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DENSITY TOPOLOGY INVOLVING MEASURE AND CATEGORY

In their papers [3], [4] Poreda, Wagner and Wilczyński introduced and examined a categorial equivalent of measure density on real line. Let us recall their definition, restricted to sets $\mathcal{B}$, having the Baire property :

Definition 1. [3] We shall say that 0 is an I-density point of a set $A \in \mathcal{B}$ if and only if $\chi_{(nA)\cap[-1,1]} \xrightarrow[n\to\infty]{I} 1$. By I we denote σ-ideal of sets of the first category ; see [3] for other denotations.

It follows from Theorem 2 in [3] that in the definition mentioned above we can replace the set A with any of the open sets G, which occur if we present A in the form $A = G \Delta P$; where G is open and P is of the first category. Especially we can require G to be the biggest of them, i.e. regular open. This leads us in natural way to the following definitions.

Definition 2. We shall say that a set $A \in \mathcal{B}$ has at point x c-density g if and only if $\lim_{h\to 0} \frac{m(G \cap [x-h,x+h])}{2h}$ exists and is equal to g. The set G is here regular open and such that $A = G \Delta P$; where P is of the first category, m is Lebesgue measure.

It means that a set $A \in \mathcal{B}$ has at point x c-density g iff the regular open set $G = A \Delta P$, where P is of the first category, has the usual measure density equal to g at point x.

D e f i n i t i o n 3. We shall say that x is a c-density point of $A \in \mathcal{B}$ if and only if A has at point x the c-density, equal to 1.

By $A \sim B$ we mean $A \Delta B \in I$ and, to simplify the notation for $A \in \mathcal{B}$, we shall use frequently G(A) for the regular open set in the unique representation $A = (G \setminus P_1) \cup P_2$ or $A = G \Delta P$ where P_1, P_2, $P \in I$.

Now we shall show that c-density as defined above is really a density, i.e. the operation $\varphi_c : \mathcal{B} \longrightarrow 2^R$ defined $\varphi_c(A) = \{x \in R : x \text{ is a c-density point of } A\}$ for $A \in \mathcal{B}$ is so called "lower density" (see [2]). Observe first that

R e m a r k 1. For every $A \in \mathcal{B}$ we have $\varphi_c(A) = \varphi_d(G(A))$ where φ_d is usual measure lower density.

We need also the following

L e m m a 1. For $A, B \in \mathcal{B}$ we have

a) If $A \sim B$, then $G(A) = G(B)$,

b) $G(A \cap B) = G(A) \cap G(B)$ in particular $G(R \setminus A) \cap G(A) = \emptyset$.

P r o o f. We can present the sets A,B and $A \cap B$ in a unique form $A = G_1 \Delta P_1$, $B = G_2 \Delta P_2$, $A \cap B = G_3 \Delta P_3$, where $G_1 = G(A)$, $G_2 = G(B)$, $G_3 = G(A \cap B)$ and P_1, P_2, P_3 are from I. To prove a) let us notice that $A \sim B$ implies $B = A \Delta P$ where P is from I and from $B = A \Delta P = (G_1 \Delta P_1) \Delta P = G_1 \Delta (P_1 \Delta P)$ we have $G_1 = G_2$.

To prove b) we have $A \cap B = (G_1 \Delta P_1) \cap (G_2 \Delta P_2) =$

$[(G_1 \Delta P_1) \cap G_2] \Delta [(G_1 \Delta P_1) \cap P_2] = [(G_1 \cap G_2) \Delta (P_1 \cap G_2)] \Delta$

$\Delta\ [(G_1 \cap P_2)\ \Delta\ (P_1 \cap P_2)] = (G_1 \cap G_2)\ \Delta\ \{\ (P_1 \cap G_2)\ \Delta$ $[(G_1 \cap P_2)\ \Delta\ (P_1 \cap P_2)]\}$. The sets $P_1 \cap P_2$, $G_1 \cap P_2$, $P_1 \cap G_2$ are from I hence $\{\ (P_1 \cap G_2)\ \Delta\ [(G_1 \cap P_2)\ \Delta\ (P_1 \cap P_2)]\}$ is from I, too. As $G_1 \cap G_2$ is a regular open set (se [2]) we have $G_3 = G_1 \cap G_2$.

T h e o r e m 1. For every $A, B \in \mathcal{B}$

1) $\varphi_c(A) \sim A$,

2) if $A \sim B$ then $\varphi_c(A) = \varphi_c(B)$,

3) $\varphi_c(\emptyset) = \emptyset$, $\varphi_c(R) = R$,

4) $\varphi_c(A \cap B) = \varphi_c(A) \cap \varphi_c(B)$.

P r o o f. We shall start with 1). The set $A = (G \setminus P_1) \cup P_2$ where $G = G(A)$ and P_1, P_2 are from I. It is clear that $G \subset \varphi_c(A)$; hence $A \setminus \varphi_c(A)$, as a subset of $A \setminus G \subset P_2$, is from I. On the other hand, $\varphi_c(A) \subset \mathrm{Cl}\ G$, and therefore $\varphi_c(A) \setminus A$ is a subset of $\mathrm{Cl}(G) \setminus A$ and hence of $\mathrm{Cl}(G) \setminus (G \setminus P_1)$. The last set is a union of nowhere dense set $\mathrm{Cl}(G) \setminus G$ and $\mathrm{Cl}(G) \cap P$ of the first category, which implies that $\varphi_c(A) \setminus A$ is of the first category too.

To prove 2) let us notice that from $A \sim B$ we have $G(A) = G(B)$ by a) of Lemma 1 and consequently $\varphi_c(A) = \varphi_c(B)$, directly from Remark 1.

As 3) is simply the consequence of the definition of φ_c, we come to 4). From b) of Lemma 1 we have $G(A \cap B) = G(A) \cap G(B)$, hence $\varphi_c(A \cap B) = \varphi_d[G(A \cap B)] = \varphi_d[G(A) \cap G(B)] = \varphi_d[G(A)] \cap \varphi_d[G(B)] = \varphi_c(A) \cap \varphi_c(B)$ in view of Remark 1 and according to properties of φ_d.

It is not difficult to prove the following

T h e o r e m 2. If x_0 is a c-density point of a set $A \in \mathcal{B}$, then the c-density of a set $R \setminus A$ at x_0 is equal to 0.

P r o o f. Since x_0 is a d-density point of $G(A)$ and from b) of Lemma 1 $G(R \setminus A) \subset R \setminus G(A)$ we have d-density of $G(R \setminus A)$ at x_0 equal to 0 - which was required.

R e m a r k 2. It is worth noticing that the above theorem cannot be converted. Let C be the Cantor set of positive measure in closed interval [0,1] and $\{(a_n, b_n)\}_{n\in N}$ the family of components of $[0,1] \setminus C$. It is not difficult to see that both sets $A = \bigcup_{n=1}^{\infty} \left(a_n, \frac{a_n + b_n}{2}\right)$ and $B = \bigcup_{n=1}^{\infty} \left(\frac{a_n + b_n}{2}, b_n\right)$ are regular open and $A \cap B = \emptyset$. Let x_0 be a d-density point of C. It can be easily seen that $G(R \setminus A) = B$, hence $C \cap [G(A) \cup G(R \setminus A)] = \emptyset$; so c-density of A, as well as of $R \setminus A$ at point x_0, is equal to 0.

One can ask now if we obtained something really different from well known concept of d-density (measure density) and I-density (category density). The following examples will give positive answer to this question.

E x a m p l e 1. There exists a set $A_1 \in \mathcal{B}$ such that 0 is a d-density point of A_1 but not a c-density point of A_1 . As a set A_1 we can take the set A from Theorem 1.6. in [2]. A is there a set of the first category such that $m(R\setminus A) = 0$.

E x a m p l e 2. There exists a set $A_2 \in \mathcal{B}$ such that 0 is a c-density point of A_2 but not a d-density point of A_2 . As a set A_2 we can take the set $R \setminus A_1$ where A_1 is from Example 1.

E x a m p l e 3. There exists a set $A_3 \in \mathcal{B}$ such that 0 is a c-density point but not an I-density point of A_3. The sequence $\frac{1}{n!}$, $n \in N$ converges to 0 from the right side.
Let ε_n be a given monotone sequence of positive reals, convergent to 0. From each interval $\left[\frac{1}{(n+1)!}, \frac{1}{n!}\right]$ let us remove the open set B_n of the form

$$B_n = \bigcup_{i=1}^{n} \left(\frac{i}{n!(n+1)} ; \frac{i+\delta_n}{n!(n+1)} \right),$$ where δ_n is a positive number small enough to ensure $m(B_n) < \varepsilon_n \left[\frac{1}{n!} - \frac{1}{(n+1)!}\right]$. As a set A_3 we can take $R \setminus Cl\left(\bigcup_{n=1}^{\infty} B_n\right)$. Really, it is not difficult to notice that A_3 is regular open and for each $h \in \left[\frac{1}{(n+1)!}, \frac{1}{n!}\right]$ we have

$$\frac{m\left(\bigcup_{n=1}^{\infty} B_n \cap [0,h]\right)}{h} \le \frac{\frac{1}{(n+1)!}\varepsilon_n + \delta_n}{\frac{1}{(n+1)!} + \delta_n} = \frac{\varepsilon_n + \delta_n(n+1)!}{1 + \delta_n(n+1)!}.$$

In view of $\delta_n(n+1)! < \varepsilon_n$ we have also

$$0 \le \frac{\varepsilon_n + \delta_n(n+1)!}{1 + \delta_n(n+1)!} < \frac{\varepsilon_n + \delta_n(n+1)!}{1} < 2\varepsilon_n$$

which implies that 0 is d-dispersion point of $\bigcup_{n=1}^{\infty} B_n$.
As $m\left[Cl\left(\bigcup_{n=1}^{\infty} B_n\right)\right] = m\left(\bigcup_{n=1}^{\infty} B_n\right)$, 0 is a d-density point of A_3; hence 0 is a c-density point of A_3.

Now we shall show that 0 is not an I-density point of

A_3. It is enough to point out a subsequence m_n of natural numbers such that $\chi_{(m_n A_3)\cap[-1,1]}$ does not converge I-almost every- where to 1. A subsequence $m_n = n!$ is good here. In fact, we have $n!(R \setminus A_3) \supset n! \cdot B_n = \bigcup_{i=1}^{n} \left[\frac{i}{n+1} , \frac{i+\delta_n}{n+1} \right]$.

Let $D_n = \bigcup_{i=1}^{n} \left[\frac{i}{n+1} , \frac{i+\delta_n}{n+1} \right]$. We have

$\limsup_n n!(R \setminus A_3) \supset \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} D_n$. It is not difficult to observe that the set $\bigcup_{m=n}^{\infty} D_n$ is open and dense in [0,1] for every m and we have $\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} D_n$ to be a residual on [0,1] and G_δ set. That implies $\liminf_n (n! \cdot A_3)$ to be a subset of the first category set and hence 0 is not an I-density point of A_3.

E x a m p l e 4. There exists a set $A_4 \in \mathcal{B}$ such that 0 is an I-density point but not a c-density point of A_4. As a set A_4 we can take the set $R \setminus E_4$ where E_4 is the set from Theorem 1 d) in [6].

Let us now list some properties of operation φ_c .

T h e o r e m 3. For any $A \in \mathcal{B}$ we have

a) Int $(\varphi_c(A)) = \emptyset$ if and only if A is of the first category,

b) $\varphi_c(A) \setminus G(A)$ is a nowhere dense set of measure 0 ,

c) $G(A) = \text{Int}\, \varphi_c(A) = \text{Int}\, \text{Cl}(G(A))$.

P r o o f. a) If $A \in I$ we have Int $\varphi_c(A) = \emptyset$ directly from Theorem 1.2). If $A \notin I$ then $G(A) \neq \emptyset$ and in view of

$G(A) \subset \varphi_c(A)$ we have $\mathrm{Int}\ \varphi_c(A) \supset G(A) \neq \emptyset$.

b) As $\varphi_c(A) \subset \mathrm{Cl}\ G(A)$ we have $\varphi_c(A) \setminus G(A) \subset \mathrm{Cl}\ G(A) \setminus G(A)$ and the last set is nowhere dense. The fact that $\varphi_c(A) \setminus G(A)$ is a set of measure zero comes directly from the Lebesgue density theorem.

c) Is obvious in view of $G(A) = \mathrm{Int}\ \mathrm{Cl}(G(A))$ and $G(A) \subset \subset \varphi_c(A) \subset \mathrm{Cl}\ G(A)$.

We shall introduce now the topology related to the concept of c-density. Put $\mathcal{T}_c = \{ \varphi_c(A) \setminus P : A \in \mathcal{B},\ P \in I \}$.

T h e o r e m 4. $\mathcal{T}_c$ is the topology on the real line.

P r o o f. Dealing with the same σ-ideal I and in view of the fact that operation φ_c is the lower density we can carry out the proof exactly the same way as proof of Theorem 3 in [3].

Let us now list some properties of $\mathcal{T}_c$-topology.

T h e o r e m 5. The following conditions on a subset A of space $(R, \mathcal{T}_c)$ are equivalent :

a) A is of the first category ,

b) A is $\mathcal{T}_c$- nowhere dense ,

c) A is $\mathcal{T}_c$- of the first category ,

d) A is $\mathcal{T}_c$- closed discrete .

P r o o f. a) $\Leftrightarrow$ b) can be proved in an analogous way to the proof of Theorem 22.6 in [2].

b) $\Leftrightarrow$ c) comes directly from a) $\Leftrightarrow$ b).

a) $\Rightarrow$ d) any set of the first category P is $\mathcal{T}_c$- closed since we have $P = R \setminus (R \setminus P)$ where $R \setminus P$ is $\mathcal{T}_c$- open.
Any subset of P being of the first category is $\mathcal{T}_c$- closed, too.

d) $\Rightarrow$ a) if closed discrete set were not a first category set it would include a set not having the Baire property hence not $\mathcal{T}_c$- closed.

C o r o l l a r y 1. $(R,\mathcal{T}_c)$ is neither separable nor locally separable at any point.

P r o o f. It is clear since we have from Theorem 5 that countable sets are closed and no point of $(R,\mathcal{T}_c)$ has countable neighbourhood.

C o r o l l a r y 2. Any $\mathcal{T}_c$- compact subset of $(R,\mathcal{T}_c)$ is finite.

P r o o f. Let K be any $\mathcal{T}_c$- compact subset of R. Suppose $A = \bigcup_{\alpha<\omega_0} \{x_\alpha\}$ is the set of all points of the first category set which has the cardinality $\aleph_0$ and is a subset of K. We have $(R \setminus A) \cup \{x_\alpha\}$, $\alpha < \omega_0$ to be $\mathcal{T}_c$- open cover of K which has no finite subcover - that contradicts that K is $\mathcal{T}_c$- compact.

C o r o l l a r y 3. In $(R, \mathcal{T}_c)$ each sequence of different points contains no convergent subsequence.

T h e o r e m 6. $(R,\mathcal{T}_c)$ is not regular.

P r o o f. Let Q be any dense in R and $\mathcal{T}_c$- closed set and a point x_0 from $R \setminus Q$. Let Q be a subset of $\varphi_c(A) \setminus P$, where $A \in \mathcal{B}$ and $P \in I$. As $Q \subset \varphi_c(A)$ we have $\varphi_c(A)$ dense in R and therefore G(A) dense in R, in view of Theorem 3 b). The set G(A) being regular open and dense in R must be equal to R - hence $\varphi_c(A) = R$. The set $R \setminus (\varphi_c(A) \setminus P) = P$ contains no nonempty set from $\mathcal{T}_c$, so x_0 and Q cannot be separated by $\mathcal{T}_c$- open sets.

T h e o r e m 7. The family of $\mathcal{T}_c$- Borel sets coincide with the family of sets having the Baire property.

P r o o f. Let A be any $\mathcal{T}_c$- open set then it has the Baire property, in view of Theorem 1 1). Hence every $\mathcal{T}_c$-Borel set has the Baire property. Conversely, if a set A has the Baire property we can present it in a form $A = (G \setminus P_1) \cup P_2$, where $G = G(A)$ and $P_1, P_2 \in I$. We have $G = \varphi_c(A) \setminus (\varphi_c(A) \setminus G)$ and $\varphi_c(A) \setminus G$ is of the first category, according to Theorem 3 b). Hence $G \setminus P$ is $\mathcal{T}_c$- open set. As P_2 is $\mathcal{T}_c$- closed we have A to be $\mathcal{T}_c$- Borel set.

C o r o l l a r y 4. Every $\mathcal{T}_c$- Borel set is the union of $\mathcal{T}_c$- open set and $\mathcal{T}_c$- closed set.

T h e o r e m 8.

1) The set $A \in \mathcal{B}$ is $\mathcal{T}_c$- regular open if and only if $A = \varphi_c(D)$ for some $D \in \mathcal{B}$.

2) Every $\mathcal{T}_c$- regularopen set is $F_{\sigma\delta}$.

P r o o f. 1) analogous to the proof of Theorem 22.8 in [2].

2) follows from the fact that every $\mathcal{T}_d$- regular open set is $F_{\sigma\delta}$ (see [5]).

T h e o r e m 9. $(R,\mathcal{T}_c)$ satisfies countable chain condition.

P r o o f. It follows from the fact that every $\mathcal{T}_c$- open set has the Baire property and the family of Baire sets satisfies countable chain condition.

T h e o r e m 10. $(R,\mathcal{T}_c)$ is a Baire space.

P r o o f. Comes directly from Theorem 5.

T h e o r e m 11. $(R,\mathcal{T}_c)$ is connected.

P r o o f. Suppose R is a union of two nonempty disjoint sets A and B, both $\mathcal{T}_c$- open. We have $A = \varphi_d(G) \setminus P$ and $B = \varphi_d(Q) \setminus S$, where $G = G(A)$, $Q = G(B)$ and P, S are of the first category. We can assume here $P \subset \varphi_d(G)$ and $S \subset$ $\subset \varphi_d(Q)$ and as $A \cup B = R$ and $A \cap B = \emptyset$ we have also $P \subset$ $\varphi_d(Q)$ and $S \subset \varphi_d(G)$, hence $P \cup S \subset \varphi_d(G) \cap \varphi_d(Q)$. The sets G and Q are regular open - thus $\varphi_d(G) = \varphi_c(G)$ and $\varphi_d(Q) =$ $= \varphi_c(Q)$.

Suppose that $P \cup S$ is nonempty. This implies $\varphi_c(G) \cap \varphi_c(Q) \neq \emptyset$, hence from Theorem 1 4) and 1) $G \cap Q$ is nonempty, hence of the second category. As $G \cap Q \subset \varphi_c(G) \cap \varphi_c(Q)$ we have $\varphi_c(G) \cap$ $\cap \varphi_c(Q)$, as well as $(\varphi_c(G) \setminus P) \cap (\varphi_c(Q) \setminus S)$ to be of the second category. The last is in contradiction with earlier supposition, that $A \cap B = \emptyset$, so we have $P \cup S$ to be empty. Thus $R = \varphi_c(G) \cup \varphi_c(Q)$ and $\varphi_c(G) \cap \varphi_c(Q) = \emptyset$. Which in view of $\varphi_c(G), \varphi_c(Q) \in \mathcal{T}_d$ is in contradiction with the fact that $(R, \mathcal{T}_d)$ is connected space.

T h e o r e m 12. $(R, \mathcal{T}_c)$ is not locally connected at any point.

P r o o f. The set $[0,1] \setminus Q$, where Q is the set of rationals is the neighbourhood of the point $x = \frac{\sqrt{2}}{2}$ in the $\mathcal{T}_c$- topology, and no subset of it is $\mathcal{T}_c$- open and $\mathcal{T}_c$- connected. Indeed, suppose that there exists $B \subset [0,1] \setminus Q$, $\mathcal{T}_c$- open and $\mathcal{T}_c$- connected and containing x. Let α be any element of the set $(\inf B, \sup B) \cap Q$. We have the sets $(0,\alpha) \cap B$ and $(\alpha, 1) \cap B$ to be $\mathcal{T}_c$- open and as B is a subset of $(0,\alpha) \cup$ $\cup (\alpha,1)$ it cannot be $\mathcal{T}_c$- connected. As σ-algebras and σ-ideals, we are dealing with, are invariant in respect to linear transformation the property holds at any point from R.

T h e o r e m 13. Every set of measure zero is contained in $\mathcal{T}_c$- open set of measure zero. Every set of the first category is contained in $\mathcal{T}_d$- open set of the first category.

P r o o f. We can represent R as the union of two disjoint sets A_1 and A_2 , where A_1 is of the first category and A_2 is of measure zero. If A is a set of measure zero then $A_2 \cup A$ is the required $\mathcal{T}_c$- open set of measure zero. If A is a set of the first category, then $A_1 \cup A$ is the required $\mathcal{T}_d$- open set of the first category.

Let us consider now some properties of continuous functions from $(R, \mathcal{T}_c)$ into R equipped with the natural toplogy.

D e f i n i t i o n 4. We shall say that a function $f : R \longrightarrow R$ is c-approximately continuous at x_0 if and only if for every $\varepsilon > 0$ the set $f^{-1}((f(x_0)-\varepsilon,\ f(x_0)+\varepsilon))$ has x_0 as a c-density point.

D e f i n i t i o n 5. We shall say that a function $f : R \longrightarrow R$ is c-approximately continuous if and only if for every interval $(y_1,\ y_2)$ the set $f^{-1}((y_1, y_2))$ belongs to $\mathcal{T}_c$.

From the above definitions we obtain immediately the following theorem.

T h e o r e m 14. A function $f : R \longrightarrow R$ is c-approximately continuous if and only if it is c-approximately continuous at every point.

We have also

T h e o r e m 15. A function $f : R \longrightarrow R$ has a Baire property if and only if it is c-approximately continuous I - a.e.

P r o o f. We shall start with the following lemma.

L e m m a 2. If the function $f : R \longrightarrow R$ is c-approximately continuous at point x_0 , then for every $\varepsilon > 0$ the set $f^{-1}((f(x_0)-\varepsilon, f(x_0)+\varepsilon))$ includes a neighbourhood of x_0 in the $\mathcal{T}_c$- topology, which has the Baire property.

P r o o f of the lemma. Let $f : R \longrightarrow R$ be c-approximately continuous at point x_0 and ε be an arbitrary positive real number. According to Def. 4. the set $D = f^{-1}((f(x_0)-\varepsilon, f(x_0)+\varepsilon))$ has x_0 as a c-density point i.e. $x_0 \in \varphi_c(D)$. Obviously, $x_0 \in D \cap \varphi_c(D)$ which is $\mathcal{T}_c$- open since $D \cap \varphi_c(D) = \varphi_c(D) \setminus (\varphi_c(D) \setminus D)$, where $\varphi_c(D) \setminus D$, is from I. From Theorem 7 we have $D \cap \varphi_c(D)$ to have the Baire property. This ends the proof of the lemma.

The theorem can be now be proved in a way similar to the proof of Theorem 7 in [3].

T h e o r e m 16. If a function $f : R \longrightarrow R$ is c-approximately continuous, then it is continuous in a.e.- topology.

P r o o f. Suppose $f : R \longrightarrow R$ is c-approximately continuous function.

Let $(a,b) \subset R$. We have $f^{-1}((a,b)) = (G \setminus H) \cup P$ where G is regular open, $P \subset \varphi_c(G) = \varphi_d(G)$ and $H, P \in I$ and we can assume $H \subset G$ and $P \cap G = \emptyset$. From Theorem 3.b it is clear that P is nowhere dense set of measure zero. Let $x \in H$. We shall show that $f(x) \in [a,b]$. Really, as f is c-approximately continuous we have from Theorem 2 that the set $f^{-1}((f(x)-\varepsilon,\ f(x)+\varepsilon))$ has common points with $f^{-1}((a,b))$ for every ε since the last set is residual in G. Hence for every ε the set (a,b) intersects with $(f(x)-\varepsilon,\ f(x)+\varepsilon)$ and therefore $f(x) \in [a,b]$. This implies that G is a subset of $f^{-1}([a,b])$.

We can repeat the above considerations for every set of the form $(a + \frac{1}{n},\ b - \frac{1}{n})$, where $n \in N$ and we have $f^{-1}((a + \frac{1}{n},\ b - \frac{1}{n})) = (G_n \setminus H_n) \cup P_n$, where G_n is regular open, $P_n \subset \varphi_c(G_n) = \varphi_d(G_n)$, H_n and P_n are from I, $H_n \subset G_n$, $P_n \cap G_n = \emptyset$, P_n is nowhere dense set of measure zero and $G_n \subset f^{-1}([a + \frac{1}{n},\ b - \frac{1}{n}])$. Hence we have $f^{-1}((a,b)) = \bigcup_{n=1}^{\infty} f^{-1}((a + \frac{1}{n},\ b - \frac{1}{n})) \subset$ $\subset \bigcup_{n=1}^{\infty} G_n \cup P_n \subset \bigcup_{n=1}^{\infty} f^{-1}([a + \frac{1}{n},\ b - \frac{1}{n}]) = f^{-1}((a,b))$, thus $\bigcup_{n=1}^{\infty} G_n \cup P_n = f^{-1}((a,b))$.

We have $\bigcup_{n=1}^{\infty} G_n \cup P_n = \bigcup_{n=1}^{\infty} G_n \cup \bigcup_{n=1}^{\infty} P_n$ where $\bigcup_{n=1}^{\infty} G_n$ is open and $\bigcup_{n=1}^{\infty} P_n$ is of the first category and of measure zero. Since for every $n \in N$ $P_n \subset \varphi_d(G_n) \subset \varphi_d(\bigcup_{n=1}^{\infty} G_n)$ then $\bigcup_{n=1}^{\infty} P_n \subset$ $\subset \varphi_d(\bigcup_{n=1}^{\infty} G_n)$ and we have $\bigcup_{n=1}^{\infty} G_n \cup P_n = f^{-1}((a,b))$ to be an a.e.-

open set.

C o r o l l a r y 5. The a.e.- topology is the coarsests topology, for which every c-approximately continuous function is continuous.

C o r o l l a r y 6. Every c-approximately continuous function is d-approximately continuous.

C o r o l l a r y 7. Every c-approximately continuous function is of the first class of Baire and has the Darboux property.

T h e o r e m 17. There is a function $f : R \longrightarrow R$ which is d-approximately continuous d-almost everywhere ; but the set of c-approximate discontinuity of f is of positive measure.

P r o o f. Let C be a Cantor set of positive measure in $[0,1]$. We put $f(x) = 0$ for $x \in R \setminus ([0,1] \setminus C)$ and to be a "hat" function of altitude 1 on every component (a_n, b_n) of $[0,1] \setminus C$, i.e. $f(a_n) = f(b_n) = 0$, $f(\frac{1}{2}(a_n + b_n)) = 1$ and f is linear on $[a_n, \frac{1}{2}(a_n + b_n)]$ and on $[\frac{1}{2}(a_n + b_n), b_n]$. Then, obviously, f is d-approximately continuous at any point of the set $(C \cap \varphi_d(C)) \cup (R \setminus C)$. Now let $x_0 \in C \cap \varphi_d(C)$. We shall show that f is not c-approximately continuous at x_0. Indeed, the set $\{x : f(x) < \frac{1}{2}\}$ has not c-density 1 at x_0. It is not difficult to see, as $G(\{x : f(x) < \frac{1}{2}\})$, equal to $(-\infty,0) \cup \bigcup_{n=1}^{\infty} (a_n, a_n + \frac{b_n - a_n}{4}) \cup \bigcup_{n=1}^{\infty} (b_n - \frac{b_n - a_n}{4}, b_n) \cup (1,\infty)$, being disjoint with C, has not d-density 1 at x_0 .

T h e o r e m 18. In $(R,\mathcal{T}_c)$ space a set is connected iff it has the form (a,b), $(a,b]$, $[a,b)$, $[a,b]$ $(a \le b$, $a,b \in R)$, or $(-\infty,\infty)$, $(-\infty,a)$, $(-\infty,a]$, (a,∞), $[a,\infty)$ $(a \in R)$.

P r o o f. As Euclidean continuous functions are c-approximately continuous and $\mathcal{T}_c$- connected sets are Euclidean connected we can follow proof of (10.1.16) from [1].

Author was recently informed that $\mathcal{T}_c$- topology can be also defined as a *-modification of a.e.- topology (see [7] for definition).

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