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OPTIMIZATION OF THE SYSTEM DESCRIBED BY FREDHOLM'S INTEGRAL EQUATION

1. Introduction

The paper deals with the optimal control problem of the system whose state is described by Fredholm's integral equation. The functions appearing in this equation and in the performance index are of class L_2 .

In order to find the necessary conditions of optimality for the above system, the results obtained by Walczak (Ref. 1) concerning the properties of cones in normed spaces will be applied. The method presented there can be considered as a generalization of Dubovitski-Milyutin theorem (Ref. 2) and enables us to investigate the extremal problem with more than one (in this paper-two) equality constraints.

For the above optimization problem the problem of existence of the solution can be solved in the case where additional assumptions concerning the linearity of the function appearing in the state equation will be accepted.

2. Formulation of the problem

Let us consider in the space $X = L_2^n \times L_2^m$ the following optimization problem :

Find the minimum of the functional

$$I(x,u) = \int_0^1 g((x(t),u(t),t)\,dt \tag{1}$$

for the system described by the state equation

$$x(t) = \int_0^1 f(x(\tau),u(t),\tau,t)\,d\tau \tag{2}$$

with imposed constraint

$$u(t) \in U \tag{3}$$

where $t \in R$, $\tau \in R$ denote the time, x,u - are the state - and control vectors of dimensions $n \times 1$ and $m \times 1$ respectively. The functions $g : R^{n+m+1+1} \longrightarrow R$, $f: R^{n+m+1} \longrightarrow R^n$ and the set U are given. The scalar function $g(x(t),u(t),t)$ is of the form :

$$g(x(t),u(t),t) = v^T(t)\, H_0(t) v(t) + p_0(t) v(t) \tag{4}$$

where $H_0(t)$ is the $(n+m) \times (n+m)$ matrix, and $v(t) = [x(t),u(t)]^T$. We assume that the elements of $H_0(t)$ are measurable and bounded, and the elements of $p_0(t)$ belong to the space L_2 .

The vector function $f(x(t),u(\tau),\tau,t)$ is of the form :

$$f(x(t),u(\tau),\tau,t) = [f_1(x,u,\tau,t),\ f_2(x,u,\tau,t), \dots, f_n(x,u,\tau,t)]^T$$

and its components are :

$$f_i(x,u,\tau,t) = m^T(\tau,t)\, H_i(\tau,t)\, m(\tau,t) + p_i(\tau,t)\, m(\tau,t) + q_i(\tau,t), \quad i = 1,2,\dots,n$$

where $H_i(\tau,t)$, $i = 1,2,\dots,n$ are $(n+m) \times (n+m)$ matrices, $p_i(\tau,t)$, $q_i(\tau,t)$, $i = 1,2,\dots,n$ are $1 \times (n+m)$ matrices,

$m(\tau,t) = [x(\tau),u(t)]^T$, $t,\tau \in [0,1]$.

We assume that the elements of the matrices $H_i(\tau,t)$, $i = 1,2,\dots,n$ are measurable and bounded and that the elements of the matrices $p_i(\tau,t)$, $q_i(\tau,t)$, $i = 1,2,\dots,n$ belong to L_2.

The set U is a closed convex set in R^m. The vectors $u(\cdot)$ and $x(\cdot)$ belong to L_2^m and L_2^n respectively.

3. The necessary condition of optimality

For the considered system we shall prove the following :

T h e o r e m 3.1. If $v_0(t) = [x_0(t),u_0(t)]^T$ is the solution of the above formulated optimization problem and if the following additional assumption holds

$$(5)\qquad \int_0^1\int_0^1 |f_x(x_0(\tau),u_0(t),\tau,t)|^2\, dt\, d\tau < 1$$

then there exists a real number $\lambda_0 \geq 0$ and the vector function $\psi(\cdot) \in L_2^n$ such that $\lambda_0 + \|\psi(\cdot)\| > 0$ and

$$(6)\qquad \psi(t) = -\lambda_0\, g_x(t) + \int_0^1 f_x^T(\mu(\tau,t),\tau,t)\, \psi(\tau)\, d\tau$$

and

$$(7)\qquad \left\langle -\int_0^1 f_u^T(\mu(t,\tau),t,\tau)\psi(\tau)dt + \lambda_0 g_u(v_0(\tau),\tau), u-u_0(\tau) \right\rangle \geq 0$$

for $\tau \in [0,1]$ a.e. where

$$v_0(\tau) = [x_0(\tau),\ u_0(\tau)]^T ,$$

$$\mu(\tau,t) = [x_0(\tau),\ u_0(t)]^T ,$$

$$\mu(t,\tau) = [x_0(t),\ u_0(\tau)]^T ,$$

$$f(\mu(\tau,t),\tau,t) = f(x_0(\tau),\ u_0(t),\tau,t) \ ,$$

$$f(\mu(t,\tau),t,\tau) = f(x_0(t),\ u_0(\tau),t,\tau) \ ,$$

$$g(v_0(\tau),\tau) = g(x_0(\tau),\ u_0(\tau),\tau) \ ,$$

$$g(v_0(t),t) = g(x_0(t),\ u_0(t),t) \ .$$

P r o o f. First we shall specify the characteristic cones, according to (Ref. 3) which enable us to obtain some Euler-Lagrange equation. We consider the cartesian product of L_2 spaces

$$X = L_2^n \times L_2^m$$

denote by Z_1, Z_2 the following sets

(8) $$Z_1 = \{\ [x,u]^T \in X : u(t) \in U \ \} \ ,$$

(9) $$Z_2 = \{\ [x,u]^T \in X : x(t) = \int_0^1 f(x(\tau),u(t),\tau,t)\ d\tau \ \}$$

and we observe that our optimization problem becomes :

Find the minimum of the functional

$$I(x,u) = \int_0^1 g(x(t),u(t),t)\ dt$$

where

$$[x,u]^T \in Z_1 \cap Z_2.$$

The cone C_0 of decrease of the functional $I(x,u)$ at the point $v(\cdot) = [x(\cdot),u(\cdot)]^T$ and the conjugate cone C_0^* are, according to ([3], Th. 7.4) and ([3], Th. 10.2) of the form

(10) $$C_0 = \{\ [\bar{x},\bar{u}]^T \in X :$$

$$\int_0^1 g_x^T(v_0(t),t)\ \bar{x}(t) + g_u^T(v_0(t),t)\ \bar{u}(t))dt < 0 \}$$

(11) $$C_0^* = \{ f_0 \in X^* : f_0(\bar{x},\bar{u}) =$$

$$= -\lambda_0 \int_0^1 (g_x^T(v_0(t),t)\bar{x}(t) + g_u^T(v_0(t),t)\ \bar{u}(t))dt;\ \lambda_0 \geq 0 \}$$

where by $\bar{x}$, $\bar{u}$ we denoted the vectors in the neighbourhood of x_0, u_0 . Next, we denote by C_1^* the set of functionals supporting the set Z_1 at the point $\mu(\cdot,\cdot)$. It is known ([3], Th. 10.5) that the cone C_1^* is the cone conjugate to a cone tangent to the set Z_1 at the point $\mu(\cdot,\cdot)$. Hence

(12) $$C_1^* = \{f_1 \in X^* : f_1(\bar{x},\bar{u}) = f_1'(\bar{u})\}$$

where f_1' is the functional supporting the set $V = \{u \in L_2^m : u(t) \in U\}$ at the point $u_0(\cdot)$.

Applying the Lusternik theorem ([3], Th. 9.1) we shall find the cone conjugent to Z_2 at the point $\mu(\cdot,\cdot)$. Let us consider the operator $P : L_2^n \times L_2^m \longrightarrow L_2^n$ of the form

(13) $$P(x,u) = x(t) - \int_0^1 f(x(\tau),u(t),\tau,t)\ d\tau.$$

It can be checked that operator $P(x,u)$ is of the class C^1 and that its Frechet derivative is of the form

(14) $$P'(x,u)\ (\bar{x},\bar{u}) = \bar{x}(t) - \int_0^1 f_x(x(\tau),u(t),\tau,t)\ \bar{x}(\tau) +$$

$$+ f_u(x(\tau),u(t),\tau,t)\ \bar{u}(t))\ d\tau.$$

We show that $P'(x,u)$ maps $L_2^n \times L_2^m$ onto the whole space L_2^n .

It means that the equation

$$(15) \qquad x(t) - \int_0^1 (f_x(x(\tau),u(t),\tau,t)\,\bar{x}(\tau) + f_u(x(\tau),u(t),\tau,t)\,\bar{u}(t))\,d\tau = a(t)$$

has the solution $(\bar{x},\bar{u})$ for any function $a(t) \in L_2^m$. If we put $\bar{u}(t) = 0$, formula (15) takes the form

$$(16) \qquad \bar{x}(t) - \int_0^1 f_x(x(\tau),u(t),\tau,t)\,\bar{x}(\tau)\,d\tau = a(t).$$

It is known ([4], Chapt. II § 10, Th. 1) that Fredholm's linear equation (16) has a unique solution $\bar{x}(t)$ for any function $a(t) \in L_2^n$ in the case where (5) is satisfied. Hence, the cone tangent to Z_2 at the point $\mu(\cdot,\cdot)$ is of the form

$$(17) \qquad C_2 = \{ (\bar{x},\bar{u}) \in X : x(t) = \int_0^1 (f_x(\mu(\tau,t),\tau,t)\,\bar{x}(\tau) + f_u(\mu(\tau,t),\tau,t)\,\bar{u}(t))\,d\tau \}$$

and the conjugate cone

$$(18) \qquad C_2^* = \{f_2 \in X^* : f_2(\bar{x},\bar{u}) = 0 \ ; \ (\bar{x},\bar{u}) \in C_2\} .$$

We observe that :

- the cone C_0 is open and convex ,
- the cones C_1 and C_2 are convex ,
- the cones C_1^* and C_2^* are of the same sense, according to Theorem 3.4 in (Ref. 1).

In Lemma 3.1 we shall prove that the intersection of cones C_1 and C_2 is a subset of a cone tangent to $Z_1 \cap Z_2$.

This enables us to apply the Euler-Lagrange equation of the form

$$(19)\qquad -\lambda_0 \int_0^1 (g_x(v_0(t),t)\ \bar{x}(t) + g_u(v_0(t),t)\ \bar{u}(t))\ dt +$$

$$+ f_1'(\bar{u}) + f_2(\bar{x},\bar{u}) = 0$$

where $f_1(\bar{u})$ - as in the formula (12).

Equation (19) is satisfied for any $(\bar{x},\bar{u}) \in X$. According to (17) for any $(\bar{x},\bar{u}) \in C_2$ we find

$$\bar{x}(t) = \int_0^1 (f_x(\mu(\tau,t),\tau,t)\ \bar{x}(\tau) + f_u(\mu(\tau,t),\tau,t)\ \bar{u}(t)\ d\tau\ .$$

Changing t to τ and τ to t we obtain

$$(20)\quad \bar{x}(\tau) = \int_0^1 (f_x(\mu(t,\tau),t,\tau)\ \bar{x}(t) + f_u(\mu(t,\tau),t,\tau)\ \bar{u}(\tau)\ dt$$

Because $f_2(\bar{x},\bar{u}) = 0$ for $(\bar{x},\bar{u}) \in C_2$ the equation (19) for any $(\bar{x},\bar{u}) \in C_2$ takes the form

$$(21)\quad f_1(\bar{u}) = \lambda_0 \int_0^1 (g_x(v_0(t),t)\bar{x}(t) + g_u(v_0(t),t)\bar{u}(t))dt =$$

$$= \int_0^1 \lambda_0\ g_x(v_0(t),t)\ \bar{x}(t)dt + \int_0^1 \lambda_0\ g_u(v_0(t),t)\ \bar{u}(t))dt.$$

According to (6) we obtain

$$(22)\qquad \lambda_0\ g_x(v_0(t),t) = -\ \psi(t) + \int_0^1 f_x^T(\mu(t,\tau),t,\tau)\ \psi(\tau)\ d\tau\ .$$

Introducing (24) into (23), changing the sequence of integration we obtain - according to the properties of the scalar product

$$(25)\quad f_1(\bar{u}) = -\int_0^1 \left[\psi(\tau)\int_0^1 f_u(\mu(t,\tau),t,\tau)\ \bar{u}(\tau)\ dt\right] d\tau + \\ + \lambda_0 \int_0^1 g_u(v_0(t),t)\ \bar{u}(t)\ dt =$$

$$= \int_0^1 < \int_0^1 -f_u^T(\mu(t,\tau),t,\tau)\ \psi(\tau)\ dt + \lambda_0\ g_u(v_0(\tau),\tau),\ \bar{u}(\tau) > d\tau$$

Denoting by $A(\tau)$ the integral

$$A(\tau) = -\int_0^1 f_u^T(\mu(t,\tau),t,\tau)\ \psi(\tau)\ dt + \lambda_0\ g_u(v_0(\tau),\tau)$$

we can write (25) in the form

$$f_1(\bar{u}) = \int_0^1 <A(\tau),\bar{u}(\tau)>\ d\tau\ .$$

This means ([3], Example 10.5)

$$(26)\qquad <A(\tau),\ u-u_0(\tau)> \ \geq 0 \quad \text{for a.e. } t.$$

Hence

$$< \int_0^1 -f^T(\mu(t,\tau),t,\tau)\ \psi(\tau)\ dt + \lambda_0\ g_u(v_0(\tau),\tau),\ u-u_0(\tau)> \ \geq 0 \qquad \text{for a.e. } t.$$

This completes the proof.

L e m m a 3.1. The intersection of cones C_1 and C_2 is a subset of a cone tangent to $Z_1 \cap Z_2$.

P r o o f. The operator P from formula (13) is differentiable and the operator $P'(x,u)$ from formula (14) maps $L_2^n \times L_2^m$ onto the whole space L_2^n . We also observe that the mapping

$$P'(x,u)\ \bar{x} = \bar{x}(t) - \int_0^1 (f_x(x(\tau),u(t),\tau,t)\ \bar{x}(\tau)d\tau$$

is isomorphic.

From the fact that for any function $a(t) \in L_2^n$ there exists a unique $\bar{x}(t) \in L_2^n$ such that with assumption (5) the following formula holds

$$\bar{x}(t) - \int_0^1 (f_x(x(\tau),u(t),\tau,t)\ \bar{x}(\tau)\ d\tau = a(t)$$

we conclude that $P'_x(x,u)$ is inversible. Hence, because L_2^n [0,1] and L_2^m [0,1] are Banach spaces and $P'_x(x,u)$ is linear and continuous, we find ([6], Th. 41.1) that the inverse operator is also linear and continuous. We conclude that P satisfies the assumptions of the implicit function theorem ([7], Chapter 0.2.3) in some neighbourhood V_0 of (x_0,u_0). Hence the set Z_2 can be represented in this neighbourhood in the form

(27) $$Z_2 = \{\ (\bar{x},\bar{u}) \in X : \bar{x} = \varphi(\bar{u})\ \}$$

where $\varphi : L_2^m \longrightarrow L_2^n$ is the C^1 class operator satysfying the condition $P(\varphi(\bar{u}),\bar{u}) = 0$ for all $\bar{u}$ such that $(\varphi(\bar{u}),\bar{u}) \in V_0$. Hence we deduce that the cone C_2 can be represented in the form

(28) $$C_2 = \{\ (\bar{x},\bar{u}) \in X : \bar{x} = \varphi_u(u_0)\bar{u})\ \}.$$

Let $(\bar{x},\bar{u})$ be an arbitrary element of the set $C_1 \cap C_2$. Then there exists an operator $v_u^2 : R \longrightarrow U$ such that

$$\lim_{\varepsilon \to 0^+} \frac{v_u^2(\varepsilon)}{\varepsilon} = 0$$

and the formula

(29) $$(x_0, u_0) + \varepsilon(\bar{x}, \bar{u}) + (v_x^2(\varepsilon), v_u^2(\varepsilon)) \in Z_1$$

holds for sufficiently small ε and any $v^2(\varepsilon)$ such that

$$\lim_{\varepsilon \to 0^+} \frac{v_x^2(\varepsilon)}{\varepsilon} = 0 .$$

Hence, according to (27) we observe that for sufficiently small ε the following formula holds

(30) $$(\varphi(u_0 + \varepsilon\bar{u} + v_u^2(\varepsilon)),\ u_0 + \varepsilon\bar{u} + v_u^2(\varepsilon)) \in Z_2 .$$

$\varphi(u)$ is differentiable operator, hence

(31) $$\varphi(u_0 + \varepsilon\bar{u} + v_u^2(\varepsilon)) = \varphi(u_0) + \varepsilon\varphi_u(u_0)\bar{u} + v_x^1(\varepsilon)$$

for some $v_x^1(\varepsilon)$ such that

$$\lim_{\varepsilon \to 0^+} \frac{v_x^1(\varepsilon)}{\varepsilon} = 0 .$$

From (27) and (28) we obtain

(32) $$(x_0, u_0) + \varepsilon(\bar{x}, \bar{u}) + (v_x^1(\varepsilon), v_u^2(\varepsilon)) \in Z_2 .$$

If we take $v_x^2(\varepsilon) = v_x^1(\varepsilon)$ we conclude from (29) and (32) that the arbitrary vector $(\bar{x}, \bar{u}) \in C_1 \cap C_2$ is tangent to the set $Z_1 \cap Z_2$. That completes the proof.

4. The existence theorem

The existence theorem will be proved with additional assumption concerning the function in the state equation (2). We assume that the function $f(x(\tau),u(t),\tau,t)$ is linear with respect to (x,u). We also assume that the set U is a compact convex set in R^m. With notation

$$A = \{(x,u) \in L_2^n \times L_2^m\ ,\ P(x,u) = 0,\ u(t) \in U \ \text{ for a.e. } t\}$$

where $P(x,u)$ is the operator (13), our optimization problem becomes :

Find the minimum of the functional

$$I(v) = \int_0^1 v^T(t)\ H_0(t)v(t) + p_0(t)v(t)\ dt$$

where $v \in A$. The form of the set A and the convexity of $U \in R$ enable us to formulate :

L e m m a 4.1. The set $A \in L_2^n \times L_2^m$ is convex.

The function $g(v) = v^T(t)\ H_0(t)v(t) + p_0(t)v(t)$ is a continuous and convex function with respect to v. That gives us :

L e m m a 4.2. The functional $I(\cdot)$ is lower semicontinuous in the weak topology of the space L_2^{n+m}.

L e m m a 4.3. The set $A \in L_2^n \times L_2^m$ is bounded in the norm of the space $L_2^n \times L_2^m$.

P r o o f. Let $(x,u) \in A$. We know that $u(t) \in U$ for a. e. t. The convexity of the set $U \in R^m$ implies the existence of real number $c > 0$ such that $|u(t)| \le c$ for a.e. $t \in [0,1]$. Hence

$$\|u\|_{L_2^m} = (\int_0^1 |u(t)|^2 \, dt)^{1/2} \le (\int_0^1 c^2 \, dt)^{1/2} = c.$$

It can be easily shown (Ref. 5) that in the set A the operator P(x,u) satisfies the assumptions of the implicit function theorem. That implies the existence of the linear operator $\varphi : L_2^m \longrightarrow L_2^n$ of the class C^1, such that the conditions $x = \varphi(u)$ and $P(x,u) = 0$ are equivalent. The boundness of $\varphi(\cdot)$ implies

$$\|x\|_{L_2^n} = \|\varphi(u)\| \le \|\varphi\| \|u\| \le \|\varphi\| c < \infty .$$

Then, there exists the ball K of the radius $\sqrt{c_x^2 + c_u^2}$ where $c_x = c$, $c_u = \|P\|c$ such that $A \subset K$. That completes the proof.

L e m m a 4.4. The set A is closed in the strong topology of the space $L_2^n \times L_2^m$.

P r o o f. Let us consider the sequence $(x_k, u_k) \in A$ such that

$$\|(x_k, u_k) - (x_0, u_0)\| \xrightarrow[k \to \infty]{} 0 .$$

We shall prove that $(x_0, u_0) \in A$.

From $\|(x_k, u_k) - (x_0, u_0)\| \xrightarrow[k \to \infty]{} 0$ we conclude that $\|x_k - x_0\| \xrightarrow[k \to \infty]{} 0$ and $\|u_k - u_0\| \xrightarrow[k \to \infty]{} 0$. Hence there exists an integer k_0 , such that for any $k \ge k_0$ we find

$$\int_0^1 |u_k(t) - u_0(t)|^2 \, dt \xrightarrow[k \to \infty]{} 0 .$$

Let us suppose that $u_0(t) \notin U$ on some measurable set of positive measure M. The set $U \in R^m$ is compact, hence there exists a real number $a > 0$ and the set C of positive measure m, such that

$$|u_k(t) - u_0(t)| > a \quad \text{for} \quad t \in C \text{ and some } k \geq k_0 .$$

Hence for some k ≥ 0 we have

$$\int_0^1 |u_k(t) - u_0(t)|^2 \, dt = \int_{C\subset[0,1]} |u_k(t) - u_0(t)|^2 \, dt \geq$$

$$\geq a^2 \operatorname{mes}(C) = a^2 m > 0 .$$

We have obtained a contradiction to (27) and we conclude that $u_0(t) \in U$ for a.e. t.

For any k is $x_k = \varphi(u_k)$, where φ is a continuous operator.

According to this result and to the definition of the set A we conclude that $(x_0, u_0) \in A$. That completes the proof.

L e m m a 4.5. The set $A \subset L_2^n \times L_2^m$ is compact in the weak topology of $L_2^n \times L_2^m$.

P r o o f. Lemmas 4.1 and 4.4 imply that A is convex and closed in the strong topology of the space $L_2^n \times L_2^m$. We deduce that A is closed in the weak topology because it is bounded and closed in the weak toplogy of the reflexive space $L_2^n \times L_2^m$.

We can formulate now :

T h e o r e m 4.1. If all above mentioned assumptions hold then there exists the point $(x_0, u_0) \in A$ such that :

1^0 $x_0(t) = \int_0^1 f(x_0(\tau), u_0(t), \tau, t) \, d\tau$,

2^0 $u_0(t) \in U$ for a. e. $t \in [0,1]$,

3^0 $I(x_0(t),u_0(t)) \le I(x,u)$ for any $(x,u) \in A$.

The proof is based on the Weierstrass theorem and the Lemmas 4.2 and 4.5.

REFERENCES

[1] S. W a l c z a k : Some properties of Cones in Normed Spaces and their Application to Investigating Extremal Problems, Journal of Optimization Theory Appl. 42, No 4, 1984.

[2] A. Ya. D u b i v i t s k i i, A. A. M i l y u t i n : Extremum Problems in the Presence of Constraints, Vyčisl. Mat. i Mat. Fiz. Vol. 5 (1965), 394-453.

[3] I. V. G i r s a n o v : Lectures on Mathematical Theory of Extremum Problems, Springer-Verlag, New York, 1972.

[4] A. P i s k o r e k : Równania całkowe (Integral Equations PWN, Warszawa 1980.

[5] J. P e ł c z e w s k i : On an Optimization Problem Described by some Integral Equation, Acta Univ. Lodziensi Folia Mathematica 2, 1987.

[6] W. K o ł o d z i e j : Wybrane rozdziały analizy matematycznej (Selected Chapters of Mathematical Analysis), PWN, Warszawa, 1982.

[7] A. D. I o f f e, V. M. T i k h o m i r o v : Theory and Extremal Problems, Academic Press, New York, 1979.

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