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SOME ESTIMATIONS OF AVERAGED MODULI OF ENTIRE FUNCTIONS IN ORLICZ SPACES

1. Preliminaries

In [1]-[3] we obtained some inequalities between two norms of trigonometric polynomials in $L^{\varphi}_{2\pi}$, where φ is a φ-function : the Luxemburg norm and the Nikolskiĭ-type norm. We thus generalized results known in the case of $L^{p}_{2\pi}$-spaces. Also, we applied these inequalities in order to estimate the averaged L^{φ}-moduli of continuity and smoothness of a bounded, measurable function by means of the norm of its derivative. This was applied to obtain converse approximations theorems for averaged moduli in Orlicz spaces.

The inequalities for trygonometric polynomials in $L^{p}_{2\pi}$ were transferred in [4]-[7] to the case of entire functions of finite expotential order, belonging to $L^{p}(-\infty,\infty)$.

The aim of this paper is to extend tha later results from spaces $L^{p}(-\infty,\infty)$, $0 < p < \infty$, to the case of Orlicz spaces $L^{\varphi} = L^{\varphi}(-\infty,\infty)$, both for convex and for concave functions φ. Also, we give estimations both for the norms and for the modulars. The resulting inequalities may be applied to problems of converse approximation theorems in the above classes of functions.

Let φ be a φ-function and let $f \in L^{\varphi}$. We write

$$\rho_{\varphi}(f) = \int_{-\infty}^{\infty} \varphi(|f(x)|)\, dx$$

and

$$\rho_{\varphi}^{h}(f) = h \sup_{-\infty<x<\infty} \sum_{j=-\infty}^{\infty} \varphi(|f(x+v_j)|), \quad h > 0,$$

where $v_j = jh$, $j = 0, \pm1, \pm2, \ldots$ These two modulars define two norms in the respective modular spaces. In the case of convex φ, this is the Luxemburg norm

$$\|f\|_{\varphi} = \inf\{u > 0 : \rho_{\varphi}(f/u) \le 1\}$$

and the Nikolskiĭ-type norm

$$\|f\|_{\varphi}^{h} = \inf\{u > 0 : \rho_{\varphi}^{h}(f/u) \le 1\}.$$

In the case of s-convex φ with $0 < s \le 1$, these are the s-homogeneous norms

$$\|f\|_{s,\varphi} = \inf\{ u > 0 : \rho_{\varphi}(f/u^{1/s}) \le 1\}$$

and

$$\|f\|_{s,\varphi}^{h} = \inf\{ u > 0 : \rho_{\varphi}^{h}(f/u^{1/s}) \le 1\}.$$

Besides the above, we shall need modulars and norms for sequences $\bar{w} = (w_j)_{j=-\infty}^{\infty}$, defined as follows :

$$\rho_{\varphi}^{(h)}(\bar{w}) = h \sum_{j=-\infty}^{\infty} \varphi(|w_j|), \quad h > 0$$

and

$$\|\bar{w}\|_{\varphi}^{(h)} = \inf\{u > 0 : \rho_{\varphi}^{(h)}(\bar{w}/u) \le 1\}$$

in case of convex φ,

$$\|\bar{w}\|_{s,\varphi}^{(h)} = \inf \{u > 0 : \rho_{\varphi}^{(h)}(\bar{w}/u^{1/s}) \le 1 \}$$

in case of s-convex φ, where $0 < s \le 1$.

2. Estimations for Nikolskiĭ-type norms

Let E_σ be the space of entire functions of expotential type $\sigma > 0$, and let $B_{\sigma,\varphi}$ be the space of all $f \in E_\sigma$ such that the restriction of f to the real axis belongs to L^φ.

The following transfers the result of [2], Lemma 1, to the case of entire functions in place of trigonometric polynomials :

L e m m a. Let φ be a convex φ-function and let $f \in B_{\sigma,\varphi}$. Let us write $f\bar{v} = (f(v_j))_{j=-\infty}^{\infty}$ with $v_j = jh$, $j = 0, \pm 1, \pm 2, \ldots$ Then there hold the following inequalities :

(a) $$\|f\bar{v}\|_{\varphi}^{(h)} \le (1 + \sigma h)\|f\|_{\varphi} ,$$

(b) if $\sigma h > 1$, then $\rho_{\varphi}^{(h)}(f\bar{v}) \le \frac{1}{2}(1 + \sigma h)\, \rho_{\varphi}(2f)$.

P r o o f. Let $|f(\eta_j)| = \min\limits_{v_j \le t \le v_{j+1}} |f(t)|$, $v_j \le \eta_j \le v_{j+1}$, $\bar{\eta} = (\eta_j)_{j=-\infty}^{\infty}$, and let $u > 0$. Then, by Jensen's inequality,

$$\rho_{\varphi}^{(h)}(f\bar{\eta}/u) \le h \sum_{j=-\infty}^{\infty} \varphi\left[\frac{1}{h}\int_{v_j}^{v_{j+1}} \frac{|f(t)|}{u}\,dt\right] \le \rho_{\varphi}(f/u).$$

Hence

$$\|f\bar{\eta}\|_{\varphi}^{(h)} \le \|f\|_{\varphi} .$$

Thus,

$$\|f\bar{\eta}\|_{\varphi}^{(h)} \le \|f\bar{v} - f\bar{\eta}\|_{\varphi}^{(h)} + \|f\|_{\varphi} .$$

Again by Jensen's inequality, we have

$$\rho_\varphi^{(h)}\left[\frac{f\bar{v} - f\bar{\eta}}{u}\right] \le h\sum_{j=-\infty}^{\infty}\varphi\left(\frac{1}{h}\int_{v_j}^{v_{j+1}}\frac{h|f'(s)|}{u}\,ds\right) \le \rho_\varphi\left[\frac{hf'}{u}\right] .$$

But, by Bernstein's inequality ([8], p.277) ,

$$\rho_\varphi(\sigma^{-1}f') \le \rho_\varphi(f) ,$$

whence

$$\rho_\varphi^{(h)}\left[\frac{f\bar{v} - f\bar{\eta}}{u}\right] \le \rho_\varphi\left[\frac{\sigma hf}{u}\right] \text{ for } u > 0.$$

Consequently,

$$\|f\bar{v} - f\bar{\eta}\|_\varphi^{(h)} \le \sigma\, h\, \|f\|_\varphi$$

and so

$$\|f\bar{v}\|_\varphi^{(h)} \le (1 + \sigma h)\, \|f\|_\varphi .$$

Next, supposing $\sigma h \le 1$, we have

$$\rho_\varphi^{(h)} \le \frac{1}{2}\,\rho_\varphi^{(h)}(2(f\bar{v}-f\bar{\eta})) + \frac{1}{2}\,\rho_\varphi^{(h)}(2f\bar{\eta}) \le \frac{1}{2}\,\rho_\varphi(2\sigma hf) + \frac{1}{2}\,\rho_\varphi(2f) \le$$

$$\le \frac{1}{2}\,(1 + \sigma h)\,\rho_\varphi(2f).$$

Applying the above Lemma, we prove now the following

T h e o r e m 1. Let φ be a convex φ-function and let $f \in B_{\sigma,\varphi}$. Then

(a) $$\|f\|_\varphi \le \|f\|_\varphi^h \le (1 + h\sigma)\|f\|_\varphi ,$$

(b) if $\sigma h \le 1$, then $\rho_\varphi(f) \le \rho_\varphi^h(f) \le \frac{1}{2}\,(1 + \sigma h)\,\rho_\varphi(2f)$.

P r o o f. Since

$$\rho_\varphi(f) = \sum_{j=-\infty}^{\infty} \int_0^h \varphi(|f(t+v_j)|)\, dt \le \rho_\varphi^h(f) ,$$

we obtain the left-hand side inequality without assumption of convexity of φ, immediately.

Let $f \in B_{\sigma,\varphi}$, $g(z) = f(x+z)$ for a fixed real x. Then $g \in B_{\sigma,\varphi}$, $\|g\|_\varphi = \|f\|_\varphi$, and applying the Lemma we obtain

$$\|g\bar{v}\|_\varphi^{(h)} \le (1 + \sigma h)\|f\|_\varphi$$

and for $\sigma h \le 1$

$$\rho_\varphi^{(h)}(g\bar{v}) \le \frac{1}{2}\,(1 + \sigma h)\ \rho_\varphi(2f).$$

Thus, taking $\sigma h \le 1$ we have

$$h \sum_{j=-\infty}^{\infty} \varphi(|f(x+v_j)|) \le \frac{1}{2}\,(1 + \sigma h)\ \rho_\varphi(2f) ,$$

for all real x. Hence we get the right-hand inequality in (b).

In order to prove the right-hand inequality in (a), we note that

$$\left\| \frac{\delta\ g\bar{v}}{(1+\sigma h)\|f\|_\varphi} \right\|_\varphi^{(h)} \le \delta$$

for every $0 < \delta < 1$, whence

$$\rho_\varphi^{(h)}\left(\frac{\delta\ g\bar{v}}{(1+\sigma h)\|f\|_\varphi} \right) \le \delta < 1$$

and taking $\delta \uparrow 1$, we obtain

$$\rho_\varphi^{(h)}\left(\frac{g\bar{v}}{(1+\sigma h)\|f\|_\varphi} \right) \le 1 .$$

This gives the desired inequality easily.

Let us remark, that Theorem 1 transforms the result of [2], Th. 1 and Corollary, form trygonometric polynomials to entire functions. It is also a generalization of Th. 3.3.1 in [4] p.122 from spaces L^p to Orlicz spaces.

Now, let us recall that a φ-function φ is said to be s-convex, if $\varphi(\alpha u+\beta v) \le \alpha^s\varphi(u) + \beta^s\varphi(v)$ for $u,v \ge 0$, $\alpha,\beta \ge 0$, $\alpha^s + \beta^s = 1$, and is said to be strongly s-convex, if $\tilde{\varphi}(u) = \varphi(u^{1/s})$ is convex $(0 < s \le 1)$. If φ satisfies the condition (Δ_2) for all $u \ge 0$, then

$$\psi(u) = \sup_{v>0} \frac{\varphi(uv)}{\varphi(v)} < \infty \text{ for all } u > 0.$$

If φ is s-convex (strongly s-convex), then so is ψ (see [2]). There holds the following

T h e o r e m 2. Let φ be a concave, s-convex φ-function with some $s \in (0,1\rangle$ and let it satisfy (Δ_2) for all $u \ge 0$. Let $f \in B_{\sigma,\varphi}$. Then there hold the inequalities

(a) $$\|f\|_{s,\varphi} \le \|f\|^h_{s,\varphi} \le C(s)\,\frac{1}{\sigma h}\,(1+\sigma h)\,\|f\|_{s,\varphi}$$

(b) $$\rho_\varphi(f) \le \rho^h_\varphi(f) \le C(s)\,\frac{1}{\sigma h}\,(1+\sigma h)\,\rho_\varphi(f)\,,$$

where $h > 0$ and $C(s)$ is a positive constant.

P r o o f. The left-hand side inequalities being shown in the proof of Theorem 1, we are going to prove the right-hand ones. Let r be the least integer such that $rs \ge 2$, and let $t_k = v_k = kh$, $k = 0,\pm1,\pm2,\dots$, $s_k = t_k - x$. By Lemma 2.1, we have

$$f(x) = \sum_{k=-\infty}^{\infty} f(t_k)\,\frac{\sin 2\sigma(x-t_k)}{2\sigma(x-t_k)}\left\{\frac{\sin \frac{\sigma}{r}(x-t_k)}{\frac{\sigma}{r}(x-t_k)}\right\}^{r-1}$$

for real x, the above quotients being replaced by 1 for $x = t_k$.

We obtain

$$h \sum_{j=-\infty}^{\infty} \varphi(|f(x+v_j)|) \le h \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \varphi \left\{ |f(t_k)| \left| \frac{\sin\frac{\sigma}{r}(x+v_j-t_k)}{\frac{\sigma}{r}(x+v_j-t_k)} \right|^r \right\} \le$$

$$\le h \sum_{k=-\infty}^{\infty} \varphi(|f(t_k)|)\ Y_k ,$$

where

$$Y_k = \left[\sum_{|v_j-s_k|\le\frac{1}{\sigma}} + \sum_{|v_j-s_k|>\frac{1}{\sigma}} \right] \psi \left(\left| \frac{\sin\frac{\sigma}{r}(v_j-s_k)}{\frac{\sigma}{r}\ (v_j-s_k)} \right|^r \right) = A_k + B_k.$$

It easily to seen that

$$A_k \le 2 \left[\frac{1}{\sigma h} + 1 \right] .$$

Moreover, we have

$$B_k \le \sum_{|v_j-s_k|>\frac{1}{\sigma}} \psi\left[\frac{1}{|\frac{\sigma}{r}(v_j-s_k)|^r} \right] \le 2\ \psi(r^r) +$$

$$+ \frac{2}{h} \int_{1/\sigma}^{\infty} \psi\left[\frac{1}{(\frac{\sigma}{r}x)^r} \right] dx = 2\ \psi(r^r) + \frac{2}{\sigma h} \int_0^1 \frac{\psi(r^r v^r)}{v^2}\, dv \le$$

$$\le 2\ \psi(r^r) \left[1 + \frac{1}{\sigma h} \frac{1}{rs-1} \right] \le 2\ \psi(r^r) \left[1 + \frac{1}{\sigma h} \right] ,$$

by s-convexity of φ. Hence

$$Y_k \le 2\ (1 + \psi(r^r))\ (1 + \frac{1}{\sigma h}).$$

Consequently,

$$h \sum_{j=-\infty}^{\infty} \varphi(|f(x+v_j)|) \le C(s) \frac{1}{\sigma} (1 + \sigma h) \sum_{k=-\infty}^{\infty} \varphi(|f(t_k)|) \ ,$$

where $C(s) = 2(1 + \psi(r^r))$. Hence

$$\rho_\varphi^h(f) \le C(s) \frac{1}{\sigma} (1 + \sigma h) \sum_{k=-\infty}^{\infty} \varphi(|f(t_k)|) \ ,$$

and replacing $f(z)$ by $f(t+z)$, we obtain

$$\rho_\varphi^h(f) \le C(s) \frac{1}{\sigma} (1 + \sigma h) \sum_{k=-\infty}^{\infty} \varphi(|f(t+t_k)|).$$

Integrating the above inequality with respect to t in $<0,t_1>$, we obtain easily the right-hand one of inequalities (b). Inequalities (a) follow from (b), immediately.

Theorem 2 in case of trigonometric polynomials was given in [2], Th. 2. In case of entire functions with L^p-norms, $0 < p \le 1$, it was obtained by R. Taberski in [5], Th. 7.1, p. 178.

3. Inequality for averaged moduli

First we shall recall the fundamental notions. Let f be a bounded, measurable function on $(-\infty,\infty)$. Then

$$\omega(f;x,\delta) = \sup_{u,v \in I_\delta(x)} |f(u) - f(v)|$$

where $I_\delta(x) = <x - \frac{\delta}{2} , x + \frac{\delta}{2}>$, $-\infty < x \le \infty$, $\delta > 0$, is measurable, and one may define the averaged moduli of smoothness of f in $L^\varphi(-\infty,\infty)$ as follows :

$$\tau^*(f;\delta) = \rho_\varphi(\omega(f;x,\delta)) \ , \quad \tau(f;\delta) = \|\omega(f;\cdot,\delta)\|_{s,\varphi} \ ,$$

where φ is an s-convex φ-function. It is easily seen that both moduli are nondecreasing and subadditive functions of $\delta > 0$.

Next results give estimations of both averaged moduli by means of the norm resp. modular of the derivative of the function f in the Orlicz space. First, we shall deal with convex φ ; in this case the result does not require f to be an entire function and is immediate. Namely, we have

T h e o r e m 3. Let φ be a convex φ-function and let $f \in L^{\varphi}$, f absolutely continuous in $(-\infty,\infty)$. Then

$$\tau^{*}(f;\delta) \le \rho_{\varphi}(\delta f') \ , \ \tau(f;\delta) \le \delta \ \|f'\|_{\varphi} \ .$$

P r o o f. We have, by Jensen's inequality,

$$\tau^{*}(f;\delta) \le \int_{-\infty}^{\infty} \varphi \left[\int_{x-\delta/2}^{x+\delta/2} |f'(t)| \ dt \right] dx \le$$

$$\le \frac{1}{\sigma} \int_{-\infty}^{\infty} \left[\int_{x-\delta/2}^{x+\delta/2} \varphi(\delta|f'(t)|) \ dt \right] dx = \rho_{\varphi}(\delta f').$$

Theorem 3 in case of trigonometric polynomials and k-th derivatives may be found in [1], Theorem 2, p. 288, and in case of entire functions and powers $p \ge 1$, in [6], Prop.2.6.p. 487. The some result in case of trigonometric polynomials and k-th derivatives, but in case of concave, strongly s-convex φ, is given in [3], Prop. 1.3. In case of entire functions and $\varphi(u) = |u|^{p}$ with $0 < p \le 1$ it was obtained by R. Taberski in [7], Lemma, p. 254. These results are generalized by the following.

T h e o r e m 4. Let φ be a concave, strongly s-convex (with some $s \in (0,1\rangle$) φ-function and let it satisfy (Δ_2) for all $u \geq 0$. Let $f \in B_{\sigma,\varphi}$ and $\alpha > 0$. Then for $0 < \delta \leq \alpha/\sigma$ we have

(a) $$\tau^*(f;\delta) \leq C^*(\alpha,s)\, \rho_\varphi(\delta f')\ ,$$

(b) $$\tau(f;\delta) \leq C(\alpha,s)\, \delta^s\, \|f'\|_{s,\varphi}$$

with some positive constants $C^*(\alpha,s)$, $C(\alpha,s)$.

P r o o f. We have (see [7], p. 255)

$$|f(u+h)-f(u)| \leq \int_u^{u+h} \sum_{k=-\infty}^{\infty} |f'(t_k)| \left| \frac{\sin \frac{\sigma}{r}(t-t_k)}{\frac{\sigma}{r}(t-t_k)} \right|^{r-1} dt$$

where $t_k = \frac{\pi}{2\sigma}k$, $k = 0,\pm1, +2,\ldots$, $h > 0$, u real, and r is the least integer such that $(r-1)s > 1$. Hence

$$\omega(f;x,\delta) \leq \int_{x-\delta/2}^{x+\delta/2} \sum_{k=-\infty}^{\infty} |f'(t_k)| \left| \frac{\sin \frac{\sigma}{r}(t-t_k)}{\frac{\sigma}{r}(t-t_k)} \right|^{r-1} dt\ .$$

Consequently, by subadditivity of φ and the definition of ψ, we have

$$\tau^*(f;\delta) \leq \sum_{k=-\infty}^{\infty} \varphi\left[\delta|f'(t_k)| \int_{-\infty}^{\infty} \psi \left\{ \frac{1}{\sigma} \int_{x-\delta/2}^{x+\delta/2} \left| \frac{\sin \frac{\sigma}{r}(t-t_k)}{\frac{\sigma}{r}(t-t_k)} \right|^{r-1} dt \right\} dx.\right.$$

But

$$\max_{t_j-\delta/2 \leq t \leq t_{j+1}+\delta/2} \left| \frac{\sin \frac{\delta}{r} t}{\frac{\delta}{r} t} \right|^{r-1} \leq C_1(\alpha,r) \frac{1}{|j|^{r-1}}$$

(see [6], p. 255). Hence

$$\int_{-\infty}^{\infty}\psi\left\{\frac{1}{\delta}\int_{x-\delta/2}^{x+\delta/2}\left|\frac{\sin\frac{\sigma}{r}(t-t_k)}{\frac{\sigma}{r}(t-t_k)}\right|^{r-1}dt\right\}dx \le \frac{\pi}{2\sigma}\sum_{j=1}^{\infty}\psi\left(\frac{C_1(\alpha,r)}{j^{r-1}}\right) \le$$

$$\le \frac{\pi}{2\sigma}\,\psi(C_1(\alpha,r))\sum_{j=1}^{\infty}\frac{1}{j^{(r-1)s}} = \frac{C_2(\alpha,s)}{\sigma} .$$

Thus

$$\tau^*(f;\delta) \le C_2(\alpha,s)\,\frac{1}{\sigma}\sum_{k=-\infty}^{\infty}\varphi(\delta|f'(t_k)|) ,$$

for $0 < \delta \le \alpha/\sigma$. By Theorem 2 applied to $\delta f'$ in place of f, with $h = \frac{\pi}{2\sigma}$, we obtain

$$\frac{1}{\sigma}\sum_{k=-\infty}^{\infty}\varphi(\delta|f'(t_k)|) \le \frac{2}{\pi}\,\rho_{\varphi}^{\pi/2\sigma}(\delta f') \le C_3(\alpha,s)\,\rho_{\varphi}(\delta f').$$

Hence

$$\tau^*(f'\delta) \le C_3(\alpha,s)\,\rho_{\varphi}(\delta f'),$$

i.e. (a). The inequality (b) follows from (a), immediately.

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