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Maria Szapiel

ON ANNULI CONTAINING THE ZEROS OF POLYNOMIALS

The paper extends some results of E. Deutsch [1] concerning bounds for the zeros of polynomials as functions of all the coefficients. The first contributors to this problem were C. F. Gauss and A. L. Cauchy. Since then many other mathematicians have taken part in the further growth of the subject, see [2], and this article is a slight contribution to the mentioned problem.

Consider a complex polynomial

(1) $$\mathbb{C} \ni z \longmapsto a_0 + a_1 z + \ldots + a_n z^n \stackrel{df}{=} f(z), \quad n \geq 2,$$

with $a_0 a_n \neq 0$ so that

(2) $$f(z) \equiv a_n(z-z_1) \ldots (z-z_n) \text{ and } z_j \neq 0 \text{ for } 1 \leq j \leq n.$$

It is trivial that each theorem saying that

(3) $$|z_j| \leq \varphi_n(a_0, a_1, \ldots, a_n) \text{ for } j = 1, \ldots, n$$

is equivalent to the following

(4) $$|z_j| \geq 1/\varphi_n(a_n, a_{n-1}, \ldots, a_0) \text{ for } j = 1, \ldots, n.$$

Indeed, in (3) or (4) put $z \longmapsto z^n f(1/z)$ instead of f. To obtain a little stronger form of (3)-(4) consider the family $\{z \longmapsto f(cz) : c \in \mathbb{C}\}$. Then we get

(3') $\quad |z_j| \le \inf \{ |c|\varphi_n(a_0, a_1c, \ldots, a_nc^n) : c \in \mathbb{C} \}$

for $j = 1, \ldots, n$,

and

(4') $\quad |z_j| \ge \sup \{ |c|/\varphi_n(a_nc^n, a_{n-1}c^{n-1}, \ldots, a_0) : c \in \mathbb{C} \}$

for $j = 1, \ldots, n$.

Moreover, if we know that $\varphi_n(a_0, a_1, \ldots, a_n) \ge 1$ for all $a_0, a_1, \ldots, a_n \in \mathbb{C}$ with $a_0a_n \ne 0$ and that the theorems (3), (4) hold for polynomials (1)-(2) and $n = 2,3,\ldots$, then for (2) we have

(5) $\quad |z_j| \le \varphi_{n+1}(a_0, a_1-a_0, \ldots, a_n-a_{n-1}, -a_n)$ if $j = 1, \ldots, n$,

and

(6) $\quad |z_j| \ge 1/\varphi_{n+1}(-a_n, a_n-a_{n-1}, \ldots, a_1-a_0, a_0)$ if $j = 1, \ldots, n$.

Really, it is enough to apply (3)-(4) with $n+1$ in place of n to the polynomial $z \longmapsto (1-z)f(z)$.

We let add that the classical results of Cauchy have the form (3), see [2]. Now we formulate the basic lemma. To that purpose consider the following real polynomials

(7) $\quad \lambda_k(x) = |a_0| - |a_1|x - \ldots - |a_k|x^k - \left(\sum_{s=k+1}^{n} |a_s| \right) x^{k+1}$

and

(8) $\mu_k(x) = \sum_{s=0}^{n-k-1} |a_s| + |a_{n-k}|x + \ldots + |a_{n-1}|x^k - |a_n|x^{k+1}$,

$k = 0, 1, \ldots, n-1$.

Each polynomial λ_k strictly decreases on $[0,\infty)$ and the polynomial $x \longmapsto x^{k+1}\mu_k(1/x)$ strictly increases on $[0,\infty)$. Hence any from the polynomials (7)-(8) has the only zero in the interval $[0,\infty)$. Denote the positive zeros of (7) and (8) by $\tilde{r}_k$ and $\tilde{\rho}_k$, respectively, and let

$$r_k = \begin{cases} \min\{1,\tilde{r}_k\} & \text{for } k = 0,1,\dots,n-2 \\ \tilde{r}_{n-1} = r & \text{for } k = n-1\ , \end{cases}$$

$$\rho_k = \begin{cases} \max\{1,\tilde{\rho}_k\} & \text{for } k = 0,1,\dots,n-2 \\ \tilde{\rho}_{n-1} = \rho & \text{for } k = n-1\ . \end{cases}$$

Then we have

L e m m a. $r_0 \le r_1 \le \dots \le r_{n-1} = r \le |z_j| \le$
$\le \rho = \rho_{n-1} \le \rho_{n-2} \le \dots \le \rho_1 \le \rho_0$
for all $j = 1,\dots,n$.

P r o o f. Since (3) is equivalent to (4), it is sufficient to check all the lower bounds for the zeros. Indeed, if we denote $\tilde{r}_k = \varphi(a_0,a_1,\dots,a_n)$, $\tilde{\rho}_k = \psi(a_0,a_1,\dots,a_n)$, then by $\lambda_k(\tilde{r}_k) = 0$, $\mu_k(\tilde{\rho}_k) = 0$ we obtain easily that $\psi(a_0,a_1,\dots,a_n) = 1/\varphi(a_n,a_{n-1},\dots,a_0)$, whence the mentioned conclusion follows.

Observe now that for $k = 0,1,\dots, n-2$ we have

$$\lambda_{k+1}(r_k) = \lambda_k(r_k) + r_k^{k+1}(1-r_k)\sum_{j=k+2}^{n}|a_j| \ge 0 = \lambda_{k+1}(\tilde{r}_{k+1}),$$

because $\lambda_k(r_k) \ge \lambda_k(\tilde{r}_k) = 0$ and $r_k \le 1$. Hence $r_k \le \min\{1,\tilde{r}_{k+1}\} =$ $= r_{k+1}$ for $k = 0,1,\dots,n-2$. It remains to show that $|z_j| \ge r_{n-1}$ for $j = 1,\dots,n$. Assume that $f(\zeta) = 0$, $|\zeta| \le 1$. Then

$$0 = |f(\zeta)| \geq |a_0| - |a_1\zeta| - \dots - |a_k\zeta^k| - |f(\zeta) - \sum_{s=0}^{k} a_s\zeta^s| \geq$$

$\geq \lambda_k(|\zeta|)$, that is $|\zeta| \geq \tilde{r}_{n-1} = r.$

R e m a r k s.

(i) The upper bounds in

(9) $$r \leq |z_j| \leq p \ , \ 1 \leq j \leq n \ ,$$

and

(10) $$r_0 = \min\left\{ 1, |a_0|/\sum_{s=1}^{n} |a_s| \right\} \leq |z_j| \leq$$

$$\leq \max\left\{1, \sum_{s=0}^{n-1} |a_s|/|a_n|\right\} = \rho_0 \ , \ 1 \leq j \leq n \ ,$$

are known as classical results due to Cauchy, see [2].

(ii) Let $M = \max\{|a_s/a_n| : s = 1,\dots,n-1\}$. Then $0 = -\mu_{n-1}(\rho) \geq$ $\geq |a_n|(\rho^n - \rho)(1-M/(\rho-1)) + |a_n|\rho - |a_0|$, whence $\rho \leq \max\{|a_0/a_n|, 1+M\}$ an one more result of Cauchy, see [2]. Thus for $j = 1,\dots,n$ we have

(11) $$\min\{|a_0/a_n|, |a_0|/(|a_0|+|a_1|),\dots,|a_0|/(|a_0|+|a_{n-1}|)\} \leq$$

$$\leq |z_j| \leq \max\{|a_0/a_n|, \ 1+|a_1/a_n|,\dots,1+|a_{n-1}/a_n|\}.$$

C o r o l l a r y 1. For any $k \in \{1,\dots,n-2\}$ and $j = 1,\dots,n$ $(n \geq 3)$ we have

$$|z_j| \leq \max\left\{\sum_{s=0}^{k} |a_s/a_n|, \ 1+|a_{k+1}/a_n|,\dots,1+|a_{n-1}/a_n|\right\}, \text{ see [1]},$$

and

$$|z_j| \geq \min\left\{|a_0|/\sum_{s=k+1}^{n}|a_s|,\ |a_0|/(|a_0|+|a_1|),\ldots,|a_0|/(|a_0|+|a_k|)\right\}.$$

P r o o f. To the polynomial (7) apply (11) with n = k+1. Then we get an estimation for $\tilde{r}_k$ from below. Similarly, applying (11) to the polynomial (8) we obtain an upper bound for ρ_k . Next use Lemma.

C o r o l l a r y 2. Let $M_1 = \sum_{s=2}^{n}|a_s|$ and $M_2 = \sum_{s=0}^{n-2}|a_s|$. Then for any $j = 1,\ldots,n$ we have

$$\min\left\{1,\ \frac{2|a_0|}{|a_1|+(|a_1|^2+4|a_0|M_1)^{1/2}}\right\} \leq |z_j| \leq$$

$$\leq \max\left\{1,\ \frac{|a_{n-1}|+(|a_{n-1}|^2+4|a_n|M_2)^{1/2}}{2|a_n|}\right\}.$$

P r o o f. This is exactly the inequality : $r_1 \leq |z_j| \leq \rho_1$ for $1 \leq j \leq n$, see Lemma.

Hence, in particular, we obtain

C o r o l l a r y 3. For all $1 \leq j \leq n$,

(12) $$|z_j| \geq \min\left\{1,\ (|a_0|/\sum_{s=2}^{n}|a_s|)^{1/2}\right\} \text{ if } a_1 = 0,$$

(13) $$|z_j| \leq \max\left\{1,\ \left(\sum_{s=0}^{n-2}|a_s/a_n|\right)^{1/2}\right\} \text{ if } a_{n-1} = 0,$$

(14) $$|z_j| \geq 1 \text{ if } |a_0| \geq \sum_{s=1}^{n}|a_s|$$ (A. Cohn, see [2]) ,

(15) $$|z_j| \leq 1 \text{ if } |a_n| \geq \sum_{s=0}^{n-1}|a_s|$$ (A. Cohn, see [2]) .

C o r o l l a r y 4. Let $N_1 = |a_n| + \sum_{s=2}^{n} |a_s - a_{s-1}|$, $N_2 = |a_0| + \sum_{s=1}^{n-1} |a_s - a_{s-1}|$. For any $j = 1,\dots,n$ we have then

(16) $$|z_j| \geq 2|a_0|/(|a_1-a_0|+(|a_1-a_0|^2 + 4|a_0|\ N_1)^{1/2})$$

and

(17) $$|z_j| \leq (|a_n-a_{n-1}|+(|a_n-a_{n-1}|^2 + 4|a_n|\ N_2)^{1/2})/(2|a_n|).$$

P r o o f. Observe first that there is the same kind of the connection between Corollaries 2 and 4 as between (3)-(4) and (5)-(6). One should only notice that

$$N_1 \geq |a_1|, \quad N_2 \geq |a_{n-1}|,$$

$$|a_1-a_0|+(|a_1-a_0|^2 + 4|a_0|\ N_1)^{1/2} \geq |a_1-a_0|+|a_1+a_0| \geq 2|a_0|$$

and, similarly,

$$|a_n-a_{n-1}|+(|a_n-a_{n-1}|^2 + 4|a_n|\ N_2)^{1/2} \geq 2|a_n|.$$

In particular, we get

C o r o l l a r y 5. For any $j = 1,\dots,n$,

(18) $$|z_j| \geq (|a_0|/N_1)^{1/2} \quad \text{if } a_0 = a_1 ,$$

(19) $$|z_j| \leq (N_2/|a_n|)^{1/2} \quad \text{if } a_n = a_{n-1} ,$$

(20) $$|z_j| \geq a_0/\max\{a_0,a_1\} \text{ if } a_0 > 0 \text{ and } a_1 \geq a_2 \geq \dots \geq a_n > 0$$

(a slight generalization of the Enestrom-Kakeya theorem, see [2]),

(21) $$a_0/\max\{a_0,a_1\} \leq |z_j| \leq (2\max\{a_0,a_1\}/a_n - 1)^{1/2} \text{ if } a_0 > 0$$

and $a_1 \geq a_2 \geq \dots \geq a_{n-1} = a_n > 0$,

(22) $|z_j| \le \max\{a_{n-1}, a_n\}/a_n$ if $a_n > 0$ and $a_{n-1} \ge a_{n-2} \ge \dots \ge a_0 > 0$,

(23) $(a_0/(2\max\{a_{n-1}, a_n\} - a_0))^{1/2} \le |z_j| \le \max\{a_{n-1}, a_n\}/a_n$

if $a_n > 0$ and $a_{n-1} \ge a_{n-2} \ge \dots \ge a_1 = a_0 > 0$.

Let now all the coefficients of (1) be positive and let $m = \min\{a_j/a_{j+1} : j=0,1,\dots,n-1\}$, $M = \max\{a_j/a_{j+1} : j=0,1,\dots,n-1\}$. Applying (20) and (22) to the polynomials $z \longmapsto f(mz)$ and $z \longmapsto f(Mz)$, respectively, we have also

C o r o l l a r y 6. $m \le |z_j| \le M$ for all $j \in \{1,\dots,n\}$, cf. [2].

E x a m p l e s.

(i) Consider first the following example of Deutsch [1]: $f(z) = 0.7 + 0.7z + 0.3z^2 + z^3$. Then

$0.35 \le |z_j| \le 1.7$ by (10), $0.5 \le |z_j| \le 1.7$ by (11), $0.5 \le |z_j| \le 1.4$ by Corollary 1 and $0.577\ldots = 3^{-1/2} \le |z_j| \le$ $\le 1.45\ldots$ by Corollary 4.

Let us add that the exactly values of the zeros are : $z_1 = -0.7079\ldots$, $|z_2| = |z_3| = 0.9943\ldots$.

(ii) $f(z) = 0.9 + 0.9z + z^2 + 0.9z^3 + z^4 + z^5$. For this polynomial we have :

$9/19 \le |z_j| \le 2$ by (10) or by Corollary 1, $3/16 \le |z_j| \le 4.7$ by (11) and $\sqrt{9/13} \le |z_j| \le \sqrt{1.2}$ by (18)-(19).

(iii) $f(z) = 2 + 2z + pz^2 + qz^3 + z^4 + z^5$, where $2 \ge p \ge q \ge 1$. In this case we have : $0.5 \le |z_j| \le 3$ by (10) and $1 \le |z_j| \le \sqrt{3}$ by (16)-(17) or by (18)-(19) or else by (21). For instance, if $p = 2$, $q = 1$, then an easy calculation implies that $|z_1| = |z_2| = 1$ and $|z_3| = |z_4| = |z_5| = 2^{1/3}$.

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DEPARTAMENT OF MATHEMATICS, TECHNICAL UNIVERSITY,
20-618 LUBLIN, POLAND

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