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PRODUCTS OF SETS IN LINEAR GROUPS

In this paper we will generalize results contained in [2]. We will use the following notations : Z(G) denotes the centrum of a group G, $\bar{g}$ denotes the conjugacy class of g in the group G, $|a_{i1},\dots,a_{in}|$ (i=1,...,n) denotes the $\det(a_{ij})$. $SL^-(n,K)$ denotes the set of matrices of GL(n,K) with determinant -1, A+B denotes the matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$, $K_m = \{g \in G : o(g) = m\}$, $V = \mathrm{diag}(v_1,\dots,v_n)$, $W = \mathrm{diag}(w_1,\dots,w_n)$ with $v_i \neq v_j$, $w_i \neq w_j$ for $i \neq j$.
The remaining notation are standard.

In the paper [1] it has been proved that $K_2K_2 \subseteq GL(n,K)$ for $n \geq 3$ and for $n = 2$ if $\mathrm{char}K \neq 2$. In this paper we will prove that if $V,W \in SL(n,K)$ or $V,W \in SL^-(n,K)$ then $SL(n,K) = (\bar{V}\,\bar{W})^2$ in the group GL(n,K) and if $V,W \in SL(n,K)$, where K is an algebraikyly closed field, then $SL(n,K) = (\bar{V}\,\bar{W})^2$ in the group SL(n,K). We will prove also that $SL(n,C) \subseteq (K_2K_2)^2$ in the group GL(n,C), where C is the field of complex numbers.

We begin with the main theorem.

T h e o r e m 1. If $V,W \in GL(n,K)$, $V = \mathrm{diag}(v_1,\dots,v_n)$, $W = \mathrm{diag}(w_1,\dots,w_n)$ with $v_i \neq v_j$, $w_j \neq w_i$ for $i \neq j$, A - the matrix in primary rational canonical form, $\det A = \det VW$, $A \notin Z(GL(n,K))$, then the matrix equation $X^{-1}VXYWY^{-1} = A$ has a solution $X, Y \in GL(n,K)$ such that det X, det Y are arbitrary elements of field the K.

To prove Theorem 1 we will use a few lemma.

L e m m a 1. If $V, W \in GL(n,K)$, A - the companion matrix of the polynomial $f(x) = x^n + a_1x^{n-1} + \ldots + a_n \in K[x]$, det VW = det A, then the matrix equation

$$(1) \qquad X^{-1}VXYWY^{-1} = A$$

has a solution $X, Y \in GL(n,K)$ such that det X, det Y are arbitrary elements $\neq 0$ of K.

P r o o f. Let us consider the equation

$$(2) \qquad VXYW = XAY .$$

The matrix equation (2) is equivalent to the system of equations

$$(3) \quad \sum_{k=1}^{n-1} x_{ik}(v_i w_j y_{kj} - y_{k+1j}) + x_{in}(v_i w_j y_{nj} + \sum_{k=0}^{n-1} a_{n-k} y_{k+1j}) = 0.$$

$i, j = 1, \ldots, n$.

The system (3) we can consider as a homogeneous linear system of equations in x_{ij} $(i,j = 1,\ldots,n)$. If we take y_{ij} = 0 for $j > 1$ and $y_{ii} = 1$ for $i > 1$, then we get a condition on non- trivial solution of (3). Namely

$$(4)\qquad \begin{vmatrix} v_1w_1y_{11}-y_{21}, & v_iw_1y_{21}-y_{31}, & \dots, & v_iw_1y_{n-1\;1}-y_{n1}, & v_iw_1y_{n1}+y_{11}a_n+a_{n-1}y_{21}+\dots+a_1y_{11} \\ -1 & v_iw_2 & & 0 & a_{n-1} \\ & -1 & \ddots & & a_{n-2} \\ & & \ddots & \ddots & \vdots \\ & & & v_iw_{n-1} & a_2 \\ & & & -1 & v_iw_n+a_1 \end{vmatrix} = 0\ ,\ i = 1,\dots,n.$$

After elemntary operations the condition (4) takes the form

$$(5)\qquad y_{11}(a_n+\sum_{s=1}^{n} a_{n-s}v_i^s\ w_1\dots w_s) + $$

$$+\sum_{k=2}^{n-1} y_{k1}(a_{n-k}v_i+\sum_{s=k+1}^{n}(a_{n-s}v_i^{s-k+1}\cdot w_{k+1}\dots w_s)).$$

$$(w_1-w_k)+y_{n1}v_i(w_1-w_n)=0,\ i=1,\dots,n,\ a_0:=1.$$

Hence, the determinant of the system (5) take the form

$$R|1,v_i,\dots,v_i^{n-1}|\ (a_n-v_1v_2\dots v_nw_1w_2\dots w_n),\ i=1,\dots,n.$$

where R denotes the product of $w_i - w_j$ $(i \neq j)$ and is equal to zero by the assumption det A = det VW.

If we consider y_{11} as a parameter, then the system (5) without the last equation has the determinant

$$e\, w_n^{n-2} w_{n-1}^{n-3} \dots w_3 \prod_{k=2}^{n} (w_1 - w_k) \prod_{k=1}^{n-1} w_k \, |v_1^{n-1} v_1^{n-2} \dots v_1, 1|, \quad e = \pm 1.$$

Therefore there exists the matrix Y, where det Y = y_{11} is an arbitrary element of K different from zero.

For our Y the system (3) takes the form

$$(6)\quad \begin{cases} \sum_{k=1}^{n-1} x_{ik}(v_i w_1 y_{k1} - y_{k+11}) + x_{in}(v_i w_1 y_{n1} + \sum_{k=0}^{n-1} a_{n-k} y_{k+11}) = 0 \\ -x_{ij-1} + x_{ij} v_i w_j + x_{in} a_{n-j+1} = 0 \text{ for } j = 2,\dots,n-1 \\ -x_{in-1} + x_{in}(a_1 + v_i w_n) = 0\,, \\ i = 1,\dots,n. \end{cases}$$

The system (6) is linearly dependent by (4), so we can omit the first equation of (6). From (6) by the recursion we have

$$(7)\quad x_{is} = (a_{n-s} a_{n-s-1} v_i w_{s+1} + a_{n-s-2} v_i^2 w_{s+1} w_{s+2} + \dots +$$

$$+ a_0 v_1^{n-s} w_{s+1} \dots w_n) x_{in}, \quad i = 1,\dots,n-1\ ;\ s = 1,\dots,n-1.$$

If we take $X = [x_{i1}, x_{i2}, \dots, x_{in}]$, $i = 1,\dots,n$ then easy calculations give

$$\det X = x_{1n}\ldots x_{nn}\, w_n^{r_n}\, w_{n-1}^{r_{n-1}} \ldots w_2^{r_2}\, |v_1^{n-1} v_1^{n-2} \ldots v_1, 1| \neq 0,$$

$$i = 1,\ldots,n\ ;\ r_i \in N.$$

Therefore the equation (1) has a solution described in Lemma 1.

L e m m a 2. If $V, W \in GL(n,K)$, $A = \mathrm{diag}(a_1,\ldots,a_n)$, $a_i \neq a_j$ $(i \neq j)$, $\det VW = \det A$, then the equation (1) has a solution $X, Y \in GL(n,K)$ such that $\det X$, $\det Y$ are arbitrary elements $\neq 0$ of K.

The proof results from Lemma 1 and from the fact that the matrix A is similar to a companion matrix of polynomial $(-1)^n(x^n - s_1x^{n-1} + \ldots + (-1)^n s_n)$ where s_i $(i = 1,\ldots,n)$ denoted the elementary symmetric functions in $a_1, a_2, \ldots, a_n$, (see [4] p.252).

L e m m a 3. If $V, W \in GL(n,K)$, A_{qq} - companion matrices of polynomials

$$f_q(x) = x^{s_q} + a_{q1}x^{s_q - 1} + \ldots + a_{qs_q},\quad q = 1,\ldots,t\ ;$$

$\det A = \det VW$, then the matrix equation (1) where $A = A_{11} + A_{22} + \ldots + A_{tt}$ has a solution $X, Y \in GL(n,K)$ such that $\det X$, $\det Y$ are arbitrary elements $\neq 0$ of K.

P r o o f. a) We consider the equation (2) for $A = A_{11} + A_{22} + \ldots + A_{tt}$, A_{ii} has a degree ≥ 2. In this case the equation (2) is equivalent to the equations system

$$(8)\quad \sum_{q=0}^{t-1}\Big[\sum_{k_q=s_q+1}^{s_{q+1}-1} x_{ik_q}(v_i w_j y_{k_q j} - y_{k_q+1j}) +$$

$$+ x_{is_{q+1}}(v_i w_j y_{s_{q+1}j} + \sum_{k_q=s_q}^{s_{q+1}-1} a_{q+1s_{q-1}-k_q}\, y_{k_q+1j})\Big] = 0,$$

$$i,\ j = 1,\dots,n\ ;\ a_{q+1} := 1\ ;\ a_0 := 0.$$

We can consider system (8) as a homogeneous linear system in x_{ij} $(i,j = 1,\dots,n)$. If we treat y_{j1} $(j = 1,\dots,n)$ as parameters and put $y_{jj} = 1$ $(j = 1,\dots,n)$, $y_{s_q s_{q+1}} = 1$ $(q = 1,\dots,t-1)$ and zeros on remain places, then after substracting s_q row from s_q+1 row $(q = 1,\dots,t-2)$ the condition on the existence of nonzero solution of the system (8) gives the following homogeneous system on y_{j1} $(j = 1,\dots,n)$:

$$
(9)\quad
\left|\begin{array}{llllllll}
b_{11}, & b_{12}, \ldots, b_{1s_1-1}, & b_{1s_1}, & b_{1s_1+1}, \ldots & \ldots, b_{1s_2-1}, & & & \\
-1 & v_iw_2 & a_{1s_1-1} & & & & & \\
 & \ddots \quad \ddots & \vdots & & & & & \\
 & v_iw_{s_1-1}, & a_{12} & & & & & \\
 & -1 & v_iw_{s_1}+a_{11} & & & & & \\
 & & v_iw_{s_1+1}-w_{s_1}, & v_iw_{s_1+1} & & & & \\
 & & & -1 \quad \ddots & \ddots & & & \\
 & & & & v_iw_{s_2-1} & & & \\
 & & & & -1 & & &
\end{array}\right.
$$

$$
\left.\begin{array}{lllll}
b_{1s_2} \ldots & b_{1s_{t-1}+1}, \ldots, & b_{1s_t-1}, & b_{1s_t} \\
a_{2s_2} & & & \\
a_{2s_2-1} & & & \\
\vdots & & & \\
a_{22} & & & \\
v_iw_{s_2}+a_{21} & v_iw_{s_{t-1}+1} & & b_{ts_t} \\
v_iw_{s_2+1}-w_{s2} & -1 \quad \ddots & & \vdots \\
 & \ddots & v_iw_{s_t-1} & a_{t2} \\
 & & -1 & v_iw_{s_t}+a_{t1}
\end{array}\right| = 0
$$

$$i = 1, \ldots, n.$$

where unmarked elements denote zeros, $b_{1s_q+1} = v_i w_1 y_{s_q+11} - y_{s_q+21}$, $b_{1s_q+2} = v_i w_1 y_{s_q+21} - y_{s_q+31}$, ...,

$b_{1s_{q+1}} = v_i w_1 y_{s_{q+1}1} + a_{q+1s_{q+1}} y_{s_q+11} + \dots + a_{q+11} y_{s_{q+1}1}$

for $q = 0,1,\dots,t-1$.

We reduce the terms $v_i w_j$ along the main diagonal and next we reduce the terms which are in columns $s_2,\dots,s_t$. Then the system (9) takes the form

$$\begin{aligned}(10)\qquad & v_i^{t-1} K_{t-1}(w_{s_{t-1}} - w_{s_{t-1}-1}) \dots (w_{s_1+1} - w_{s_1}) - \\ & - v_i^{t-2} K_{t-2}(w_{s_{t-2}} - w_{s_{t-2}-1}) \dots (w_{s_1+1} - w_{s_1}) r_{t-1} + \\ & + v_i^{t-3} K_{t-3}(w_{s_{t-3}} - w_{s_{t-3}-1}) \dots (w_{s_1+1} - w_{s_1}) r_{t-1} r_{t-2} + \\ & + \dots + (-1)^{t-1} K_0\, r_{t-1}\, r_{t-2} \dots r_0 = 0\,, \quad i = 1,\dots,n.\end{aligned}$$

where

$$\begin{aligned}K_q = {} & y_{s_q+11}\Big(a_{q+1s_{q+1}} + \sum_{p=s_q+1}^{s_{q+1}} a_{q+1s_{q+1}-p}\, v_i^p\, w_1 w_{s_q+2} \dots w_{s_{q+1}}\Big) + \\ & + \sum_{k_q=s_q+2}^{s_{q+1}-1} a_{q+1s_{q+1}-k_q} v_i + \sum_{p=k_q+1}^{s_{q+1}} a_{q+1s_{q+1}-p}\, v^{p-k_q+1}\, w_{k_q+1} \dots \\ & w_p (v_{s_q+1} - v_{k_q}) + y_{q+11} v_i (w_{s_q+1} - w_{s_{q+1}})\,, \quad q = 0,1,\dots,t-1\,;\end{aligned}$$

$a_{q+10} := 1$, $s_0 := 0$,

$$r_0 = a_{1s_1-1} + a_{1s_1-2}v_1w_2 + a_{1s_1-3}v_1^2 w_2w_3 + \dots + a_{11}v_1^{s_1-2} w_2 \dots$$

$$\dots w_{s_1-1} + v_1^{s_1-1} w_2 \dots w_s ,$$

$$r_q = a_{q+1s_{q+1}} + a_{q+1s_{q+1}-1}v_1w_{s_q+1} + a_{q+1s_{q+1}-2} v_1^2 w_{s_q+1}w_{s_q+2} +$$

$$+ \dots + a_{q+11} v_1^{s_{q+1}-1} w_{s_q+1}\dots w_{s_{q+1}-1} + v_1^{s_{q+1}}w_{s_q+1}\dots w_{s_{q+1}} ,$$

$$q = 0,1,\dots,t-1.$$

We will show that the determinant Δ of linear system (10) with respect y_{j1} $(j = 1,\dots,n)$ is equal to zero. Indeed, after operations the same as in the proof of Lemma 1, the determinant Δ takes the form

$$\Delta = R\ |v_1^{t-1}\bar{K}_{t-1}, v_1^{t-2}\bar{K}_{t-2}r_{t-1}, v_1^{t-3}\bar{K}_{t-3}r_{t-1}r_{t-2}\dots,\bar{K}_0r_{t-1}\dots r_0|.$$

where R is a product of terms $w_i - w_j$ $(i \neq j)$,

$$\bar{K}_q = \left[v_1, v_1^2,\dots, v_1^{s_{q+1}-1}, v_1^{s_{q+1}}w_1w_{s_q+2} \dots w_{s_{q+1}} + a_{q+1s_{q+1}}\right],$$

$i = 1,\dots,n$; $q = 0,1,\dots,t-1$.

Let us consider s_t+1 first columns. Substracting the s_t+1 column from s_t column, we get

$$v_1^{s_t+t-1} w_1w_{s_t+2} \dots w_{s_t} + a_{ts_t}v_1^{t-1} - a_{ts_t}v_1^{t-1} - a_{ts_t}v_1^{t} w_{s_{t-1}+1} -$$

$$- a_{ts_t-2}v_1^{t+1}w_{s_{t-1}+1} - w_{s_{t-1}+2} - \dots - a_{t1}v_1^{s_t+t-2} w_{s_{t-1}+1}\dots w_{s_t-1} -$$

$$- v_1^{s_t+t-1} w_{s_{t-1}+1} \dots w_{s_t}.$$

After a reduction of similar terms and subtracting the s_t-1 first columns multiplied by proper terms from the s_t column, we obtain

$$v_1^{s_t+t-1}(w_1 - w_{s_{t-1}+1}) w_{s_{t-1}+2} \dots w_{s_t}$$

in the s_t column.

In this way we can reduce the terms which contains the powers v_1^t, v_1^{t+1}, ..., $v_1^{s_t+t-1}$ in next columns. If we will continue this process, then the detrminant gives the form

$$\Delta = R R_1 | v_1, v_1^2, \dots, v_1^{n-1}, v_1^n w_1 \dots w_n + a_{ts_t} a_{t-1s_{t-1}} \dots a_{1s_1} | ,$$

where R_1 is a product of terms w_i and $w_i - w_j$ $(i \neq j)$. Now after easy transformations we have

$$\Delta = R R_1 (v_1 \dots v_n w_1 \dots w_n - a_{ts_t} a_{t-1s_{t-1}} \dots a_{1s_1}) | 1, v_i, \dots, v_i^{n-1} | ,$$

$i = 1,\dots,n$. Therefore, $\Delta = 0$, by the assumption det A = det VW. If we treat y_{11} as a parameter, then the linear system (10) in y_{j1} $(j = 2,\dots,n)$ without the last equation has the determinant Δ_1 different from zero, because now Δ_1 do not contain of column by y_{11}. Observe that det Y = y_{11}.

To calculate the matrix X we use the system (8). Since for our matrix Y the determinant of system (8) is equal to zero, hence that system has a non-zero solution. The system (8) without first equation and with x_{in} $(i = 1,\dots,n)$ as parameters has the determinant equal to

$$v_1^{t-1}\,(-1)^{n-t-1}\,(w_{s_1+1}-w_{s_1})\,\ldots\,(w_{s_{t-1}+1}-w_{w_{t-1}})\neq 0.$$

This system will be denoted by (8'). If we treat x_{1s_t} as a parameters from the system (8') we have recursively

(11) $$x_{1s_q}=\left[v_1^{s_{q+1}-s_q}w_{s_q+1}\ldots w_{s_{q+1}}+a_{q+11}v_1^{s_{q+1}-s_q-1}w_{s_q+1}\ldots w_{s_{q+1}-1}+\ldots+a_{q+1s_{q+1}}v_1w_{s_q+1}+a_{q+1s_{q+1}}\right]\left[w_{s_q}-w_{s_q+1}\right]^{-1}v_1^{-1}x_{1s_{q+1}},$$

$$q=1,\ldots,t-1.$$

(12) $$x_{1s_q+k}=\Big[v^{s_{q+1}-s_q-k}w_{s_q+1+k}\ldots w_{s_{q+1}}+$$
$$+a_{q+11}v_1^{s_{q+1}-s_q-k}w_{s_q+1+k}\ldots w_{s_{q+1}-1}+$$
$$+\ldots+a_{q+1s_{q+1}-k-1}v_1w_{s_q+1+k}+a_{q+1s_{q+1}}\Big]x_{1s_{q+1}},$$

$k=1,\ldots,s_{q+1}-s_q-1$; $q=0,1,\ldots,t-1$; $s_0:=0$. Hence det X has the form

$$\det X=|x_{11},\ldots,x'_{1s_1},\ldots,x'_{1s_2},\ldots,x'_{1s_{t-1}},x_{1s_{t-1}+1},\ldots,x_{1s_t}|R,$$

where R is a product $w_{s_q}-w_{s_q+1}$ $(q=1,\ldots,t-1)$,

$$x'_{s_q}=x_{1s_q}(w_{s_q}-w_{s_q+1})^{-1}.$$

After easy transformation the last $s_t-s_{t-1}-1$ columns we obtain

$$\det X = |x_{11},\dots,x'_{1s_1},\dots,x'_{1s_{t-1}},v_1^{s_t-s_{t-1}-1}x_{1s_t},v_1^{s_t-s_{t-1}-2}x_{1s_t},$$

$$\dots,v_1x_{1s_t},x_{1s_t}\ |\ R\ w_{s_{t-1}+2}\dots w_{s_t}.$$

Eliminating $v_1,\dots,v_n$ before the sign of det X we obtain

$$\det X = v_1\dots v_n\ |v_1x_{11},\dots,v_1x'_{1s_1},\dots,v_1x'_{1s_{t-1}},v_1^{s_t-s_{t-1}}x_{1s_t},$$

$$v_1^{s_t-s_{t-1}-1}x_{1s_t},\dots,v_1^2x_{1s_t},v_1x_{1s_t}\ |\ R\ w_{s_{t-1}+2}\dots w_{s_t}.$$

The term $v_1x'_{1s_{t-1}}$ is a polynomial of degree s_t-s_{t-1} with the free term $a_{ts_t}\neq 0$. Substracting the last $s_t-s_{t-1}-1$ column multiplied by proper factors from s_{t-1} column we obtain

$$\det X = v_1\dots v_n\ |v_1x_{11},\dots,v_1x'_{1s_1},\dots,v_1x_{1s_{t-1}-1},x_{1s_t},$$

$$v_1^{s_t-s_{t-1}}x_{1s_t},v_1^{s_t-s_{t-1}-1}x_{1s_t},\dots,v_1^2x_{1s_t},v_1x_{1s_t}\ |Rw_{s_{t-1}+2}\dots w_{s_t}a_{ts_t}.$$

Continuing this proces we get

$$\det X = R_1|v_1^{n-1},v_1^{n-2},\dots,v_1,1|x_{1n}x_{2n}\dots x_{nn}\ ,$$

where R_1 denotes a product of factors different from zero. Hence a) of Lemma 3 follow.

b) Proof of the case when there is A_{qq}, $1 \le q \le t$, of degree 1. Without loss of generality we can assume that A_{qq}

of dimension 1 stay on the end of main diagonal and equals b.

In this case we put

$$Y_1 = \begin{bmatrix} \begin{bmatrix} Y \end{bmatrix} & & 0 & \\ & 1 \dots & & 1 \\ & 1 & & \\ 0 & & . & 0 \\ & & & . \\ & & & & 1 \end{bmatrix} s_t \text{ row}$$

where Y denotes the matrix of dimension s_t from the case a). Now the condition (10) takes the form

$$(13) \qquad \Big[v_1^{t-1} K_{t-1} (w_{s_{t-1}} - w_{s_{t-1}-1}) \dots (w_{s_1+1} - w_{s_1}) -$$

$$v_1^{t-2} K_{t-2} (w_{s_{t-2}} - w_{s_{t-2}-1}) \dots (w_{s_1+1} - w_{s_1}) r_{t-1} +$$

$$v_1^{t-3} K_{t-3} (w_{s_{t-3}} - w_{s_{t-3}-1}) \dots (w_{s_1+1} - w_{s_1}) r_{t-1} r_{t-2} +$$

$$+ \dots + (-1)^{t-1} K_0 r_{t-1} r_{t-2} \dots r_0 \Big] e_{1s_t+1} \dots e_{1n} +$$

$$+ \sum_{p=s_t+1}^{n} w_p y_{1p} e_{1p}^{-1} e_{11} e_{1s_t+1} \dots e_{11} v_1^t = 0 ,$$

$i = 1,\dots,n$; $e_{ij} = v_i w_j - b$, where w_p denotes a product of $w_i - w_j$ $(i \neq j)$. The formula in square bracked of (13) can be trasformed as in the case a). Next we substract the s_t+1 column from the columns s_t+2, $s_t+3,\dots,n$. After this operations we substract the s_t+2 column from the columns $s_t+3, \ldots, n$, e.t.c. This process gives at last the determinant Δ_2 of system (13) in the following form

$$\Delta_2 = |v_1^n w_1 \ldots w_n + (-1)^t a_{ts_t} \ldots a_{1s_1} b^{n-s_t}, v_1^{n-1} \ldots v_1^{t-1}, v_1^t,$$
$$v_1^{t-1}, v_1^t, v_1^{t+1}, \ldots, v_1^{t+n-s_t}| .$$

From the assumption det A = det VW there results that $\Delta_2 = 0$. If we take y_{11} as a parameter then the system (13) of equations on y_{j1} ($j = 1, \ldots, n-1$) without the last equation has the determinant different from zero. Thererore in this case there also exists a matrix Y with det Y = $y_{11} \neq 0$.

To calculate the matrix X we consider two cases : b_1) $e_{1j} \neq 0$ ($j=s_t+1, \ldots, n$) and b_2) for certain i_0, $e_{i_0 s_t+1} = 0$ (the cases $e_{i_1 j} = e_{i_2 j} = 0$ or $e_{1j_1} = e_{1j_2} = 0$ are impossible).

Ad b_1) If we treat x_{1s_t} as parameter then for x_{1j}, $j=1,\ldots,s_t-1$ we have formulas (11) and 12) and

(14) $\qquad x_{1j} = v_1 (w_1 - w_{s_t}) x_{1s_t} e_{1j}^{-1}$, $\quad j = s_t+1, \ldots, n.$

Then after easy transformation we obtain

(15) $$\det X = \prod_{j=s_t+1}^{n} (w_j - w_{s_t}) c_1 c_2 \ldots c_n |x_{11} c_1, \ldots, x_{1s_1} c_1, \ldots,$$
$$x_{1s_t-1} c_1, x_{1s_t} v_1 e_{1s_t+1}^{-1} c_1, \ldots, x_{1s_t} v_1 e_{1n}^{-1} c_1 | ,$$

where $c_i = e_{is_t+1} \ldots e_{in}$, $i = 1, \ldots, n.$

If we use the method calculating det X from the case a) and the method of calculating the determinant of the system

(10) then we obtain det $X \neq 0$, as required.

Ad b_2) In this case $x_{i_0j} = 0$ for $j \neq s_t + 1$ and x_{i_0j} $(j = s_t+1)$ - parameters. If we use the expansion on i_0 row of det X, then we will get the determinant of the form (15). Lemma 3 is thus completed.

The proof of Theorem 1 results from Lemma 1,2 and 3.

From Theorem 1 and from Corollary 4.7 ([3],p.360) we obtain

T h e o r e m 2. If $V,W \in GL(n,K)$, $V = \mathrm{diag}(v_1,\dots,v_n)$, $W = \mathrm{diag}(w_1,\dots,w_n)$, $v_i \neq v_j$, $w_i \neq w_j$ for $i \neq j$, $A \notin Z(GL(n,K))$, $\det A = \det VW$, then $A \in \overline{V}\cdot\overline{W}$.

We will use yet the next two lemmas.

L e m m a 4. (see [1]) Let G be a group. An element g is in K_2^m $(m \geq 2)$ if and only if there is an element $t \in K_2^{m-1}$, $t \neq g^{-1}$ such that $(gt)^2 = 1$.

L e m m a 5. Let M is a subset of group G such that $M = M^{-1}$. If for each $x \in G$, $xM \cap M \neq 0$, then $G = MM$.

The proof of Lemma 5 is obvious.

T h e o r e m 3. If $V, W \in SL(n,K)$ or $W \in SL^{-}(n,K)$, $V = \mathrm{diag}(v_1,\dots,v_n)$, $W = \mathrm{diag}(w_1,\dots,w_n)$, $v_i \neq v_j$, $w_i \neq w_j$ for $i \neq j$, then $SL(n,K) = (\overline{V}\,\overline{W})^2$ in the group $GL(n,K)$.

P r o o f. By Theorem 2,

$$(16) \qquad SL(n,K) - Z(SL(n,K)) \subseteq \overline{V}\,\overline{W}\ .$$

Let $M = SL(n,K) - Z(SL(n,K))$. Then for each $x \in SL(n,K)$ we have $xM \cap M \neq 0$. If not, then there exists x_0 such that for each $m \in M$, $x_0m = aE$, i.e. $m = ax_0^{-1}$ but this is impossible. Naturally, $M = M^{-1}$. Therefore $SL(n,K) = MM \subseteq \overline{V}\,\overline{W}$ by (16) and Lemma 5.

If K is an algebraicaly closed field, then each matrix of the group SL(n,K) is similar to a matrix in primary rational canonical form in the group SL(n,K). Hence we have

C o r o l l a r y 3.1. If K is algebraicaly closed field, V, $W \in SL(n,K)$, then $SL(n,K) = (\bar{V}\ \bar{W})^2$ in the group SL(n,K).

T h e o r e m 4. If C is the field of comlex numbers, then $SL(n,C) \subseteq (K_2K_2)^2$ in the group GL(n,C).

P r o o f. Let $D_1 = \mathrm{diag}(1,e,\dots,e^{n-1})$, (e-$n^{th}$ primitive root of unit), $D_2 = \mathrm{diag}(1,a^{-1},\dots,a^{-(n-1)})$, $D_3 = \mathrm{diag}(a,a^2,\dots,a^n)$, $(a^n = 1)$. One can verify that $T^{-1}D_iT = D_i^{-1}$ (i=1,2), $S^{-1}D_3S = D_3^{-1}$ where

$$T = \begin{bmatrix} 1,0, & \dots & ,0 \\ 0 & \cdot & 1 \\ & \cdot & \\ \cdot & 0 & \\ 0,1, & \dots & ,0 \end{bmatrix}, \qquad S = \begin{bmatrix} 0, & \dots & ,1,0 \\ 0 & \cdot & 0 \\ & \cdot & \\ 1, & \dots & ,0,0 \\ 0, & \dots & ,0,1 \end{bmatrix}$$

Therefore $D \in KK$ (i = 1,2,3) by lemma 4. From (16) for $W = V = D_1 = \mathrm{diag}(1,e,\dots,e^{n-1})$ we have $SL(n,C) \subseteq (K_2K_2)^2 \cup \cup Z(SL(n,C))$. On the other hand, $D_2D_3 = aE \in Z(SL(n,C))$. Then $SL(n,C) \subseteq (K_2K_2)^2$. This ends the proof of the theorem.

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