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ON SOME PROJECTIVELY RECURRENT SPACES

1. Introduction

An n-dimensional ($n \geq 3$) Riemannian space (M,g) not necessarily of definite metric is called a projectively recurrent space, when the Weyl's projective curvature tensor W^h_{ijk} of the space defined by

$$(1.1) \qquad W^h_{ijk} = R^h_{ijk} - \frac{1}{n-1}\left[\delta^h_k R_{ij} - \delta^h_j R_{ik}\right]$$

is recurrent, that is, the W^h_{ijk} satisfies $\nabla_l W^h_{ijk} = k_l W^h_{ijk}$ for a non-zero vector k_l which is said a recurrence vector, where ∇_i denotes covariant differentiation with respect to the metric connection of the space, R^h_{ijk} and R_{ij} denote respectively the curvature tensor and Ricci tensor of the space.

The W^h_{ijk} is zero if and only if (M,g) is of constant curvature, i.e.,

$$R^h_{ijk} = \frac{R}{n(n-1)}\left[\delta^h_k g_{ij} - \delta^h_j g_{ik}\right]$$

where R is the scalar curvature of (M,g). It is obvious that a recurrent space is a projectively recurrent. E. Głodek [1] and T. Miyzawa [3] proved that every projectively recurrent space is necessarily a recurrent space.

M.Matsumoto was proved ([2], Theorem 3.6) that in projective recurrent space the recurrence vector k_l is a gradient vector.

Let M be an n-dimensional Riemannian space with metric tensor g_{ij} and a semi-symmetric metric connection $\mathring{\Gamma}^h_{ij}$. It is known (see [6]) that a semi-symmetric metric connection $\mathring{\Gamma}^h_{ij}$ is given by

$$\mathring{\Gamma}^h_{ij} = \Gamma^h_{ij} + \delta^h_i\, p_j - p^h\, g_{ij}$$

where Γ^h_{ij} denotes the Christoffel symbol and p_i is a gradient vector and $p^h = g^{hr} p_r$, and that the curvature tensors $\mathring{R}^h_{ijk}$ of $\mathring{\Gamma}^h_{ijk}$ and R^h_{ijk} of Γ^h_{ij} are related is such a way that

$$\mathring{R}^h_{ijk} = R^h_{ijk} - \delta^h_k\, A_{ij} + \delta^h_j\, A_{ik} - g_{ij}\, A^h_k + g_{ik}\, A^h_j \, , \tag{1.2}$$

where A_{ij} is a tensor field of type (0,2) defined by

$$A_{ij} = \nabla_j p_i - p_j p_i + \frac{1}{2}\, p^r\, p_r\, g_{ji} \, , \tag{1.3}$$

and $A^h_k = g^{hr} A_{rk}$ .From (1.3) we have

$$\mathring{R}_{ij} = \mathring{R}^r_{ijr} = R_{ij} - (n-2)\, A_{ij} - A\, g_{ij} \, , \tag{1.4}$$

where

$$A = \nabla_r p^r + \frac{n-2}{2}\, p^r\, p_r \, . \tag{1.5}$$

If we now put

$$\mathring{W}^h_{ijk} = \mathring{R}^h_{ijk} - \frac{1}{n-1}\left[\, \delta^h_k\, \mathring{R}_{ij} - \delta^h_j\, \mathring{R}_{ik} \,\right]$$

at each point of M, then by means of (1.1), (1.2) and (1.4), we see that

$$(1.6)\qquad \overset{\circ}{W}{}^{h}_{ijk} = W^{h}_{ijk} - \frac{1}{n-1}\Big[\delta^{h}_{k}\,(A_{ij} - A\,g_{ij}) - \delta^{h}_{j}\,(A_{ik} - A\,g_{ik})\Big] + + g_{ik}\,A^{h}_{j} - g_{ij}\,A^{h}_{k}\,.$$

The tensor $\overset{\circ}{W}{}^{h}_{ijk}$ we shall call the projective curvature tensor corresponding to the connection $\overset{\circ}{\Gamma}{}^{h}_{ij}$. We have the following theorem (see [4]).

T h e o r e m A. If the vector field p_i in (1.3) is concircularly, i.e., satisfies the condition

$$\nabla_m p_i = p_m p_i + F\,g_{mi}\,,$$

then $\overset{\circ}{W}{}^{h}_{ijk} = W^{h}_{ijk}$.

D e f i n i t i o n. An n-dimensional Riemannian space is called special projectively recurrent (S.P.R. in short) with respect to the metric semi-symmetric connection $\overset{\circ}{\Gamma}{}^{h}_{ij}$, when the projective curvature tensor corresponding to this connection has the following properties

$$(1.7)\qquad \overset{\circ}{W}{}^{h}_{ijk} = W^{h}_{ijk}\,,$$

$$(1.8)\qquad \overset{\circ}{\nabla}_{m}\overset{\circ}{W}{}^{h}_{ijk} = a_m\,\overset{\circ}{W}{}^{h}_{ijk}\,,$$

where $\overset{\circ}{\nabla}_m$ denotes the covariant differentiation with respect to the connection $\overset{\circ}{\Gamma}{}^{h}_{ij}$ and a_m is a vector field.

The purpose of this note is to study the S.P.R. spaces with respect to the connection $\overset{\circ}{\Gamma}{}^{h}_{ij}$.

All spaces under considerations are assumed to be not of constant curvature, connected and analytic. The metric are not assumed to be definite.

2. Preliminary

In the sequel we need the following well-known results.

L e m m a 1. [5] The curvature tensor of a Riemannian manifold (M,g) satisfies the identity

$$\nabla_s\nabla_m R_{hijk} - \nabla_m\nabla_s R_{hijk} + \nabla_j\nabla_k R_{mshi} - \nabla_k\nabla_j R_{mshi} +$$
$$+ \nabla_i\nabla_h R_{jkms} - \nabla_h\nabla_i R_{jkms} = 0 .$$

L e m m a 2. [5] If T_{bc} , w_a are numbers satisfying $T_{bc} = T_{cb}$,

$$w_a T_{bc} + w_c T_{ba} + w_b T_{ca} = 0$$

for a,b,c = 1,2,...,n, then all the T_{bc} are zero or all w_a are zero.

T h e o r e m 1. Suppose that (M,g) admits a concircularly vector field. The projective curvature tensor corresponding to the connection $\overset{\circ}{\Gamma}{}^h_{ij}$ is equal to the projective curvature tensor corresponding to the Levi-Civita connection if and only if

$$\nabla_j p_i = p_j\, p_i + F\, g_{ji} \tag{2.1}$$

where F is a scalar function.

P r o o f. If we put $\overset{\circ}{W}{}^h_{ijk} = W^h_{ijk}$, then (1.6) is of the form

$$\delta^h_k\,(A_{ij} - A\,g_{ij}) - \delta^h_j\,(A_{ik} - A\,g_{ik}) + \tag{2.2}$$
$$+ (n-1)(A^h_k\, g_{ij} - A^h_j\, g_{ik}) = 0 .$$

Transvecting (2.2) with g^{ij}, we get

$$(n-2)(n\, A^h_k - A\, \delta^h_k) = 0$$

whence, by making use of (1.3) and (1.5) we obtain (2.1). Now, Theorem A completes the proof.

T h e o r e m 2. If (M,g) admits the connection $\overset{o}{\Gamma}{}^h_{ij}$ such that M is S.P.R. with respect to this connection, then F is a constant.

P r o o f. Differentiating (2.1) covariantly with respect to ∇_k and using Ricci-identity, we obtain

$$(2.3) \qquad p_r\, R^r_{ijk} = (F\, p_j - F_j)\, g_{ik} - (F\, p_k - F_k)\, g_{ij}\ ,$$

where $F_j = \delta_j F$. Transvecting (2.3) with p^k and using $p^r\, p^s\, R_{rijs} = p^r\, p^s\, R_{rjis}$, we find

$$(2.4) \qquad \begin{cases} \text{a) } F\, p_j - F_j = \lambda\, p_j\ , \\ \text{b) } p_r\, R^r_{ijk} = \lambda(p_j\, g_{ik} - p_k\, g_{ij})\ , \\ \text{c) } p^r\, R_{rk} = \lambda\, (1-n)\, p_k\ , \end{cases}$$

where λ is a scalar function. From (1.1) and (2.4), we have

$$(2.5) \begin{cases} \text{a) } p^r\, W_{rijk} = \lambda(p_j\, g_{ik} - p_k\, g_{ij}) - \frac{1}{n-1}(p_k R_{ij} - p_j R_{ik}), \\ \text{b) } p^r\, W_{hrjk} = 0, \\ \text{c) } p^r\, W_{hirk} = p_h(\lambda\, g_{ik} + \frac{1}{n-1}\, R_{ik})\ , \\ \text{d) } p^r\, W_{hijr} = -p_h(\lambda\, g_{ij} + \frac{1}{n-1}\, R_{ij})\ , \end{cases}$$

where $W_{hijk} = g_{hr}\, W^r_{ijk}$. From (1.7) and (1.8) it follows

(2.6) $\nabla_m W_{hijk} = a_m W_{hijk} + p_h W_{mijk} + p_i W_{hmjk} + p_j W_{himk} +$

$+ p_k W_{hijm} - p^r[g_{mh} W_{rijk} + g_{mi} W_{hrjk} + g_{mj} W_{hirk} + g_{mk} W_{hijr}]$.

Differentiating (2.5) b) covariantly with respect to ∇_m and using (2.1), (2.5) and (2.6), we get $(F + p^r p_r) W_{hijk} = 0$. But $W_{hijk} \neq 0$, hence we have

(2.7) $$F + p^r p_r = 0 .$$

Therefore $2p^r \nabla_j p^r + F_j = 0$. Using (2.1) we have

$$2(p^r p_r + F)p_j + F_j = 0 \quad \text{and from (2.7)} \quad F_j = 0.$$

L e m m m a 3. Suppose that (M,g) admits the connection $\overset{0}{\Gamma}{}^h_{ij}$ such that M is S.P.R. with respect to this connection. Then

(2.8)
$$\begin{cases} \text{(i)} \quad \nabla_m R_{ik} = a_m[R_{ik} + (n-1)F g_{ik}] + \\ \qquad\qquad + p_i[R_{mk} + (n-1)F g_{mk}] + p_k[R_{im} + (n-1)F g_{im}], \\ \text{(ii)} \quad R_m = [R + n(n-1)F]a_m , \\ \text{(iii)} \quad a_r p^r + 2F = 0 . \end{cases}$$

P r o o f. Applying ∇_m to the both sides of (2.5) c) and using (2.1), (2.5), (2.6), (2.7) and $F = \lambda = \text{const.}$, we get (2.8)(i), whence, by (2.4) and contraction with g^{ik}, we obtain (ii). Contracting now (2.8)(i) with g^{mi} and using (ii), we get (iii).

3. Main results

T h e o r e m 3. If M is S.P.R. with respect to the connection $\overset{g}{\Gamma}{}^{h}_{ij}$, then the vector a_m is gradient, that is, it satisfies

$$\nabla_m a_s = \nabla_s a_m \,. \tag{3.1}$$

P r o o f. Substituting (1.1), (2.5) and (2.8) into (2.6), we get

$$\nabla_m R_{hijk} = a_m B_{hijk} + p_h B_{mijk} + p_i B_{hmjk} + p_j B_{himk} + p_k B_{hijm}, \tag{3.2}$$

where

$$B_{hijk} = R_{hijk} + F(g_{hk}\, g_{ij} - g_{hj}\, g_{ik}) \,. \tag{3.3}$$

From the above we infer that $\nabla_m B_{hijk} = \nabla_m R_{hijk}$. Applying ∇_s to the both sides of (3.2) and using (3.2) and (3.3), we get

$$\begin{aligned} \nabla_s\nabla_m R_{hijk} - \nabla_m\nabla_s R_{hijk} = {} & (\nabla_s a_m - \nabla_m a_s)\, B_{hijk} + \\ & + F[g_{hs}B_{mijk} - g_{hm}B_{sijk} + g_{is}B_{hmjk} - g_{im}B_{hsjk} + \\ & + g_{js}\, B_{himk} - g_{jm}\, B_{hisk} + g_{ks}\, B_{hijm} - g_{km}\, B_{hijs}] \,. \end{aligned} \tag{3.4}$$

The tensor B_{hijk} has the same algebraic properties as the curvature tensor. Therefore (3.4) and Lemma 1 imply

$$\begin{aligned} & (\nabla_s a_m - \nabla_m a_s)\, B_{hijk} + (\nabla_k a_j - \nabla_j a_k)\, B_{mshi} + \\ & + (\nabla_i a_h - \nabla_h a_i)\, B_{jkms} = 0. \end{aligned} \tag{3.5}$$

If we now put $W_{ms} = \nabla_s a_m - \nabla_m a_s$, from (3.5) follows

$$W_{ms} B_{hijk} + W_{jk} B_{mshi} + W_{hi} B_{jkms} = 0$$

which is of the form of Lemma 2 since $B_{mshi} = B_{hims}$, indices a,b,c being replaced by pairs ms, hi, jk. Thus from Lemma 2, we have $\nabla_s a_m = \nabla_m a_s$ or $B_{hijk} = 0$. Since B_{hijk} cannot be zero, our theorem is proved.

T h e o r e m 4. If M is S.P.R. with respect to the connection $\overset{g}{\Gamma}{}^h_{ij}$, then F = 0, i.e.,

$$(3.6) \qquad \nabla_j p_i = p_j \, p_i \; .$$

P r o o f. The equation (3.4), by making use of (3.1) and Ricci-identty, can be written in the form

$$(3.7) \qquad R_{rijk} R^r_{hms} + R_{hrjk} R^r_{ims} + R_{hirk} R^r_{jms} + R_{hijr} R^r_{kms} =$$
$$= F \, [g_{hs} B_{mijk} - g_{hm} B_{sijk} + g_{is} B_{hmjk} - g_{im} B_{hsjk} +$$
$$+ g_{js} B_{himk} - g_{jm} B_{hisk} + g_{ks} B_{hijm} - g_{km} B_{hijs}] \; .$$

Applying ∇_z to the both sides of (3.7) and using the Ricci-identity and (3.2), we find

$$(3.8) \qquad F \, [B_{hizk}(p_s \, g_{jm} - p_m \, g_{js}) + B_{hijz}(p_s \, g_{mk} - p_m \, g_{ks})] = 0.$$

Contracting now (3.8) with g^{sk} , we easily obtain

$$(2- n) \, F \, p_m \, B_{hijz} = 0.$$

Since the space is not of constant curvature by assumption, the las formula gives F = 0, which completes the proof.

The following theorem is an immediate consequence of (2.7) and Theorem 4.

T h e o r e m . 5. A S.P.R. space with respect to the connection $\overset{2}{\Gamma}{}^h_{ij}$ with definite metric can not exist.

If (M,g) admits a vector field p_i such that (1.7) and (1.8) hold, then from (3.6) we have $\nabla_m \varphi_i = 0$, where $\varphi_i = e^{-p} p_i$ and $\partial_i p = p_i$.

From the above form of φ_i it is easily seen that $\varphi_r \varphi^r = 0$.

The above discussion leads us to consider a space (M,g) which admits a null paralell vector field. We now have the following theorem (for detail see [5], p.40).

T h e o r e m 6. If (M,g) admits a vector field p_i such that (1.7) and (1.8) hold, then coordinates can be chosen so that the metric takes the form

$$2dx^1 dx^n + g_{\alpha\beta}\, dx^\alpha\, dx^\beta , \quad \alpha,\beta = 1,2,\dots,n-1 ,$$

where the $g_{\alpha\beta}$ are independent of x^n . The vector field p_i is equal $\delta^i_n e^p$.

T h e o r e m 7. If (M,g) admits a vector field p_i such that (1.7) and (1.8) hold, then $w_r R^r_h = \frac{R}{2} w_h$ and

$$(3.9) \quad w^r w_r R_{hijk} = w_h w_k R_{ij} - w_h w_j R_{ik} - w_i w_k R_{hj}$$

where $w_i = a_i - 2p_i$.

P r o o f. Substituting F = 0 into (3.2), we get

$$(3.10) \quad \nabla_m R_{hijk} = a_m R_{hijk} + p_h R_{mijk} + p_i R_{hmjk} + p_j R_{himk} + p_k R_{hijm} .$$

Summing the equality (3.10) cyclically in m, h, i and using the first and second Bianchi identity, we obtain

$$(3.11)\qquad w_m R_{hijk} + w_i R_{mhjk} + w_h R_{imjk} = 0\ ,$$

where $w_i = a_i - 2p_i$. Transvecting (3.11) with p^m and using $p^r R_{rijk} = 0$, $R_{hijk} \neq 0$, we find $p^r w_r = 0$, which, by $p^r p_r = 0$, implies $p^r a_r = 0$ and $w^r w_r = a^r a_r$. Contracting (3.11) with g^{mk} and then with g^{ij} , we find

$$(3.12)\qquad w_r R^r_{jih} = w_h R_{ij} - w_i R_{hj}\ ,\qquad w_r R^r_h = \frac{R}{2} w_h\ .$$

Transvecting (3.11) with w^m we find

$$w^r w_r R_{hijk} = w^r(w_h R_{rijk} - w_i R_{rhjk})\ .$$

By (3.12), this yelds (3.9). This completes the proof.

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