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ON A FORMULA CONCERNING STIELTJES STOCHASTIC DIFFERENTIALS

1. Introduction

An analogue of Itô formula is shown for Stieltjes stochastic integral $\int_a^b f(P_{t-})dP_t$, where f is a real analyticfunction and (P_t), $t \geq 0$, is a Poisson stochastic process.

Let $P = (P_t)$, $t \geq 0$, be a standard Poisson process with a fixed parameter $a > 0$. Its paths $t \longmapsto P_t(\omega)$, are a. e. non-decreasing and right-continuous. Hence, for sufficiently large class of stochastic processes $\{(f_t) : t \geq 0\}$, which are left-continuous, we can put (cf. [3], [4], [5])

$$\left[\int_0^t f_s dP_s\right](\omega) = \int_0^t f_s(\omega)dP_s(\omega) \quad , \quad \text{a.e., for all } t \geq 0 ,$$

where the integral on the right-hand side is the Lebesgue-Stieltjes integral. The integral on the left-hand side is called the Stieltjes stochastic integral (with respect to the Poisson process P, [3], [4], [5]). The Poisson process is a representant of an important class of processes, so called point processes. The theory of Stieltjes stochastic integrals with respect to point processes, or generally, with respect to the discontinuous martingales was developed in [3], [4] and [5].

The Stieltjes stochastic integrals play in this theory as important role as the Itô stochastic integrals ([1]), in the theory of diffusion processes.

This paper deals with the special topic in the field of point processes : differential and integral calculus based on the Poisson process. We show, roughly speaking, it is the same as the customary calculus of smooth functions, except that taking the differential of an analytic function f of the Poisson process path $t \longmapsto P_{t-}$, it is necessary to keep all terms in the Taylor expansion :

$$(1.1) \qquad df(P_t) = \sum_{n=1}^{+\infty} (1/n!)\, f^{(n)}(P_{t-})dP_t \ ,$$

or, which is the same , $\sum_{n=1}^{+\infty} (1/n!) \int_0^t f^{(n)}(P_{s-})dP_s = \int_0^t df(P_s)$.

The formula (1.1) shows how to differentiate in the Stieltjes stochastic sense and it is analogue of the famous Itô formula ([2]), for Itô integrals. Using formula (1.1) it is easy to calculate a lot of fundamental Stieltjes stochastic integrals, by the methods of the elementary differential calculus. We would like to underline the fact of economy and speed of computations when we use the formula (1.1), cf. Section 3, Example 2.

Although in this paper we consider only Stieltjes stochastic integrals with respect to Poisson processes, however, the formula (1.1) is true for Stieltjes stochastic integrals with respect to arbitrary point processes.

2. The change of variable formula for Stieltjes stochastic integrals and Stieltjes stochastic differentials

Let $P = (P_t)$, $t \geq 0$, be a standard Poisson process with some parameter $a > 0$, defined on some probability space (Ω, Prob). We define a new process $P_- = (P_{t-})$, $t \geq 0$, by the formula : $P_{t-}(\omega) = \lim_{s \to \infty} P_s(\omega)$, for a. e. $\omega \in \Omega$.

The process P_- has the non-decreasing, left-continuous paths (whereas P has the right-continuous ones), with jumps equal to one. By the definition of the Stieltjes stochastic integral we have (cf. [4], [5])

$$(2.1) \quad \int_0^t P_{s-} dP_s = \sum_{s \leq t} P_{s-} \cdot [P_s - P_{s-}] = P_t^2/2 - P_t/2 \ , \quad t \geq 0.$$

The formula (2.1) is essentially different from the well- known formulas

$$(\text{R}) \quad \int_0^t s ds = t^2/2 \qquad \text{(the Riemann integral)}$$

$$(\text{I}) \quad \int_0^t w_s ds = w_t^2/2 - t/2 \quad \text{(the Itô integral)},$$

where $w = (w_t)$, $t \geq 0$, is the standard Brownian motion.

The formulas (R) and (I) are obviously consequences of the following "change of variable" formulas ; the Newton-Leibnitz formula for the Riemann integral

$$(2.2.a) \quad \int_a^b f'(s) ds = f(b) - f(a) \ , \quad f \in C^1([a,b]) \ ,$$

and the Itô formula ([2]) for Itô integral

$$\int_a^b f'(w_s)dw_s + \frac{1}{2}\int_a^b f''(w_s)dw_s = f(w_b) - f(w_a), \qquad (2.3.a)$$

$$f \in C^2([a,b]).$$

In the sequel we shall formulate and prove the change of variable formula for Stieltjes stochastic integrals, which implies (2.1).

Thus, let $A(\mathbb{R}_+)$ denote the set of all real analytic functions on $\mathbb{R}_+ = [0,+\infty)$. Obviously, a function f belongs to $A(\mathbb{R}_+)$ if and only if

(A1) f has all derivatives outside zero and right-derivatives at zero , and

(A2) at each point $a \in \mathbb{R}_+$, f has a full power series (Taylor) expansion at a, which absolutely converges to f on $\mathbb{R}_+$.

It is obvious that $\int_0^t f(P_{s-})dP_s$ $(t \geq 0)$ has sense for arbitrary $f \in A(\mathbb{R}_+)$.

T h e o r e m 1. (The change of variable formula for Stieltjes stochastic integral). Let f be a function from $A(\mathbb{R}_+)$. Then

$$f(P_t) - f(P_0) = \sum_{n=1}^{+\infty} (1/n!) \int_0^t f^{(n)}(P_{s-})dP_s , \qquad (2.4)$$

for all $t \geq 0$.

P r o o f. Since $f \in A(\mathbb{R}_+)$, then for all $x,a \in \mathbb{R}_+$ (cf. (A1),(A2)) we have

$$f(x) - f(a) = \sum_{n=1}^{+\infty} (1/n!)\, f^{(n)}(a)\, (x-a)^n . \qquad (2.5)$$

It is known that for a. e. $\omega \in \Omega$ the path $s \longmapsto P_s(\omega)$, $s \leq t$, has the form given at Fig. 1. By (2.5) and Fig. 1 we have

$$f(P_t) - f(P_0) = f(P_r) - f(P_0) = \sum_{k=1}^{r} (f(P_{t_k}) - f(P_{t_{k-1}})) =$$

$$= \sum_{k=1}^{r} (f(P_{t_k}) - f(P_{t_k-})) = \sum_{k=1}^{r} \left[\sum_{n=1}^{+\infty} (1/n!) f^{(n)}(P_{t_k-})(P_{t_k} - P_{t_k-})^n \right].$$

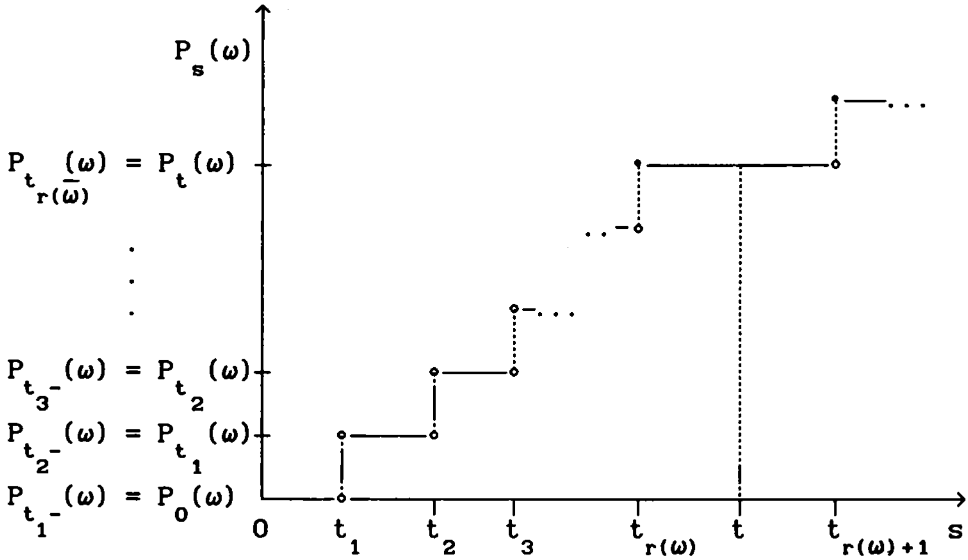

Fig. 1

But, $P_{t_k} - P_{t_k-} = 1 = (P_{t_k} - P_{t_k-})^n$, for all $n \geq 1$, and $k = 1, 2, \ldots, r(\omega)$. Thus, we obtain

$$f(P_t) - f(P_0) = \sum_{k=1}^{r} \left[\sum_{n=1}^{+\infty} (1/n!) f^{(n)}(P_{t_k-})(P_{t_k} - P_{t_k-}) \right] =$$

$$= \sum_{n=1}^{+\infty} \left[\sum_{k=1}^{r} (1/n!) f^{(n)}(P_{t_k-})(P_{t_k} - P_{t_k-}) \right] = \sum_{n=}^{+\infty} (1/n!) \int_0^t f^{(n)}(P_{s-}) dP_s,$$

which completes the proof.

It is well known that using the *Newton Leibnitz differential* d_N and Ito stochastic differential d_I ([2]), the formulas (2.2.a) and (2.3.a) may be written in the following forms :

(2.2.b) $\quad d_N(f)(t) = f'(t)\, dt$, $\quad f \in C^1([a,b])$

(2.3.b) $d_I(f(w_t)) = f'(w_t)dw_t + (1/2)f''(w_t)dt$, $f \in C^2([a,b])$.

Similarly, by Theorem 1, we can introduce the *Stieltjes* stochastic differential d_S by the formula

(2.6) $$d_S(f(P_t)) = \sum_{n=1}^{+\infty} (1/n!)\, f^{(n)}(P_{t-})\, dP_t \ , \quad f \in A(\mathbb{R}_+) \ .$$

R e m a r k 1. Heuristically, the formulas (2.2.b), (2.3.b) and (2.6) may be expressed as follows : $(dt)^n = 0$, for all $n \geq 2$; $(dw_t)^2 = dt$, $(dw_t)^n = 0$, for all $n \geq 3$; and $(dP_t)^n = dP_t$, for all $n \geq 1$.

3. Some simple application of the Stieltjes differential formula

We will illustrate an application of the formula (2.6) by four examples.

E x a m p l e 1. Applying the formula (2.6) to the function $f(x) = x^2/2$ we obtain

$$d_S(P_t^2/2) = P_{t-}dP_t + (1/2)\, dP_t \ , \text{ or equivalently}$$

$$\int_0^t P_{s-}dP_s = P_t^2/2 - P_t/2 \ , \ (\text{cf. } (2.1)).$$

E x a m p l e 2. Let $f(x) = x^3/3$. Then by (2.6)

$$d_S(P_t^3/3) = P_{t-}^2 dP_t + P_{t-}dP_t + (1/3)\, dP_t \ .$$

By the above formula and Example 1 we get

$$\int_0^t P_{s-}^2 dP_s = P_t(1 - 3P_t - 2P_t^2)/6 .$$

Observe that using (2.6) we can easily find the sums of the form $\sum_{k=1}^{l} k^n$, $l,n \geq 1$; if n is not too large. Moreover, it seems that the calculation of $I_n = \int_0^t P_{s-}^n dP_s$ by the formula (2.6) is simpler than the calculation of I_n by the above sums.

E x a m p l e 3. Applying the formula (2.6) to the functions $f(x) = \sin x$ and $f(x) = \cos x$, we have

$$\begin{cases} d_S(\sin P_t) = \cos P_{t-} dP_t \cdot \sin 1 + \sin P_{t-} dP_t \cdot (\cos 1 - 1) \\ d_S(\cos P_t) = \cos P_{t-} dP_t \cdot (\cos 1 - 1) - \sin P_{t-} dP_t \cdot \sin 1 . \end{cases}$$

Hence

$$\int_0^t \sin P_{s-} dP_s = (\cos P_t - 1)(\sin(1)/2(\cos 1 - 1)) - (\sin P_t)/2,$$

and

$$\int_0^t \cos P_{s-} dP_s = -(\sin P_t)(\sin 1)/2\ (\cos 1 - 1)) - (\cos P_t - 1)/2.$$

E x a m p l e 4. We have

$$d_S(\exp(P_t)) = \sum_{n=1}^{+\infty} (1/n!)\ \exp(P_{t-}) dP_t = (e-1)\ e^{P_{t-}}\ dP_t .$$

Hence, $\int_0^t \exp(P_{s-}) dP_s = (\exp(P_t) - 1)/(e-1).$

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