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# SOME SUFFICIENT CONDITIONS FOR UNIVALENCE OF HOLOMORPHIC FUNCTIONS

The following theorem is well known (cf. [1], [2], [3]) :

Theorem A. If D is a convex domain in the complex plane C, $f : D \longrightarrow C$ is holomorphic in D, and $\operatorname{re} f'(z) > 0$ for $z \in D$, then f is univalent in D.

In the present paper we shall deal with some generalizations of this theorem. The essence of those generalizations consists in replacing the condition $\operatorname{re} f'(z) > 0$ by the condition $\operatorname{re} f'(z) + \varphi(\operatorname{im} f(z)) \operatorname{im} f'(z) > 0$ where $\varphi$ is some real function of a real variable.

1. First, we shall prove the following

Theorem 1. If $D \subset C$ is a convex domain, $f : D \longrightarrow C$ is holomorphic in D, $\varphi : R \longrightarrow R$ is a continuous function in R, and

(1) $$\operatorname{re} f'(z) + \varphi(\operatorname{im} f(z)) \operatorname{im} f'(z) > 0 , \quad z \in D,$$

then f is univalent in D.

Proof. Let $z_1, z_2 \in D$, $z_1 \neq z_2$. We may assume that $\alpha \overset{df}{=} \operatorname{Arg}(z_2 - z_1) \in \langle 0, \Pi)$ since the contrary case reduces to this one in consequence of changing $z_1$ to $z_2$ and $z_2$ to $z_1$. Let $p(t) = z_1 + t(z_2 - z_1)$ , $F(t) = f(p(t))$, $t \in \langle 0, 1\rangle$.

If $\alpha = 0$, then, denoting by g any of the primitives of the function $\varphi$ in R and putting

$$s(t) = \text{re } F(t) + g(\text{im } F(t)) \ , \ t \in \langle 0,1\rangle \ ,$$

we have

$$(2) \quad s'(t) = (z_2-z_1)[\text{re } f'(p(t)) + \varphi(\text{im } f(p(t)))\text{im } f'(p(t))], \quad t \in \langle 0,1\rangle.$$

Hence and from (1) it follows that $s' > 0$. So, $s(0) \neq s(1)$ and, in consequence, $f(z_1) \neq f(z_2)$.

Assume now that $\alpha \in (0,\Pi)$, i.e. that $\text{im}(z_2 - z_1) > 0$. The following two cases are possible : I) $\varphi(\text{im } F(t)) \neq \text{ctg}\alpha$ for $t \in \langle 0,1\rangle$ , II) $\varphi(\text{im } F(t_1)) = \text{ctg}\alpha$ for some $t_1 \in \langle 0,1\rangle$.

Ad I. Let $a = \min_{t\in\langle 0,1\rangle} \text{im } F(t)$, $b = \max_{t\in\langle 0,1\rangle} \text{im } F(t)$.

Of course, $\varphi(x) \neq \text{ctg}\alpha$ for $x \in \langle a,b\rangle$. Let g denote any of the primitives of the function

$$(3) \qquad \frac{1 + \varphi \ \text{ctg}\alpha}{\text{ctg}\alpha - \varphi}$$

in the interval $\langle a,b\rangle$ when $a < b$ and let g denote a function equal to 0 at the point a when $a = b$.

Put

$$s(t) = \text{re } F(t) + g(\text{im } F(t)), \quad t \in \langle 0,1\rangle \ .$$

Evidently,

$$s'(t) = \text{re } F'(t) + \frac{1 + \varphi(\text{im } F(t))\text{ctg}\alpha}{\text{ctg}\alpha - \varphi(\text{im } F(t))} \text{im } F'(t) \ , \ t \in \langle 0,1\rangle.$$

Hence and from the equalities

$$(4) \quad \text{re } F'(t) = \text{im}(z_2 - z_1) \ [\text{re } f'(p(t)) \ \text{ctg}\alpha - \text{im } f'(p(t))], \quad t \in \langle 0,1\rangle \ ,$$

(4') $\operatorname{im} F'(t) = \operatorname{im}(z_2 - z_1)\,[\operatorname{re} f'(p(t)) + \operatorname{im} f'(p(t))\operatorname{ctg}\alpha\,]$,
$t \in \langle 0,1\rangle$,

after easy calculations we obtain

$$(5)\qquad s'(t) = \frac{(1+\operatorname{ctg}^2\alpha)\,\operatorname{im}(z_2-z_1)}{\operatorname{ctg}\alpha - \varphi(\operatorname{im} F(t))}\cdot$$

$$\cdot\,[\operatorname{re} f'(p(t)) + \varphi(\operatorname{im} F(t))\operatorname{im} f'(p(t))]$$

for $t \in \langle 0,1\rangle$. The denominator of the above expression, as a function continuous and non-vanishing in $\langle 0,1\rangle$, has a constant sign in $\langle 0,1\rangle$. Furthermore, taking account of (1), we see that s' has a constant sign in $\langle 0,1\rangle$. So, $s(0) \neq s(1)$ and, in consequence, $f(z_1) \neq f(z_2)$.

Ad II. Assume first that $t_1 \in \langle 0,1\rangle$, where $t_1$ is such as in the definition of this case. From (1) it follows that re $f'(p(t_1)) + \operatorname{ctg}\alpha \operatorname{im} f'(p(t_1)) > 0$. This and (4') imply that $\operatorname{im} F'(t_1) > 0$. Consequently, there exist $t' \in (0,t_1)$ and $t'' \in (t_1,1)$ such that

$$(6)\qquad \operatorname{im} F(t) > \operatorname{im} F(t_1) \quad \text{for } t \in (t_1,t'')\ ,$$

$$(6')\qquad \operatorname{im} F(t) < \operatorname{im} F(t_1) \quad \text{for } t \in (t',t_1)\ .$$

In order to prove that $f(z_1) \neq f(z_2)$, it suffices to show that $\operatorname{im} f(z_2) > \operatorname{im} F(t_1) > \operatorname{im} f(z_1)$.

Let us first suppose that $\operatorname{im} f(z_1) \geq \operatorname{im} F(t_1)$. This and (6') imply that there exists $t \in \langle 0,t'\rangle$ such that $\operatorname{im} F(t) = \operatorname{im} F(t_1)$. Let $\tau_1 = \max\{t \in \langle 0,t'\rangle\ ;\ \operatorname{im} F(t) = \operatorname{im} F(t_1)\}$. Obviously, $\operatorname{im} F(t) < \operatorname{im} F(t_1) = \operatorname{im} F(\tau_1)$ for $t \in (\tau_1,t_1)$. Consequently,

$$\text{(7)} \qquad \operatorname{im} F'(\tau_1) = \lim_{t \to \tau_1^+} \frac{\operatorname{im} F(t) - \operatorname{im} F(\tau_1)}{t - \tau_1} \le 0 .$$

On the other hand, from the fact that $\varphi(\operatorname{im} F(\tau_1)) = \operatorname{ctg}\alpha$ and from (1) and (4') it follows that $\operatorname{im} F'(\tau_1) > 0$, which contradicts (7).

Suppose now that $\operatorname{im} f(z_2) \le \operatorname{im} F(t_1)$. This and (6) imply that there exists $t \in \langle t'', 1\rangle$ such that $\operatorname{im} F(t) = \operatorname{im} F(t_1)$. Let $\tau_2 = \min \{t \in \langle t'', 1\rangle ; \operatorname{im} F(t) = \operatorname{im} F(t_1)\}$. Of course, $\operatorname{im} F(t) > \operatorname{im} F(t_1) = \operatorname{im} F(\tau_2)$ for $t \in (t_1, \tau_2)$.
Consequently,

$$\text{(8)} \qquad \operatorname{im} F'(\tau_2) = \lim_{t \to \tau_2^-} \frac{\operatorname{im} F(t) - \operatorname{im} F(\tau_2)}{t - \tau_2} \le 0.$$

On the other hand, from the fact that $\varphi(\operatorname{im} F(\tau_2)) = \operatorname{ctg}\alpha$ and from (1) and (4') it follows that $\operatorname{im} F'(\tau_2) > 0$, which contradicts (8).

If $t_1 = 0$, then, analogously as before, we prove that $\operatorname{im} f(z_2) > \operatorname{im} F(0)$, whereas if $t_1 = 1$, then, analogously as before, we prove that $\operatorname{im} f(z_1) < \operatorname{im} F(1)$, which completes the proof of the theorem.

R e m a r k. If $\alpha \in R$, and $f$ is a complex function, then $f$ is univalent if and only if $e^{i\alpha}f$ is univalent. Theorems A and 1 can therefore be strengthened by replacing the conditions $\operatorname{re} f' > 0$ and (2), respectively, by $\operatorname{re} e^{i\alpha}f' > 0$ and $\operatorname{re} e^{i\alpha}f'(z) + \varphi(\operatorname{im} e^{i\alpha}f(z)) \operatorname{im} e^{i\alpha}f'(z) > 0$, $z \in D$, for some $\alpha \in R$. In particular, with $\alpha = -\Pi/2$, condition (2) can be replaced by $\operatorname{im} f'(z) + \varphi(\operatorname{re} f(z)) \operatorname{re} f'(z) > 0$, $z \in D$, since the function $\varphi$ can also be replaced by $-\varphi(-x)$, $x \in R$.

2. As an application of Theorem 1 let us consider a function $f(z) = -z \operatorname{Log} z$, $\operatorname{re} z > 0$. Fix any number $A > e$ and put $\varphi(x) = -Ax$, $x \in R$. After easy calculations we obtain

$$H(z) \overset{df}{=} \operatorname{re} f'(z) + \varphi(\operatorname{im} f(z)) \operatorname{im} f'(z) =$$
$$= -(1 + \operatorname{Log} r) - Art(\cos t + \sin t \operatorname{Log} r)$$

where $r = |z|$, $t = \operatorname{Arg} z$, $\operatorname{re} z > 0$.

Since $t(\cos t + \sin t \operatorname{Log} r) \le 0$ for $t \in (-\Pi/2, \Pi.2)$, $r \in (0, e^{-1})$, therefore

$$H(z) \ge -(1 + \operatorname{Log} r) - t(\cos t + \sin t \operatorname{Log} r)$$

for $t \in (-\Pi/2, \Pi/2)$, $r \in (A^{-1}, e^{-1})$. For $t \in (-\Pi/2.\Pi/2)$, $r > 0$ the right-hand side of the above inequality is greater than zero if and only if

$$r < \exp\left(-\frac{1 + t^2\cos t}{1 + t\sin t}\right).$$

Consequently, putting

$$p(t) = \exp\left(-\frac{1 + t^2\cos t}{1 + t\sin t}\right),$$

$$D = \{\, z = re^{it} \,;\, t \in (-\Pi/2, \Pi/2),\ r \in (A^{-1}, p(t))\},$$

we see that $H(z) > 0$ for $z \in D$. It is also easy to verify that $p(t) > e^{-1}$ for $t \in (-\Pi/2, \Pi/2)$.

Denote by $\Gamma$ a curve with the following equation in polar coordinates

$$r = p(t)\ ,\quad t \in (-\Pi/2, \Pi/2).$$

From the theory of implicit functions it easily follows that, in some neighbourhood of the point $(e^{-1}, 0)$, the graph of the curve $\Gamma$ is a graph of some function $g$ of the variable $y$. Since, as can easily be checked, $g''(0) = -e < 0$, there exists $\delta > 0$ such that $g''(y) < 0$ for $y \in (-\delta,\delta)$. Consequently, the function g is concave in the interval $(-\delta,\delta)$. Hence it follows that the set

$$D_\delta \overset{df}{=} \{z = re^{it} ;\ t \in (-\alpha,\alpha),\ \mathrm{re}\, z > A^{-1},\ r < p(t)\},$$

where $\alpha = \mathrm{Arg}(g(\delta) + i\delta)$, is convex. Since $D_\delta \subset D$, therefore $H(z) > 0$ for $z \in D_\delta$ . Thus, in virtue of Theorem 1, the function f is univalent in the domain $D_\delta$ .

Let us still notice that

$$f'(z) > 0 \text{ for } z \in (0,e^{-1}) ,$$

$$\mathrm{Arg}\, f'(z) \in (\Pi/2,\Pi) \text{ for } |z| > e^{-1},\ \mathrm{Arg}\, z \in (-\Pi/2,0),$$

$$\mathrm{Arg}\, f'(z) \in (-\Pi,-\Pi/2) \text{ for } |z| > e^{-1},\ \mathrm{Arg}\, z \in (0,\Pi/2).$$

Hence it follows that the set $f'(D)$ is contained in none of the half-planes $P_\gamma = \{z : \mathrm{re}\, e^{i\gamma}z > 0 \}$, $\gamma \in \langle 0,2\Pi)$. So, the univalence of the function f in the set $D_\delta$ cannot be ascertained on the basis of Theorem A or its modified version in which $\mathrm{re}\, f' > 0$ is replaced by $\mathrm{re}\, e^{i\gamma}f' > 0$ for some $\gamma \in \langle 0,2\Pi)$.

3. The assumption about the continuity of the function $\varphi$, occuring in Theorem 1, can be weakened. For this purpose, let us denote by $\Phi$ the set of all functions $\varphi$ for which there exists a finite or infinite sequence $\ldots < x_{-1} < x_0 < x_1 < \ldots$ of real numbers, such that $\varphi : R - \bigcup_i \{x_i\} \longrightarrow R$, $\varphi$ is continuous and, at all points $x_i$, there exist finite limits

$$q_l^{(1)} \overset{df}{=} \lim_{x \to x_l^-} \varphi(x) , \qquad q_l^{(2)} \overset{df}{=} \lim_{x \to x_l^+} \varphi(x) .$$

T h e o r e m 2. If $D$ is a convex domain in $C$, $f : D \longrightarrow C$ is holomorphic in $D$, $\varphi \in \Phi$ and

(9) $\quad \operatorname{re} f'(z) + \varphi(\operatorname{im} f(z)) \operatorname{im} f'(z) > 0$ for $z \in D$, $\operatorname{im} f(z) \notin \bigcup_l \{x_l\}$ ,

(10) $\operatorname{re} f'(z) + q_l^{(k)} \operatorname{im} f'(z) > 0$ for $k = 1,2$ and those $z \in D$ for which there exists $l$ such that $\operatorname{im} f(z) = x_l$,

then $f$ is univalent in $D$.

P r o o f. With no essential loss of generality we may assume that sequence $\dots x_{-1}, x_0, x_1, \dots$ is one-element and consists of the element $x_0$ . Let $z_1, z_2 \in D$, $z_1 \neq z_2$ . We may assume that $\alpha \overset{df}{=} \operatorname{Arg}(z_2 - z_1) \in \langle 0, \Pi)$. Let $p(t) = z_1 + t(z_2 - z_1)$, $F(t) = f(p(t))$, $t \in \langle 0, 1\rangle$.

Assume first that $\alpha = 0$. From the assumptions concerning the function $\varphi$ it follows that there exists a function $g$ , $g: R \longrightarrow R$ such that $g'(x) = \varphi(x)$ for $x \neq x_0$, $g'_+(x_0) = q_0^{(2)}$, $g'_-(x_0) = q_0^{(1)}$.

Put $s(t) = \operatorname{re} F(t) + g(\operatorname{im} F(t))$, $t \in \langle 0, 1\rangle$. The functions is, of course, continuous.

In order to demonstrate that $f(z_1) \neq f(z_2)$, it is sufficient to prove that $s$ is increasing; to that end, it is enough to show that, at any point $t \in \langle 0, 1\rangle$ the lower Darboux derivative of the function $s$ at the point $t$, which will be denoted by $s'_d(t)$, is greater than zero. So, let us take any $t \in \langle 0, 1\rangle$. If $\operatorname{im} F(t) \neq x_0$, then (2) holds. Consequently, taking (9) into account, we see that $s'_d(t) = s'(t) > 0$. Assume now that $\mathrm{im}\, F(t) = x_0$. There exists a sequence $(t_n)_{n\in N}$ of elements of the interval <0,1> different from t, converging to t and such that

$$s'_d(t) = \lim_{n\to\infty} \frac{s(t_n)-s(t)}{t_n-t}.$$

From the sequence $(t_n)_{n\in N}$ one can choose a subsequence $(t_{n_k})_{k\in N}$ such that $\mathrm{im}\, F(t_{n_k}) > x_0$ for $k \in N$ or $\mathrm{im}\, F(t_{n_k}) < x_0$ for $k \in N$ or $\mathrm{im}\, F(t_{n_k}) = x_0$ for $k \in N$. Then we have, respectively,

$$(11)\quad \begin{cases} s'_d(t) = (z_2-z_1)[\mathrm{re}\, f'(p(t)) + q_0^{(2)}\, \mathrm{im}\, f'(p(t))], \\ s'_d(t) = (z_2-z_1)[\mathrm{re}\, f'(p(t)) + q_0^{(1)}\, \mathrm{im}\, f'(p(t))], \\ s'_d(t) = (z_2-z_1)\, \mathrm{re}\, f'(p(t)) = \\ \qquad = (z_2-z_1)\, [\mathrm{re}\, f'(p(t)) + q_0^{(1)}\, \mathrm{im}\, f'(p(t))] \end{cases}$$

because, in this last case, $\mathrm{im}\, f'(p(t)) = 0$. Since the right-hand sides of the above expressions are, by (10), greater than zero, therefore $s'_d(t) > 0$.

Assume now that $\alpha \in (0,\Pi)$. Let

$A = \{t\in <0,1>;\ \mathrm{im}\, F(t) \neq x_0\}$, $B = \{t \in <0,1>;\ \mathrm{im}\, F(t) = x_0\}$,

$$a = \min_{t\in<0,1>} \mathrm{im}\, F(t), \quad b = \max_{t\in<0,1>} \mathrm{im}\, F(t).$$

There must occur one of the following three cases :

I) $\left[ B \neq \emptyset \wedge (q_0^{(1)} - \mathrm{ctg}\alpha)(q_0^{(2)} - \mathrm{ctg}\alpha) \leq 0 \right]$

$$\vee \exists_{t_1 \in A} \varphi(\mathrm{im}\, F(t_1)) = \mathrm{ctg}\alpha ,$$

II) $B = \emptyset \wedge \forall_{t \in \langle 0,1 \rangle} \varphi(\mathrm{im}\, F(t)) \neq \mathrm{ctg}\alpha$ ,

III) $B \neq \emptyset \wedge (q_0^{(1)} - \mathrm{ctg}\alpha)(q_0^{(2)} - \mathrm{ctg}\alpha) > 0$

$$\wedge \forall_{t \in A} \varphi(\mathrm{im}\, F(t)) \neq \mathrm{ctg}\alpha .$$

Ad I. If the second part of alternative I holds, we may proceed in the same way as in case II of the proof of Theorem 1. Assume now that the first part of alternative I holds. Let $t \in B$. From the inequalities $\mathrm{re}\, f'(p(t)) + q_0^{(2)} \cdot \mathrm{im}\, f'(p(t)) > 0$, $\mathrm{re}\, f'(p(t)) + q_0^{(1)} \mathrm{im}\, f'(p(t)) > 0$, $(q_0^{(1)} - \mathrm{ctg}\alpha)(q_0^{(2)} - \mathrm{ctg}\alpha) \leq 0$ it easily follows that $\mathrm{re}\, f'(p(t)) + \mathrm{ctg}\alpha\, \mathrm{im}\, f'(p(t)) > 0$ . This and (4') imply that

$$\mathrm{im}\, F'(t) > 0 . \tag{12}$$

Let us fix $t_1 \in B$. Further, one can repeat the considerations included in case II of the proof of Theorem 1. The only change will be the justification of the inequalities $\mathrm{im}\, F'(t_1) > 0$, $\mathrm{im}\, F'(\tau_1) > 0$, $\mathrm{im}\, F'(\tau_2) > 0$ which follow from (12).

Ad II. We proceed in the same manner as in case I of the proof of Theorem 1.

Ad III. It is not difficult to notice that the difference $\varphi(x) - \mathrm{ctg}\alpha$ has a constant sign in $\langle a,b \rangle - \{x_0\}$

equal to the sign of the numbers $q_0^{(k)} - \mathrm{ctg}\alpha$ , $k = 1,2$. Without any loss of generality let us assume that these signes are positive. Consequently, there exists a real function g defined in some open interval containing <a,b> and such that

$$g'(x) = \frac{1 + \varphi(x)\ \mathrm{ctg}\alpha}{\mathrm{ctg}\alpha - \varphi(x)}, \quad x \in \langle a,b\rangle - \{x_0\},$$

$$g'_+(x_0) = \frac{1 + q_0^{(2)}\mathrm{ctg}\alpha}{\mathrm{ctg}\alpha - q_0^{(2)}}, \quad g'_-(x_0) = \frac{1 + q_0^{(1)}\mathrm{ctg}\alpha}{\mathrm{ctg}\alpha - q_0^{(1)}}.$$

Put $s(t) = \mathrm{re}\ F(t) + g(\mathrm{im}\ F(t))$, $t \in \langle 0,1\rangle$. Obviously, s is continuous. Proceeding in the same way as in the proof of equality (5), we assertain that $s'(t)$ is expressed by formula (5) for $t \in A$. Consequently, $s'_d(t) = s'(t) > 0$ for $t \in A$. Whereas proceeding similarly as in the proof of equalities (11), we easily find that, at any point $t \in B$, the lower derivative $s'_d(t)$ is equal to one of the three numbers of which the first two are the following

$$(13)\quad \frac{(1+\mathrm{ctg}^2\alpha)\ \mathrm{im}(z_2-z_1)}{\mathrm{ctg}\alpha - q_0^{(k)}}\left[\mathrm{re}\ f'(p(t))+q_0^{(k)}\mathrm{im}\ f'(p(t))\right],\ k=2,1$$

while the third one, corresponding to the case, $(\mathrm{im}\ F(t_{n_k}) = x_0)$ for $k \in N$, is equal to $\mathrm{re}\ F'(t)$. Since, in this case, $\mathrm{im}\ F'(t) = 0$, therefore from (4) and (4') it easily follows that $\mathrm{re}\ F'(t)$ is equal to numbers (13). Since numbers (13) are greater than zero, therefore $s'_d(t) > 0$, which ends the proof.

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