

Let M be a 3-dimensional pseudo-umbilical isotropic submanifold in a 9-dimensional space form $N^9(C)$, where C denotes the sectional curvature of $N^9(C)$. If M is constant isotropic, we may apply Theorem 2 to it and have both b and c constant. And then using the same method as in Section 4, we may find that if $b \neq 0$, then

$$(5.1) \qquad \theta_4^5 = \theta_2^3 , \quad \theta_4^6 = -\theta_5^8 = \theta_1^3 ,$$
$$\theta_5^6 = -\frac{1}{2}\theta_4^8 = \theta_1^2 , \quad \theta_4^7 = \theta_4^9 = 0 .$$

On the other hand, from (3.3) and (2.4), we have

$$(5.2) \qquad \theta_1^5 = \theta_2^6 = b\theta^3 , \quad \theta_1^4 = \theta_3^6 = b\theta^2 , \quad \theta_2^7 = c\theta^2 ,$$
$$\theta_3^7 = \frac{c^2 - b^2}{c}\theta^3 , \quad \theta_3^4 = \theta_2^5 = \theta_3^8 = \theta_2^9 = 0 .$$

Take the exterior differentiation of $\theta_4 = \theta_2$, then by using (5.1), (5.2), (2.2) and (2.4), we have

$$(5.3) \qquad \lambda^2 - 4b^2 + C = 0 ,$$

which implies

(5.4) $$3\|H\|^2 - 2\lambda^2 + C = 0 .$$

This equation gives us some information on M and $N^9(C)$. Namely, with the help of the formula (5.4) and the assertions of Theorem 1 one can easily deduce the following:

Let M be a 3-dimensional pseudo-umbilical and constant λ-isotropic submanifold in a space form $N^9(C)$, then M must be totally geodesic, or totally umbilical, or

i) when C = 0, M is a submanifold in E^9 with $\|H\|^2 = \frac{2}{3}\lambda^2$;

ii) when C > 0, M is a submanifold in S^9 with $\lambda^2 \geq \frac{1}{2} C$, the equality holds if and only if M is minimal ;

iii) when C < 0, M is a submanifold in $H^9(C)$ with $\lambda^2 > -C$.

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