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CONNECTIONS BETWEEN THE GENERALIZED FOURIER TRANSFORM AND THE FOURIER GAP SERIES

In this paper some two theorems concerning absolute convergence of Fourier gap series are given.

Denote by L^p ($1 \le p \le \infty$) the Banach space of all measurable functions, Lebesgue-integrable with p-th power on interval $(-\infty, \infty)$.

The norm of $f \in L^p$ is defined as

$$\|f\|_{L^p} = \left\{ \int_{-\infty}^{\infty} |f(t)|^p \, dt \right\}^{1/p}.$$

Let λ be a continuous and odd function increasing in the interval $(-\infty, \infty)$ such that $\lambda(0) = 0$, and $|\lambda(x)| \longrightarrow \infty$ as $x \longrightarrow \pm\infty$. Moreover, let the inverse function λ^{-1} be continuously differentiable in $(-\infty, \infty)$ and $(\lambda^{-1}(t)) \le M$ for all real t, where M is a positive constant.

Let additionally $n_k = \lambda(k)$ for integer k, where n_k are integers as well.

D e f i n i t i o n 1. (see [2] p. 65).

Let $f \in L^1$. A function given by

$$\hat{f}^{\lambda}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t)\, e^{-i\lambda(x)t} \, dt$$

will be called the generalized Fourier transform.

D e f i n i t i o n 2. (see [2] p. 73)

If $f \in L^2$, by Schwarz's inequality, f is locally integrable and following integrals exist

$$\hat{f}^{\lambda}_{R}(x) = \frac{1}{2\pi} \int_{|y|\le R} f(y)\, e^{-i\lambda(x)y}\, dy\ , \quad R > 0\ .$$

The sequence $\left\{ \hat{f}^{\lambda}_{R} \right\}$ defined as above tends with $R \longrightarrow \infty$, to an element f of space L^2.This limit in L^2 will be called the generalized Fourier transform of the function f. This definition is similar one to that of Fourier transform in ordinary sense and argumentation is of the same type as given in [3] p. 39.

If $f \in L^1$ with period 2π, then

$$c_{\lambda(k)} = \hat{f}^{\lambda}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-i\lambda(k)t}dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-in_k t}dt = c_{n_k}$$

are Fourier gap coefficients of the function f and its Fourier series with gaps can be written in the form :

$$S[f] = \sum_{k=-\infty}^{\infty} c_{n_k} e^{in_k y}\ .$$

If $f \in L^1$, $\hat{f}^{\lambda} \in L^1$ then the Inversion Formula (see [1], p. 19)

$$(1) \qquad f(t) = \int_{-\infty}^{\infty} \hat{f}^{\lambda}(x)\, e^{i\lambda(x)t}\lambda'(x)\, dx = \int_{-\infty}^{\infty} \hat{f}(x)\, e^{ixt}\, dx$$

holds almost everywhere ; if in addition, f is continuous, then (1) holds everywhere ($\hat{f}(x)$ - is the ordinary Fourier transform).

T h e o r e m 1. Suppose f is a bounded function with period 2π, $0 < \delta < \pi$ and $f(x) = 0$ for $\pi-\delta \le x \le \pi$.

Let g be defined for all x by

$$g(x) = \begin{cases} f(x) & \text{for } |x| \le \pi \\ 0 & \text{for } |x| > \pi . \end{cases}$$

Let $\varphi(t)$ be an even, positive function, non-decreasing for $t > 0$ and such that

$$\varphi(2t) \le C\, \varphi(t) \tag{2}$$

for some constant C, and λ be as above. Then

$$g(x) = \int_{-\infty}^{\infty} \hat{g}^{\lambda}(t) e^{ix\lambda(t)} \lambda'(t)\, dt = \int_{-\infty}^{\infty} \hat{g}(t) e^{ixt}\, dt$$

where

$$\int_{-\infty}^{\infty} | \hat{g}^{\lambda}(t) |\, \varphi(t)\, dt < \infty$$

if

$$f(x) = \sum_{k=-\infty}^{\infty} c_{n_k} e^{in_k x}, \text{ where } \sum_{k=-\infty}^{\infty} |c_{n_k}| \varphi(k) < \infty .$$

P r o o f. Condition (2) implies that $\varphi(t) < C_1(|t|^p+1)$ for some positive constants C_1 and p.

Let h(x) be a function with infinitely many derivatives (see [6]) such that $h(x) \equiv 1$ for $|x| \le \pi - \delta$ and $h(x) = 0$ for $|x| \ge \pi$. Since the function h has a compact support then $h \in S$ where S denotes the space of the quickly decreasing functions, hence the next estimate is true

$$| \hat{h}^{\lambda}(t) | = \frac{1}{2\pi} \left| \int_{-\infty}^{\infty} h(x) e^{-i\lambda(x)t} dt \right| < \frac{C_2}{|t|^{p+2} + 1} \tag{3}$$

where C_2 is some positive constant.

Since $g(x) = f(x)h(x)$, we have

$$\hat{g}^{\lambda}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(x) e^{-i\lambda(t)x}\, dx =$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \Big[\sum_{k=-\infty}^{\infty} c_{n_k} e^{-ix(\lambda(t)-\lambda(k))} \Big] h(x)\, dx = \sum_{k=-\infty}^{\infty} c_{n_k} \hat{h}(\lambda(t)-\lambda(k)).$$

As the assumption of $\sum_{k=-\infty}^{\infty} |c_{n_k}| \varphi(k) < \infty$ leads to a conclusion of absolute and uniform convergence of $\sum_{k=-\infty}^{\infty} c_{n_k} e^{-ix(\lambda(t)-\lambda(k))}$ we could integrate the series term after term. Thus

$$\int_{-\infty}^{\infty} | \hat{g}^{\lambda}(t) | \, \varphi(t)\, dt \le \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} |c_{n_k}| \, |\hat{h}(\lambda(t)-\lambda(k))| \, \varphi(t)\, dt =$$

$$= \sum_{k=-\infty}^{\infty} |c_{n_k}| \int_{-\infty}^{\infty} |\hat{h}(\lambda(t)-\lambda(k))| \, \varphi(t)\, dt.$$

For $k > 0$ we make the following estimates

$$\int_{-\infty}^{\infty} | \hat{h}(\lambda(t)-\lambda(k)) | \, \varphi(t)\, dt = \int_{-\infty}^{0} + \int_{0}^{2k} + \int_{2k}^{\infty} = I_1 + I_2 + I_3 .$$

By (2), (3) and the following estimate for the function λ :

$$|\lambda(t)-\lambda(k)| \ge \frac{1}{M} |t - k| .$$

We have

$$I_1 < C_1 C_2 \int_{-\infty}^{0} \frac{(|t|^p + 1)}{|\lambda(t)-\lambda(k)|^{p+2} + 1}\, dt \le$$

$$\le M^{p+2} C_1 C_2 \int_{-\infty}^{0} \frac{(|t|^p + 1)}{|t - k|^{p+2} + M^{p+2}}\, dt < \infty .$$

We can easily see that I_1 and similarly I_3 are uniformly bounded.

We consider now

$$\int_0^{2k} \left| \hat{h}(\lambda(t)-\lambda(k))\right| dt = \int_{-\lambda(k)}^{\lambda(2k)-\lambda(k)} \left|\hat{h}(t)\right| \left|\lambda^{-1}(t+\lambda(k))'\right| dt \le$$

$$\le M \int_{-\lambda(k)}^{\lambda(2k)} \left|\hat{h}(t)\right| dt < \infty .$$

Then, we have $I_2 = O(\varphi(2k) = O(\varphi(k))$.

For $k < 0$ we have analogous estimates and thus

$$\int_{-\infty}^{\infty} \left| \hat{g}^{\lambda}(t)\right| \varphi(t)\, dt < C \sum_{k=-\infty}^{\infty} \left|c_{n_k}\right| \varphi(k) .$$

The above inequality proves Theorem 1.

T h e o r e m 2. Let us suppose that the assumptions of Theorem 1 are satisfied. We assume, additionally that the function λ is such that $\lambda'(x) \le N$, where N is a positive constant. Then

$$f(x) = \sum_{k=-\infty}^{\infty} c_{n_k} e^{in_k x} , \text{ where } \sum_{k=-\infty}^{\infty} \left|c_{n_k}\right| \varphi(k) < \infty$$

if and only if

$$g(x) = \int_{-\infty}^{\infty} \hat{g}^{\lambda}(t) e^{ix\lambda(t)}\lambda'(t)\, dt = \int_{-\infty}^{\infty} \hat{g}(t)\, e^{ixt}\, dt,$$

where

$$\int_{-\infty}^{\infty} \left| \hat{g}^{\lambda}(t)\right| \varphi(t)\, dt < \infty .$$

P r o o f. By Theorem 1, the above condition is necessary. We are going now to prove its sufficiency. We have

$$c_{n_k} = \hat{f}^{\lambda}(k) = (h\cdot g)^{\lambda}(k) = \widehat{h\cdot g}(\lambda(k)) = (\hat{g} * \hat{h})(\lambda(k)) =$$

$$= \int_{-\infty}^{\infty} \hat{g}(t)\hat{h}(\lambda(k)-t)\,dt = \int_{-\infty}^{\infty} \hat{g}(\lambda(t))\hat{h}(\lambda(k)-\lambda(t))\,\lambda'(t)\,dt\,.$$

We have made use of the theorem $\widehat{h\cdot g} = \hat{g} * \hat{h}$ (see [5], p.74) for ordinary Fourier transform, which is not true for the generalized Fourier transform. Next

$$\sum_{k=-\infty}^{\infty} |c_{n_k}|\,\varphi(k) \le \frac{1}{N}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty} |\hat{g}^{\lambda}(t)|\,|\hat{h}(\lambda(k)-\lambda(t))|\,\varphi(k)\,dt =$$

$$= \frac{1}{N}\int_{-\infty}^{\infty} |\hat{g}^{\lambda}(t)| \sum_{k=-\infty}^{\infty} |\hat{h}(\lambda(k)-\lambda(t))|\,\varphi(k)\,dt\,.$$

Applying the above argumentation to the sum we have

$$\sum_{k=-\infty}^{\infty} |\hat{h}(\lambda(k)-\lambda(t))\,\varphi(k) = \sum_{k=-\infty}^{0} + \sum_{k=0}^{2t} + \sum_{k=2t}^{\infty} = J_1 + J_2 + J_3$$

and similarly as I_1, I_2, I_3 we have

$$J_1 \le \sum_{k=-\infty}^{0} \frac{C_1C_2\,(|k|^p + 1)}{|\lambda(t)-\lambda(k)|^{p+2} + 1} \le M^{p+2}C_1C_2 \sum_{k=-\infty}^{0}\frac{(|k|^p + 1)}{|t-k|^{p+2}+M^{p+2}} \le$$

$$\le M^{p+2}C_1C_2 \sum_{k=-\infty}^{0}\frac{(|k|^p + 1)}{|k|^{p+2} + M^{p+2}} < \infty\,,$$

because $|\lambda(t) - \lambda(k)| \ge \frac{1}{M}\,|t-k|$ for $t > 0$

$$J_3 \le \sum_{k=2t}^{\infty} \frac{M^{p+2}C_1C_2(|k|^p+1)}{|t-k|^{p+2} + M^{p+2}} \le M^{p+2}C_1C_2 \sum_{k=2t}^{\infty} \frac{(|k|^p + 1)}{|k|^{p+2}+M^{p+2}} < \infty,$$

$$J_2 \le M^{p+2} \sum_{k=0}^{2t} \frac{\varphi(2t)\, C_2}{|t-k|^{p+2} + M^{p+2}} \le M^{p+2}\, C_2\, \varphi(2t) \sum_{k=0}^{2t} \frac{1}{k^{p+2} + 1} =$$

$$= O(\varphi(2t)) = O(\varphi(t)).$$

We see that the expressions J_1 and J_2 are uniformly bounded. Hence

$$\sum_{k=-\infty}^{\infty} |\hat{h}(\lambda(k)-\lambda(t))|\ \varphi(k) = O(\varphi(t)) \text{ , and}$$

$$\sum_{k=-\infty}^{\infty} |c_{n_k}|\ \varphi(k) \le C \int_{-\infty}^{\infty} |\hat{g}^{\lambda}(t)\ \varphi(t)\ dt\ .$$

Taking into account the above esimate we complete the proof.

Let us show examples of the function λ fullfilling conditions as in Theorem 1

1) $$\lambda(x) = \begin{cases} a^x - 1 & \text{for } x \ge 0 \ (a > 1) \\ -a^{-x} + 1 & \text{for } x < 0 \end{cases}$$

2) $$\lambda(x) = \sinh x \quad \text{for } x \in (-\infty, \infty)$$

and the example of the function fullfilling conditions as in Theorem 2

$$\lambda(x) = x + \sin \frac{1}{2} x \qquad \text{for } x \in (-\infty, \infty)\ .$$

Finally, we would like to quote two ours theorems (see [3]), which are sufficient conditions for the integral

$$(4)\quad \int_{-\infty}^{\infty}\varphi(x)\,|\hat{f}^{\lambda}(x)|^{\gamma}dx,\ \text{ where } \varphi(x)=\begin{cases}|x|^{\beta} & \text{for } x\in(-\infty,\infty)\cup(0,\infty)\\ 1 & \text{for } x=0\end{cases}$$

to be finite.

Assuming that the function λ satisfies all the assumptions as in Theorem 1, we obtain

T h e o r e m 1'. Let $f \in L^1 \cap L^2$ and for some $\beta \ge 0$, $0 \le \gamma < 2$, there is

$$\sum_{k=1}^{\infty} |\lambda^{-1}(\pm 2^k)|^{\beta+1-\gamma/2}\,[\omega^{(2)}(f;2^{-k})]^{\gamma} < \infty$$

Then the integral (4) is finite.

T h e o r e m 2'. Let f and λ be as in Theorem 1', and such that, for some $\beta \ge 0$, $0 < \gamma < 2$, $1 \le r < 2$ there is

$$\sum_{k=1}^{\infty} 2^{-k\gamma/2}\,|\lambda^{-1}(\pm 2^k)|^{\beta+1-\gamma/2}\,[\omega(f;2^{-k})]^{(2-r)\gamma/2} < \infty$$

and $V_r(f) < \infty$[1]. Then the integral (4) is convergent.

Hence the conditions in Theorems 1' and 2' give suficient conditions (compare Theorem 1) for the Fourier transform with gaps to be absolutely convergent.

([1]) $V_r f$ is the r-th variation of the function f as defined in [5].

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