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NON-STRONGLY CONVERGING APPROXIMATION METHODS

1. Introduction

Commonly, a large class of approximation methods for solving the operator equation $Ax = y$ can be interpreted as follows :

Choose projection operators P_τ and operators A_τ : im P_τ $\longrightarrow$ im P_τ and consider a approximation equation $A_\tau x_\tau = P_\tau y$. If there is a τ_0 so that, for each $\tau \geq \tau_0$ and for each right side y, these equations have a unique solution x_τ, and if x_τ converges to a solution x of the equation $Ax = y$, then one says that the approximation method $\Pi\ \{A_\tau\}$ converges for the operator A (see [4],[5]). In some special but interesting cases (think on integral operators on spaces of bounded measurable functions as in the papers of Anselone/Sloan, Chandler/Graham, de Hoog/Sloan, and Silbermann) the usual convergence $x_\tau \longrightarrow x$ cannot be guaranteed but it turns out that the functions x_τ converge uniformly to x on each compact interval. This observation leads to a weaker notion of convergence, the convergence with respect to a family of projections. In this paper we study the corresponding "weakly" converging approximation methods from a point of view which includes the standard theory.

Thereby we make essentially use of Simonenko and Kozak's techniques in the theory of operators of local type. As an application, we show how the results of [1]-[3], [7] can be almost at once derived from our theory. Other possible applications would be equations in finite differences in spaces of bounded functions and two-dimensional Wiener-Hopf equations. Moreover, it should be pointed out that the methods presented here apply to the matrix case without changes. It is the authors'aim to return to this circle of problems in a further comprehensive publication.

2. Convergence with respect to a family of projections

Let X denote a Banach space and $\mathbb{R}^+$ the set of non-negative real numbers. By $\mathcal{M}$ we denote the set of all finte unions of left-sided closed and right-sided open intervals of $\mathbb{R}^+$. Notice that for $U, V \in \mathcal{M}$, we have $U \cup V \in \mathcal{M}$, $U \cap V \in M$ and $\mathbb{R}^+ \setminus U \in \mathcal{M}$.

Assume that to each $U \in \mathcal{M}$ a projection operator R_U is associated such that

$$(1) \qquad \begin{cases} R_U + R_{\mathbb{R}^+ \setminus U} = I \quad (I \text{ - identity operator}), \\ R_U R_V = R_V R_U = R_{U \cap V}, \\ R_U + R_V = R_{U \cup V} \quad \text{if } U \cap V = \emptyset. \end{cases}$$

$$(2) \qquad \sup_{U \in \mathcal{M}} \|R_U\| =: c < \infty$$

(3) $$\bigcap_{w\in\mathbb{R}^+} \ker R_{[0,w)} = \{0\}.$$

Further, put $\mathcal{R} = \{R_U : U \in \mathcal{M}\}$, and let T stand for a certain unbounded subset of $\mathbb{R}^+$.

We say that the (generalized) sequence $\{x_\tau\}_{\tau\in T}$ of elements $x_\tau \in X$ converges with respect to $\mathcal{R}$ to $x \in X$ (and write $x_\tau \xrightarrow{\mathcal{R}} x$ or $x = \mathcal{R}\text{-}\lim\limits_{\tau\to\infty} x_\tau$ in that case) if the set $\{x_\tau\}$ is bounded and if for each $w \in \mathbb{R}^+$

$$\|R_{[0,w)}(x_\tau - x)\| \longrightarrow 0 \text{ as } \tau \longrightarrow \infty.$$

The condition (3) ensures that the $\mathcal{R}$-limit of a given sequence is unique if it exists. Indeed, if $x_\tau \xrightarrow{\mathcal{R}} x$ and $x_\tau \xrightarrow{\mathcal{R}} y$ then, by definition, $R_{[0,w)}x = R_{[0,w)}y$, i.e. $x-y \in \ker R_{[0,w)}$ for each $w \in \mathbb{R}^+$.

Now consider the set $\mathcal{A}$ of all (generalized) sequences $\{A_\tau\}_{\tau\in T}$ of bounded operators on X which are subject of the following conditions :

(4) $$\sup_{\tau\in T} \|A_\tau\| < \infty .$$

There is an $A \in L(X)$ such that for each $w \in \mathbb{R}^+$ and $x \in X$:

(5) $R_{[0,w)}A_\tau x \to R_{[0,w)}Ax$ and $A_\tau R_{[0,w)}x \to AR_{[0,w)}x$ as $\tau \to \infty$.

This will be abbreviated to $A_\tau \xrightarrow{\mathcal{R}} A$ or $A = \mathcal{R}\text{-}\lim A_\tau$.

Notice that the $\mathcal{R}$-limit of operators is unique.

Given two subsets $U, V \in \mathbb{R}^+$ put $\rho(U,V) = \inf\{|u-v|,\ u\in U, v\in V\}$, and for $h \in \mathbb{R}^+$ set $\varphi_{(A_\tau)}(h) = \sup\{\|R_U A_\tau R_V\| : \tau \in T, U, V \in \mathcal{M} \text{ with } \rho(U,V) > h\}$.

Then

(6) $$\lim_{x \to \infty} \varphi_{\{A_\tau\}}(h) = 0.$$

P r o p o s i t i o n 1. $\mathcal{A}$ is a Banach algebra when provided with the norm $\|\{A_\tau\}\| := \sup_\tau \|A_\tau\|$ and with elementwise operations.

P r o o f. Let $\{A_\tau\}$, $\{B_\tau\} \in \mathcal{A}$. Then, obviously, $\{A_\tau\} + \{B_\tau\} = \{A_\tau + B_\tau\} \in \mathcal{A}$. Now assume that $A_\tau \xrightarrow{\mathcal{R}} A$ and $B_\tau \xrightarrow{\mathcal{R}} B$. The identity

$$R_{[0,w)}(A_\tau B_\tau x - ABx) = R_{[0,w)}(A_\tau B_\tau x - A_\tau Bx + A_\tau Bx - ABx) =$$

$$= R_{[0,w)} A_\tau R_{[0,v)}(B_\tau - B)x + R_{[0,w)} A R_{[v,\infty)}(B_\tau - B)x + R_{[0,w)}(A_\tau - A)Bx$$

shows (choose v large enough) that $R_{[0,w)} A_\tau B_\tau \longrightarrow R_{[0,w)} AB$ strongly. Similarly, the other assertion of (5) follows. Further, (6) is a consequence of

$$\|R_U A_\tau B_\tau R_V\| \le \|R_U A_\tau\| \, \|R_W B_\tau R_v\| + \|R_U A_\tau R_{\mathbb{R}^+ \setminus W}\| \, \|B_\tau R_V\| ,$$

and standard arguments show that $\mathcal{A}$ is a Banach algebra.

C o n v e n t i o n. If the constant sequence $\{A_\tau\}_{\tau \in T}$ belongs to $\mathcal{A}$ we write for brevity $A \in \mathcal{A}$ and call A an operator of local type. Condition (1) shows that $I \in \mathcal{A}$ and $R_U \in \mathcal{A}$ for $U \in \mathcal{M}$.

P r o p o s i t o n 2. If $\{A_\tau\}_{\tau \in T} \in \mathcal{A}$ with $\mathcal{R}$-lim $A_\tau = A$ then $A_\tau \in \mathcal{A}$ and $A \in \mathcal{A}$.

P r o o f. It is evident that $\{A_\tau\} \in \mathcal{A}$ implies $A_\tau \in \mathcal{A}$ for each $\tau \in T$. Now consider $R_U A R_V$. A little thought shows that we can assume $U = [0,u)$, $V = [v,\infty)$ or $U = [u,\infty)$, $V = [0,v)$

without loss of generality. In the first case we obtain

$R_U A_\tau R_V = R_{[0,u)} A_\tau R_V \longrightarrow R_{[0,u)} A R_V = R_U A R_V$ strongly, hence,

$$\|R_U A\, R_V\| \le \sup_\tau \|R_U A_\tau R_V\| .$$

For the other case the proof is analogous. Thus,

$$\varphi_A(h) := \varphi_{\{A\}}(h) = \sup \{ \|R_U A R_V\|,\ U,\ V \in \mathcal{M},\ \rho(U,V) > h\} \le$$

$$\le \sup \{ \|R_U A_\tau R_V\|,\ \tau \in T,\ U,\ V \in \mathcal{M},\ \rho(U,V) > h \} = \varphi_{\{A_\tau\}}(h),$$

as we are done.

P r o p o s i t i o n 3. Let $\{A_\tau\} \in \mathcal{A}$ and put $A := \mathcal{R}\text{-lim } A_\tau$. The sequence $\{A_\tau\}$ is invertible in $\mathcal{A}$ if and only if the operators A and A_τ are invertible for each τ and if $\sup_\tau \|A_\tau^{-1}\| < \infty$. Thereby, $\{A_\tau\}^{-1} = \{A_\tau^{-1}\}$, and $\mathcal{R}\text{-lim } A_\tau^{-1} = A^{-1}$.

P r o o f. First we verify the "if " - part :
Obviously, (4) holds true for the sequence $\{A_\tau^{-1}\}$. The proof that (6) is valid for $\{A_\tau^{-1}\}$ bases essentially on Simonenko and Kozak's proof that the inverse of an operator of local type is again of local type: We show that if $\{A_\tau\}$ is subject of the assumptions of the Proposition, then

$$\varphi_{\{A_\tau^{-1}\}}(t) \le \frac{c^4 \sup_\tau \|A_\tau\| \ \sup_\tau \|A_\tau^{-1}\|^2}{n} + 4c^2 \sup_\tau \|A_\tau^{-1}\|^2 \ \varphi_{\{A_\tau\}}\left(\frac{t}{4n-1}\right)$$

for all $t \in \mathbb{R}$ and all $n \in \mathbb{Z}^+$. This implies $\lim_{t \to \infty} \varphi_{\{A_\tau^{-1}\}}(t) = 0$.

Indeed, given any $\varepsilon > 0$ there is an n_0 such that

$$c^4 \sup_\tau \|A_\tau\| \ \sup_\tau \|A_\tau^{-1}\|^2 / n_0 < \varepsilon/2$$

and a t_0 such that

$4c^2 \sup_\tau \|A_\tau^{-1}\|^2 \varphi_{\{A_\tau\}}(t/(4n_0-1)) < \varepsilon/2$ for all $t > t_0$.

Let $U, V \in \mathcal{M}$ and $\rho(U,V) := \inf\{|u-v|,\ u \in U,\ v \in V\} = r > t$. Let h, r_1, r_2, r_3, r_4 be any real numbers satysfying $0 \le r_1 < r_2 < r_3 < r_4 \le r$, $0 < h < r_2 - r_1$, $h < r_3 - r_2$, $h < r_4 - r_3$, and put $U_1 = \{ w \in \mathbb{R}^+ : r_1 \le \rho(w,U) \le r_3\}$, $V_1 := \{ w \in \mathbb{R}^+ : r_2 \le \rho(w,U) \le r_4\}$. We claim that

$$\text{(7)} \qquad R_U A_\tau^{-1} R_V = - R_U A_\tau^{-1} R_{U_1} A_\tau R_{V_1} A_\tau^{-1} R_V + E ,$$

where $\|E\| \le 3c^2 \sup_\tau \|A_\tau^{-1}\|^2 \varphi_{\{A_\tau\}}(h)$.

Put $V' := \{w \in \mathbb{R}^+ : \rho(w,U) < r_2\}$, $U' := \{w \in \mathbb{R}^+ : \rho(w,U) \le r_3\}$. Then we have

$R_U A_\tau^{-1} R_V = R_U R_{V'} A_\tau^{-1} R_V = R_U A_\tau^{-1} R_{U'} A_\tau R_{V'} A_\tau^{-1} R_V + e_1$, where

$\|e_1\| = \|R_U A_\tau^{-1} R_{\mathbb{R}^+\setminus U'} A_\tau R_{V'} A_\tau^{-1} R_V\| \le c_2 \sup_\tau \|A_\tau^{-1}\|^2 \varphi_{\{A_\tau\}}(h)$,

because $\rho(\mathbb{R} \setminus U', V') > h$. Further

$R_U A_\tau^{-1} R_{U'} A_\tau R_{V'} A_\tau^{-1} R_V = - R_U A_\tau^{-1} R_{U'} A_\tau R_{\mathbb{R}^+\setminus V'} A_\tau^{-1} R_V$, since

$R_U A_\tau^{-1} R_{U'} A_\tau A_\tau^{-1} R_V = 0$. Finally,

$R_U A_\tau^{-1} R_{U'} A_\tau R_{\mathbb{R}^+\setminus V'} A_\tau^{-1} R_V = R_U A_\tau^{-1} R_{U_1} A_\tau R_{V_1} A_\tau^{-1} R_V + e_2$, where

$\|e_2\| \le \|R_U A_\tau^{-1} R_{U'\setminus U_1} A_\tau R_{V_1} A_\tau^{-1} R_V\| + \|R_U A_\tau^{-1} R_{U'} A_\tau R_{(\mathbb{R}^+\setminus V')\setminus V_1} A_\tau^{-1} R_V\| \le$

$\le 2c^2 \sup_\tau \|A_\tau^{-1}\|^2 \varphi_{\{A_\tau\}}(h)$,

because $\rho(U'\setminus U_1, V_1) > h$, $\rho(U', (\mathbb{R}^+\setminus V')\setminus V_1) > h$.

Putting these things together, we get our claim (7). Now let n be any positive integer. Put $h := t/(4n-1)$, $l = r/(4n-1)$,

$$U_i := \{w \in \mathbb{R}^+ : (4i-4)l \le \rho(w,U) \le (4i-2)l\} ,$$

$$V_i := \{w \in \mathbb{R}^+ : (4i-3)l \le \rho(w,U) \le (4i-1)l\} ,$$

where $i = 1, \dots, n$. From what has just been proved ($r_1 = (4i-4)l$, $r_2 = (4i-3)l$, $r_3 = (4i-2)l$, $r_4 = (4i-1)l$) we obtain

$$(8) \qquad R_U A_\tau^{-1} R_V = - R_U A_\tau^{-1} R_{U_i} A_\tau R_{V_i} A_\tau^{-1} R_V + E_i$$

with $\|E_i\| \le 3c^2 \sup_\tau \|A_\tau^{-1}\|^2 \varphi_{\{A_\tau\}}(h)$ for $i = 1,\dots,n$.

Adding the n equalities (8) we arrive at the equality

$$nR_U A_\tau^{-1} R_V = - \sum_{i=1}^{n} R_U A_\tau^{-1} R_{U_i} A_\tau R_{V_i} A_\tau^{-1} R_V + \sum_{i=1}^{n} E_i =$$

$$= - R_U A_\tau^{-1} R_{U_1 \cup \dots \cup U_n} A_\tau R_{V_1 \cup \dots \cup V_n} A_\tau^{-1} R_V + E + \sum_{i=1}^{n} E_i ,$$

where $E := \sum_{i=1}^{n} R_U A_\tau^{-1} R_{U_i} A_\tau R_{V_1 \cup \dots \cup V_{i-1} \cup V_{i+1} \cup \dots \cup V_n} A_\tau^{-1} R_V$.

Since $\rho(U_i, V_j) > h$ for $i \ne j$ it follows that

$$\|E\| \le nc^2 \sup_\tau \|A_\tau^{-1}\|^2 \varphi_{\{A_\tau\}}(h) ,$$

and thus

$$\|R_U A_\tau^{-1} R_V\| \le \frac{c^4 \sup_\tau \|A_\tau\| \sup_\tau \|A_\tau^{-1}\|^2}{n} + 4c^2 \sup_\tau \|A_\tau^{-1}\|^2 \varphi_{\{A_\tau\}}(h) ,$$

and we are done with verifying (6) for $\{A_\tau^{-1}\}$.

Finally, we prove (5) by showing that $A_\tau^{-1} \xrightarrow{\mathcal{R}} A^{-1}$:

$$R_{[0,w)}(A_\tau^{-1}x - A^{-1}x) = R_{[0,w)}A_\tau^{-1}(Ay - A_\tau y) =$$

$$= R_{[0,w)}A_\tau^{-1}\, R_{[0,v)}(Ay - A_\tau y) + R_{[0,w)}A_\tau^{-1}\, R_{[v,\infty)}(Ay - A_\tau y)$$

with $y = A^{-1}x$. Hence, if $v > w$,

$$\|R_{[0,w)}(A_\tau^{-1}x - A^{-1}x)\| \le c \sup_\tau \|A_\tau^{-1}\|\, \|R_{[0,v)}(Ay - A_\tau y)\| +$$
$$+ \varphi_{\{A_\tau^{-1}\}}(v-w) \sup_\tau \|A-A_\tau\|\, \|y\|.$$

Choose $v_0 > w$ so that $\varphi_{\{A_\tau^{-1}\}}(v_0-w) < \varepsilon/(2 \sup_\tau \|A-A_\tau\|\, \|y\|)$.
Then we can find a $\tau_0 \in T$ so that for $\tau > \tau_0$

$$\|R_{[0,v_0)}(Ay - A_\tau y)\| < \varepsilon/(2c \sup_\tau \|A_\tau^{-1}\|) ,$$

what yields the first assertion of (5). The proof of the second is similarly, and this completes the proof of the "if"-part.

Conversely, assume that $\{A_\tau\}$ is invertible in $\mathcal{A}$. Then there exists a sequence $\{B_\tau\} \in \mathcal{A}$ such that $\{B_\tau A_\tau\} = \{A_\tau B_\tau\} = \{I\}$. Hence, the operators A_τ must be invertible and $A_\tau^{-1} = B_\tau$. The uniform boundedness of $\|A_\tau^{-1}\|$ is obvious since $\{A_\tau^{-1}\} \in \mathcal{A}$ by assumption. Let $B := \mathcal{R}\text{-lim } B_\tau$. Then, since $A_\tau B_\tau = B_\tau A_\tau = I$, the passage to the $\mathcal{R}$-limit gives $AB = BA = I$, i.e. A proves to be invertible, and $B = A^{-1} = \mathcal{R}\text{-lim } A_\tau^{-1}$.

C o r o l l a r y 1. If A is an operator of local type then A is continuous in the $\mathcal{R}$-topology, i.e. if $x_\tau \xrightarrow{\mathcal{R}} x$ then $Ax_\tau \xrightarrow{\mathcal{R}} Ax$.

P r o o f. The corollary follows immediately from
$R_{[0,w)}A(x-x_\tau) = R_{[0,w)}AR_{[0,v)}(x-x_\tau) + R_{[0,w)}AR_{[v,\infty)}(x-x_\tau)$.

For the next corollaries put $\mathcal{R}^* = \{R^* : R \in \mathcal{R}\}$. Obviously, if $\{A_\tau\} \in \mathcal{A}$ then the sequence $\{A_\tau^*\}$ fulfills (6) with respect to $\mathcal{R}^*$.

C o r o l l a r y 2. a) Let $\{A_\tau\} \in \mathcal{A}$ with $\mathcal{R}\text{-}\lim\limits_{\tau\to\infty} A_\tau = A$. If A_τ is invertible for all $\tau \in T$ and if $\sup\limits_\tau \|A_\tau^{-1}\| < \infty$ then $A_\tau^{-1}A \xrightarrow{\mathcal{R}} I$, and A is one-to-one.

b) If the $\mathcal{R}^*$-limit of $\{A_\tau^*\}$ equals A^*, and if this $\mathcal{R}^*$-limit is uniquely determined then A^* is one-to-one.

P r o o f. a) The proof of Proposition 3 shows that the hypotheses of Corollary imply that $\varphi_{\{A_\tau^{-1}\}}(h) \longrightarrow 0$ as $h \longrightarrow \infty$.

Hence,

$$R_{[0,w)}(A_\tau^{-1}Ax - x) = R_{[0,w)}(A_\tau^{-1}Ax - A_\tau^{-1}A_\tau x) =$$

$$= R_{[0,w)}A_\tau^{-1}R_{[0,v)}(Ax - A_\tau x) + R_{[0,w)}A_\tau^{-1}R_{[v,\infty)}(Ax - A_\tau x)$$

becomes as small as desired if v-w is large enough. On the other hand, $A_\tau^{-1}AR_{[0,w)}x - R_{[0,w)}x = A_\tau^{-1}(A - A_\tau)\, R_{[0,w)}x$, and this becomes small as τ is large enough. Thus, $A_\tau^{-1}A \xrightarrow{\mathcal{R}} I$, and this shows, moreover, that A must be one-to-one.

b) Apply a) to $\{A_\tau^*\}$, A^*.

C o r o l l a r y 3. Assume that $R_{[0,w)} \longrightarrow I$ strongly.

a) Then $\mathcal{R}^*$ fulfills (1) - (3).

b) Let A be an operator of local type and put for $\tau \in T$ $A_\tau := R_{[0,\tau)}AR_{[0,\tau)}$. If the operators $A_\tau : \operatorname{im}R_{[0,\tau)} \longrightarrow \operatorname{im}R_{[0,\tau)}$

are invertible and if $\sup_\tau \|A_\tau^{-1}\| < \infty$ then A is invertible.

P r o o f. a) Obviously, $\mathcal{R}^*$ satisfies (1), (2). To see that (3) is fulfilled take $y \in \bigcap_{w\in\mathbf{R}^+} \ker R^*_{[0,w)}$. Then $\langle R_{[0,w)}x, y \rangle = \langle x, R^*_{[0,w)}y\rangle = 0$ for each $w \in \mathbb{R}^+$ and for each $x \in X$. Since the set $\{R_{[0,w)}x : w \in \mathbb{R}^+, x \in X\}$ is dense in X, the element $y \in X^*$ must be the zero functional.

b) If A is of local type with respect to $\mathcal{R}$ then A^* is of local type with respect to $\mathcal{R}^*$. Since obviously $\{R^*_{[0,\tau)}\}_{\tau\in T}$ is in $\mathcal{A}$ (with respect to $\mathcal{R}^*$), and since $\mathcal{A}$ is an algebra we obtain $\{R^*_{[0,\tau)}A^*R^*_{[0,\tau)}\} \in \mathcal{A}$. Now Corollary 2 applies to $\{R^*_{[0,\tau)}A^*R^*_{[0,\tau)} + R^*_{[\tau,\infty)}\}$ what gives that $\ker A^* = \{0\}$ and the assertion follows.

3. Weakly convergent approximation methods

Besides the family $\mathcal{R}$ of projections which defines the convergence we consider another family $\mathcal{P} = \{P_\tau\}_{\tau\in T}$ of projection operators on X which is related to $\mathcal{R}$ by the condition

(9) $$\{P_\tau\} \in \mathcal{A}\ , \text{ and } \mathcal{R}\text{-}\lim_{\tau\to\infty} P_\tau = I.$$

(Examples : a) $P_\tau = R_{[0,\tau)}$ for each τ ,

b) $P_\tau = I$ for each τ).

Let A be an bounded operator on X and consider the equation $Ax = y$. Let $\{A_\tau\}_{\tau\in T}$, $A_\tau : \text{im } P_\tau \longrightarrow \text{im } P_\tau$, be a sequence of operators which converges with respect to $\mathcal{R}$ to A as $\tau \to \infty$, and assume that there is a $\tau_0 \in T$ so that the approximate equation $A_\tau x_\tau = P_\tau y$ has a unique solution x_τ

for all right sides y and for all $\tau \in T$ with $\tau \geq \tau_0$.

We shall say that the approximation method $\Pi_{\mathcal{R}}\{A_\tau\}$ converges for the operator A (and write $A \in \Pi_{\mathcal{R}}\{A_\tau\}$ in that case) if the sequence $\{x_\tau\}_{\tau\geq\tau_0}$ converges with respect to $\mathcal{R}$ to a solution x of the equation $Ax = y$. In case that $A_\tau = P_\tau A P_\tau$ we write simply $\Pi_{\mathcal{R}}\mathcal{P}$ instead $\Pi_{\mathcal{R}}\{P_\tau A P_\tau\}$; this special approximation method is also called the finite section method.

P r o p o s i t i o n 4. Let $A_\tau : \operatorname{im} P_\tau \longrightarrow \operatorname{im} P_\tau$ and $\{A_\tau\} \in \mathcal{A}$ with $\mathcal{R}\text{-}\lim A_\tau = A$. Then $A \in \Pi\{A_\tau\}$ if and only if A is invertible and if there exists a $\tau_0 \in T$ such that $A_\tau : \operatorname{im} P_\tau \longrightarrow \operatorname{im} P_\tau$ is invertible for $\tau \in T$, $\tau \geq \tau_0$, and $\sup_{\tau\geq\tau_0} \|A_\tau^{-1}\| < \infty$.

P r o o f. Let $A \in \Pi_{\mathcal{R}}\{A_\tau\}$. Then, by definition, the operators $A_\tau : \operatorname{im} P_\tau \longrightarrow \operatorname{im} P_\tau$ are invertible for $\tau \geq \tau_0$. Denote by $A_\tau^{-1} : \operatorname{im} P_\tau \longrightarrow \operatorname{im} P_\tau$ the inverse of A_τ . Since $A_\tau^{-1} y \xrightarrow{\mathcal{R}} x$, the supremums $\sup_{\tau\geq\tau_0} \|A_\tau^{-1} y\|$ are bounded for each $y \in X$. The uniform-boundedness-principle shows that then $\sup_{\tau\geq\tau_0} \|A_\tau^{-1}\|$ must be bounded.

Next we verify the invertibility of A. By the definition of the approximation method, A is onto. Put $Q_\tau := I - P_\tau$, $\tilde{A}_\tau := A_\tau + Q_\tau$ for $\tau \geq \tau_0$ and $\tilde{A}_\tau = I$ for $\tau < \tau_0$. By (9), the sequence $\{\tilde{A}_\tau\}$ is in $\mathcal{A}$ and $\mathcal{R}\text{-}\lim \tilde{A}_\tau = A$. It is immediate from what has been proved above that $\tilde{A}_\tau$ is invertible for all

$\tau \in T$ and that $\sup_{\tau} \|\tilde{A}_\tau^{-1}\| < \infty$. Hence, by Corollary 2, A must be one-to-one.

The other direction is an evident consequence of Proposition 3 applied to the sequence $\{\tilde{A}_\tau\}$.

Our final goal in this section is a perturbation theorem for the approximation method $\Pi_{\mathcal{R}}\{A_\tau\}$. To that end let $\mathcal{C}$ stand for the set of all sequences $\{C_\tau\} \in \mathcal{A}$ with $\|C_\tau\| \longrightarrow 0$ as $\tau \longrightarrow \infty$. The set $\mathcal{C}$ forms a closed two-sided ideal of the algebra $\mathcal{A}$.

P r o p o s i t i o n 5. Let $\{A_\tau\} \in \mathcal{A}$, $\mathcal{R}\text{-lim } A_\tau = A$. Then the following statements are equivalent :

a) A is invertible, and the operators A_τ are invertible for τ large enough (say $\tau \ge \tau_0$), and $\sup_{\tau\ge\tau_0} \|A_\tau^{-1}\| < \infty$.

b) The coset $\{A_\tau\} + \mathcal{C}$ is invertible in the quotient algebra $\mathcal{A}/\mathcal{C}$.

P r o o f. a) $\Rightarrow$ b) Apply Proposition 3.

b) $\Rightarrow$ a) Assume there are sequences $\{B_\tau\} \in \mathcal{A}$ and $\{C_\tau\}$, $\{D_\tau\} \in \mathcal{C}$ such that $A_\tau B_\tau = I + C_\tau$, $B_\tau A_\tau = I + D_\tau$. A little thought shows that then the operators A_τ must be invertible for $\tau \ge \tau_0$ and that the norms of their inverses are uniformly bounded. Moreover, taking the $\mathcal{R}$-limit we obtain AB = BA = I (where $B = \mathcal{R}\text{-lim } B_\tau$), i.e. A is invertible.

C o r o l l a r y 4. Let $\{A_\tau\} \in \mathcal{A}$, A_τ: im $P_\tau \longrightarrow$ im P_τ and $\mathcal{R}\text{-lim } A_\tau = A$. If $A \in \Pi_{\mathcal{R}}\{A_\tau\}$ then $A \in \Pi_{\mathcal{R}}\{A_\tau + P_\tau C_\tau P_\tau\}$ for each sequence $\{C_\tau\} \in \mathcal{C}$.

R e m a r k. It should be pointed out that the "usual" approximation methods are included in our general approach.

To see this, put $R_U = I$ if $U \in \mathcal{M}$ contains an interval of the form $[0,w)$ and put $R_U = 0$ elsewhere. It is easy to check that (1)-(3) are fulfilled and that the $\mathcal{R}$-convergence of sequences of elements $\{x_\tau\}$ and of operators $\{A_\tau\}$ is nothing else than the usual convergence of $\{x_\tau\}$ and the strong convergence of $\{A_\tau\}$, respectively. Further, (6) holds true for arbitrary sequences $\{A_\tau\}$ since $R_{[v,\infty)} = 0$ for $v > 0$ and, consequently, the algebra $\mathcal{A}$ is the algebra of all strongly convergent sequences $\{A_\tau\}$. Particularly, Propositions 1-3 are almost evident. Condition (9) means that the projections P_τ converge strongly to the identity operator. In this setting, Propositions 4 and 5 are wellknown (see, e.g. [4] or [5]).

Finally we remember the fact that, generally, the class of all possible perturbations includes not only small perturbations (i.e. the ideal $\mathcal{L}$) but, moreover, the ideal of the compact operators. It would be not too hard to construct a perturbation ideal larger than $\mathcal{C}$ in our general context, too. But to avoid undue confusions we prefer to define such a larger ideal only for the special approximation method considered in the next section.

4. The finite section method for Wiener-Hopf integral equations in spaces of measurable functions

Before we are going to study finite sections of Wiener-Hopf integral operators we specify our general approach to the finite section method.

Put $T = \mathbb{R}^+$, and for given $U \in \mathcal{M}$ with $U = \bigcup_{i=1}^{n} [a_i, b_i)$ (where $0 \le a_1 < b_1 < a_2 < b_2 < \dots < a_n \le \infty$) we define $R_U = \sum_{i=1}^{n} (P_{a_i} - p_{b_i})$. Thereby we assume for definitness that $P_0 = I$, $P_\infty = 0$. Particularly, $R_{[0,\tau)} = P_\tau$. In that case, condition (9) is automatically fulfilled, i.e. the strong convergence is no longer needed.

In what follows denote by B the algebra of all operators of local type and by R the set of all bounded operators K for which $\|KQ_\tau\| \longrightarrow 0$ and $\|Q_\tau K\| \longrightarrow 0$ as $\tau \longrightarrow \infty$, where $Q_\tau = I - P_\tau = R_{[\tau,\infty)}'$.

P r o p o s i t i o n 6. a) $\mathcal{R} \subseteq \mathcal{B}$, and the set $\mathcal{R}$ forms a closed two-sided ideal of B.

b) Let $A \in B$, choose $\mathcal{P}$ and $\mathcal{R}$ as above and let (1)-(3) be fulfilled. If $A \in \Pi_{\mathcal{R}}\mathcal{P}$, $K \in \mathcal{R}$, and if $A + K$ is invertible then $A + K \in \Pi_{\mathcal{R}}\mathcal{P}$.

P r o o f. a) The inclusion $\mathcal{R} \subseteq B$ is almost obvious. Let us prove that, e.g., $\mathcal{R}$ is a left-sided ideal of B. For arbitrary $B \in B$ and $K \in \mathcal{R}$ we have $\|BKQ_\tau\| \longrightarrow 0$, and $\|Q_\tau BK\| \le$ $\le \|Q_\tau B\ P_\mu\|\ \|K\| + \|Q_\tau B\|\ \|Q_\mu K\|$ which becomes as small as desired if μ is choosen suitably and if τ is small enough.

b) Kozak's formula

$$(10) \qquad (P_\tau BP_\tau)^{-1} = P_\tau B^{-1}P_\tau - P_\tau B^{-1}Q_\tau(Q_\tau B^{-1}Q_\tau)^{-1}Q_\tau B^{-1}P_\tau$$

holding for arbitrary invertible operators B shows that the operators $P_\tau BP_\tau$ are invertible for τ large enough and that the norms of their inverses are uniformly bounded if and only if the operators $Q_\tau B^{-1}Q_\tau$ are invertible for τ large enough and if only if the norms of their inverses are uniformly bounded.

Taking into account Proposition 4 we have only to verify that the operators $Q_\tau(A + K)^{-1}Q_\tau$ are invertible for τ large enough and that the norms of their inverses are uniformly bounded.

To see this notice that again by Proposition 4 the operator A must be invertible and that by Proposition 3 the operators A^{-1} and $(A+K)^{-1}$ are of local type. The identity

$$(A + K)^{-1} = A^{-1} - (A + K)^{-1}K A^{-1}$$

yields

$$Q_\tau(A+K)^{-1}Q_\tau = Q_\tau A^{-1}Q_\tau - Q_\tau(A+K)^{-1}KP_\omega A^{-1}Q_\tau - Q_\tau(A+K)^{-1}KQ_\omega A^{-1}Q_\tau.$$

Now choose an $w \in T$ such that $\|Q_\tau(A+K)^{-1}KQ_w A^{-1}Q_\tau\|$ becomes as small as desired. Fixing this w, the norm $\|Q_\tau(A+K)^{-1}KP_w A^{-1}Q_\tau\|$ becomes small as $\tau \longrightarrow \infty$ since A^{-1} is of local type. This guarantees the invertibility of $Q_\tau(A+K)^{-1}Q_\tau$ for τ large enough, and we are done.

Now we turn our attention to an example which was studied in [1], [3], and [7] : the finite section method for Wiener-Hopf operators in spaces with uniform convergence. Thereby, we only quote the results and give sketches of the proofs and refer to [6] and [7] for a detailed treatment where, moreover, estimations of the speed of convergence are given.

Let M resp. C the Banach spaces of all bounded measurable resp. continuous functions on the real half axis $\mathbb{R}^+$, and let M_0, C_0 stand their subspaces

$$M_0 = \{x \in M : \operatorname*{ess\,sup}_{s \geq a} |x(s)| \longrightarrow 0 \text{ as } a \longrightarrow \infty\}$$

and

$$C_0 = \{x \in C : \lim_{s \to \infty} x(s) = 0\} .$$

If G is one of the spaces mentioned, let G^{μ} be the set of all functions x for which $(1+s)^{\mu}x(s) \in G$. When endowed with the norm $\|x\|_{G^{\mu}} := \|(1+s)^{\mu}x(s)\|_G$, G^{μ} becomes a Banach space. In the sequel we denote by E^{μ} one of the spaces M^{μ} or C^{μ}.

Put $T = \mathbb{R}^+$, let the projections P_τ specialize as

$$(P_\tau x)(s) = \begin{cases} x(s) & \text{if } s \leq \tau \\ 0 & \text{if } s > \tau , \end{cases}$$

set $\mathcal{P} = \{P_\tau\}_{\tau \geq 0}$ and choose $\mathcal{R}$ as above.

Further let a be a function whose Fourier transform φ is in $L^{1,|\mu|}(\mathbb{R}) := (L^1(\mathbb{R}))^{\mu}$ and define the Wiener-Hopf integral operator W_φ by

$$(11) \qquad (W_\varphi x)(s) := x(s) - \int_0^{\infty} a(s-t)\, x(t)\, dt.$$

It is a simple matter to verify that the operator (11) is of local type (i.e. that it belongs to B with respect to $\mathcal{R}$). Moreover, the condition $\varphi \in L^{1,|\mu|}(\mathbb{R})$ guarantees that, if W_φ is invertible, $(W_\varphi)^{-1} = W_{\varphi^{-1}} + K$ with a compact operator K acting from M^{μ} into C_0^{μ} (see [7]), what implies that $K \in \mathcal{R}$.

P r o p o s i t i o n 7. Let $\varphi \in L^{1,|\mu|}(\mathbb{R})$. If the operators W_φ and $W_{\varphi^{-1}}$ are invertible then $W_\varphi \in \Pi_{\mathcal{R}}\mathcal{P}$.

P r o o f. By Proposition 4 and by Kozak's formula (10) it suffices to verify the invertibility of $Q_\tau W_\varphi^{-1} Q_\tau$ for τ large enough and the uniform boundedness of the norms of the inverses. Since $W_\varphi^{-1} - W_{\varphi^{-1}} \in \mathcal{R}$ we have essentially to deal with the invertibility of $Q_\tau W_{\varphi^{-1}} Q_\tau$.

Let $W_{\varphi-} W_{\varphi+}$ stand for the Wiener-Hopf factorization of $W_{\varphi^{-1}}$ which exists by the invertibility assumptions. Then $Q_\tau W_{\varphi^{-1}} Q_\tau = Q_\tau W_{\varphi-} Q_\tau W_{\varphi+} Q_\tau$. Thereby, the operators $Q_\tau W_{\varphi\pm} Q_\tau$ are invertible, and their inverses are the operators $Q_\tau (W_{\varphi\pm})^{-1} Q_\tau$, respectively. Thus, the invertibility of $Q_\tau W_{\varphi^{-1}} Q_\tau$ is proved, and the uniform boundedness is almost evident.

R e m a r k. In the scalar case the invertibility of W_φ implies that of $W_{\varphi^{-1}}$. In the system case this is no longer true.

Now we are going to characterize some suitable perturbations for the finite section method $\Pi_{\mathcal{R}}\mathcal{P}$. i.e. we quote some operators belonging to the ideal $\mathcal{R}$.

Define an operator $K : E^\mu \longrightarrow E^\mu$ by

$$(12) \qquad (Kx)(s) = \int_0^\infty k(s,t)\, x(t)\, dt \ , \ s \in \mathbb{R}^+, \ x \in E^\mu ,$$

and put $k_s(t) := k(s,t)$. For the remainder of this section we require that

$$(13a) \qquad k_s \in L^{1,-\mu}(\mathbb{R}^+) \quad \text{for all } s \in \mathbb{R}^+ ,$$

(13b) $$\sup_s \|(1+s)^{\mu} k_s\|_{1,-\mu} < \infty ,$$

(13c) $$\begin{cases} \|(1+s')^{\mu} k_{s'} - (1+s)^{\mu} k_s\|_{1,-\mu} \longrightarrow 0 \text{ as } s' \longrightarrow s \\ \text{uniformly with respect to } s \in \mathbb{R}^+, \end{cases}$$

(13d) $$\|(1+s)^{\mu} k_s\|_{1,-\mu} \longrightarrow 0 \text{ as } s \longrightarrow \infty .$$

P r o p o s i t i o n 8. If (13a-d) are fulfilled, the operator K defined by (12) belongs to $\mathcal{R}$.

P r o o f. We only verify that $\|KQ_\tau\| \longrightarrow 0$. To that end put

$$v_a(s) := (1+s)^{\mu} \int_a^{\infty} |k(s,t)| \, (1+t)^{-\mu} \, dt \qquad (a,s \in \mathbb{R}^+).$$

Obviously, $v_a \in C(\mathbb{R}^+)$, $\{v_a\}_{a\in\mathbb{R}^+}$ is a bounded and equicontinuous function set, and $v_a \xrightarrow{\mathcal{R}} 0$ as $a \longrightarrow \infty$. The last assertion bases on the following facts :

a) For fixed s, $v_a(s)$ tends to zero as $a \longrightarrow \infty$.

b) The convergence of equicontinuous functions to a continuous function is uniform on compact intervals.

For $\tau \in \mathbb{R}^+$, the operator $(KP_\tau x)(s) = \int_0^{\tau} k(s,t)\, x(t)\, dt$ is compact as acting from M^{μ} into C^{μ}. The assumption (13d) implies that $v_\tau(s) \longrightarrow 0$ as $s \longrightarrow \infty$ uniformly with respect to τ. Given $\varepsilon > 0$ there exists an s_0 such that $\sup_{s\ge s_0} v_\tau(s) < \varepsilon$ for all $\tau \ge 0$. Because $v_\tau \xrightarrow{\mathcal{R}} 0$, we can find a τ_0 such that

$\|v_\tau\|_{s_0} < \varepsilon$ whenever $\tau \ge \tau_0$. From the obvious inequality

$$\|KQ_\tau x\| \le \sup_s v_\tau(s) \sup_{s\ge\tau} |(1+s)^\mu x(s)|$$

we finally deduce $\|KQ_\tau x\| \le \sup_s v_\tau(s) < 2\varepsilon$ for all $\tau \ge \tau_0$, and this proves the assertion.

P r o p o s i t i o n 9. Let $\varphi \in L^{1,|\mu|}$, and let K be the operator (12) fulfilling (13a-d). If $W_\varphi + K$ and $W_{\varphi^{-1}}$ are invertible then $W_\varphi + K \in \Pi_R \mathcal{P}$.

P r o o f. Combine the Proposition 6, 7 and 8.

5. A composite quadrature rule method for Wiener-Hopf integral operators

In this section we apply our approach to a composite quadrature rule method (the Nystrom method) which was proposed by Chandler/Graham in [2] for solving the integral equation $(I-K)x = y$ in spaces of continuous functions. Thereby K denotes the operator

$$(Kx)(s) = \int_0^\infty k(s,t)\, x(t)\, dt\ , \quad s \ge 0$$

in the space C_0.

The Nystrom method is understood as follows : For integers n suppose a mesh $0 = z_0^{(n)} < \ldots < z_n^{(n)} = \infty$ is selected. Let $I_i^{(n)} = (z_{i-1}^{(n)}, z_i^{(n)})$ and $h_i^{(n)} = z_i^{(n)} - z_{i-1}^{(n)}$.

For any function v, $v_i^{(n)}$ denotes the restriction $v|_{I_i^{(n)}}$.

Further, assume that $h_i^{(n)} < \gamma$ for some constant γ independent of i and n.

Moreover, suppose that we are given an m-point quadrature rule on [0,1] :

$$\int_0^1 v(\zeta)\, d\zeta \rightsquigarrow \sum_{j=1}^{m} w_j v(\zeta_j) \ , \quad (v \in C)$$

where w_j and ζ_j are the quadrature weights and points, respectively. Now define $\zeta_{ij}^{(n)} = z_{i-1}^{(n)} + h_j^{(n)} \zeta_j$ for $i=1,\dots,n-1$. For $\beta = \beta(n)$ we define the approximation operator K_n by

$$(14) \qquad (K_n x)(s) := \sum_{(i,j) \in Q(n)} w_j^{(n)} k(s, \zeta_{ij}^{(n)})\ x(\zeta_{ij}^{(n)})\ h_i^{(n)} \ ,$$

where $Q(n) = \{(i,j) : z_i < \beta(n),\ 1 \le j \le m \}$, $x \in C$.

Given this definition of K_n , the Nystrom solution x_n of the equation $(I-K)x = y$ is defined by

$$(15) \qquad x_n(s) - (K_n x_n)(s) = y(s), \ s \in \mathbb{R}^+.$$

We shall now discuss the precise choice of the meshes $\{z_i^{(n)}\}$ and of the upper bounds $\beta(n)$ here and refer to [2] for details. We only want to point out that the Nystrom method considered, for instance, in the space C of all bounded continuous functions is subject of our general approach with only slight modifications.

First notice that according to (15) the projections P_τ must equal the identity operator I for each $\tau \in T \subseteq \mathbb{Z}^+$. Then, of course, condition (9) from section 3 is satisfied. A problem arises when the convergence-generating projections $R_{[0,\tau)}$ must be chosen, since they don't map C into itself.

On the other hand, the concept of section 4 where we embedded C into M fails here, too, since the quadrature rule (14) cannot be defined for arbitrary $x \in M$. But it turns out that the Banach space PC of all functions f on $\mathbb{R}^+$ which possess one-sided limits at each point $s \in \mathbb{R}^+$ when endowed with the norm $\|f\|_\infty = \sup_{\mathbb{R}^+} |f(t)|$ is a natural candidate to overcome these difficulties at least partially, because

a) C is a closed subspace of PC .

b) The projections $R_{[0,\tau)}$ chosen as in section 4 map PC into itself.

c) Under suitable conditions (see (16) below) the operators K and K_n are bounded on PC.

On the other hand the approximation operators K_n map actually PC into C, but we don't know whether they converge in any sense to K. For this reason we modify our generalapproach as in section 2 as follows : Assume that X and X_1 are Banach spaces and that X_1 is a closed subspace of X. Let the projection family $\mathcal{R}$ be given on X as in section 2, but consider the following weaker notion of convergence of operators $A_\tau \in L(X)$ to $A \in L(X)$:

A_τ converges to A with respect to $\mathcal{R}$ and X_1 if for each $w \in \mathbb{R}^+$ and for each $x \in X_1$

$$R_{[0,w)}A_\tau x \longrightarrow R_{[0,w)}Ax.$$

Notice that the $(\mathcal{R},X_1)$-limit is only uniquely determined on the supspace X_1 of X.

The following modification of Proposition 4 is sufficient for our aims.

P r o p o s i t i o n 4'. Let $A \in L(X)$, $A_\tau : \text{im } P_\tau \longrightarrow \text{im} P_\tau$ be operators such that

(i) $\sup_\tau \|A_\tau\|_X < \infty$,

(ii) $A_\tau \xrightarrow{(\mathcal{R},X_1)} A$,

(iii) $\{A_\tau\}$ and A are of local type on X in the sense of (6),

(iv) A is invertible in L(X) ,

(v) A_τ : im $P_\tau \longrightarrow$ im P_τ is invertible for $\tau \geq \tau_0$ and

$\sup_\tau \|A_\tau^{-1}\|_X < \infty$.

If X_1 is invariant for A, A^{-1}, A_τ, A_τ^{-1} ($\tau \geq \tau_0$) then $A' \in \Pi_{\mathcal{R}}\{A'_\tau\}$ where A', A'_τ denotes the restriction of A and A_τ onto X_1.

The proof runs paralelly to that of Proposition 4.

Now put X = PC, X_1 = C, write $K = K^0 + K^1$ where

$$(K^0x)(s) = \int_0^\infty k^0(s-t)\ x(t)\ dt$$

and

$$(K^1x)(s) = \int_0^\infty k^1(s,t)\ x(t)\ dt\ .$$

Hereby we assume that

$$\text{(16a)} \qquad \int_{-\infty}^\infty |k^0(t)|\ dt < \infty\ , \quad \int_{-\infty}^\infty \left|\frac{d}{dt}\ k^0(t)\right|\ dt < \infty\ .$$

For all integer n ≥ 0,

$$\text{(16b)} \quad \begin{cases} \sup\left\{ \left\| \dfrac{\partial}{\partial s^m} \dfrac{\partial}{\partial t^n}\ k(s,t) \right\|_{L^1} : s \in \mathbb{R}^+ \right\} < \infty\ , \\[2ex] \sup\left\{ \left\| \dfrac{\partial}{\partial s^m} \dfrac{\partial}{\partial t^n}\ k^1(s,t) \right\|_{L^1} : s \in \mathbb{R}^+ \right\} < \infty\ . \end{cases}$$

(16c)
$$\lim_{s_1, s_2 \to \infty} \left\| k^1_{s_1} - k^1_{s_2} \right\|_{L^1} = 0 .$$

These conditions ensure that K^0 is bounded both on C and PC and that K^1 is compact.

P r o p o s i t i o n 10. If (16a-c) are fulfilled then

(4')
$$\sup_{n \in \mathbb{Z}^+} \|K_n\| < \infty .$$

(5')
$$R_{[0,\tau)} K_n x \longrightarrow R_{[0,\tau)} K x \quad \text{for } x \in C.$$

(6')
$$\begin{cases} \sup_{\tau,\mu,n} \left\| R_{[0,\tau)} K_n R_{[\mu,\infty)} \right\|_\infty \longrightarrow 0 \quad \text{as} \quad \mu - \tau \longrightarrow \infty , \\ \sup_{\tau,\mu,n} \left\| R_{[\mu,\infty)} K_n R_{[0,\tau)} \right\|_\infty \longrightarrow 0 \quad \text{as} \quad \mu - \tau \longrightarrow \infty . \end{cases}$$

P r o o f. For a proof of (4') see [2], Lemma 1 for m=0. Next we show the first assertion of (6') : Let $x \in PC$. Then

$$\left\| R_{[0,\tau)} K_n R_{[\mu,\infty)} x \right\| =$$

$$= \sup_{s \le \tau} \Big| \sum_{\substack{(i,j) \in Q(n) \\ \zeta_{ij} \ge \mu}} w_j^{(n)} \, k(s, \zeta_{ij}^{(n)}) \, x(\zeta_{ij}^{(n)}) \, h_i^{(n)} \Big| \le$$

$$\le \|x\|_\infty \sup_{s \le \tau} \sum_{\substack{(i,j) \in Q(n) \\ \zeta_{ij} \ge \mu}} h_i^{(n)} \, w_j^{(n)} \left| k(s, \zeta_{ij}^{(n)}) \right| \le$$

$$\leq \|x\|_\infty \cdot C \sup_{s\leq\tau} \sum_{i:z_i<\beta(n)} \left[\left\| k(s,\cdot)_i^{(n)} \right\|_{L_1(\mu,\infty)} + \right.$$

$$\left. + h_i^{(n)} \left\| \frac{\partial}{\partial t} k(s,\cdot)_i^{(n)} \right\|_{L_1(\mu,\infty)} \right]$$

by Lemma 1, (iii) of [2]. Hence,

$$\left\| R_{[0,\tau)} K_n R_{[\mu,\infty)} \right\|_\infty \leq$$

$$\leq C \left(\sup_{s\leq\tau} \int_\mu^\infty |k(s,t)| \, dt + \gamma \sup_{s\leq\tau} \int_\mu^\infty \left| \frac{\partial}{\partial t} k(s,t) \right| d\zeta \right) .$$

The first supremum tends to zero since the operator K is of local type under the assumptions (15a-c) ; the proof for the second supremum runs completely analogously. Similarly, the second assertion of (6') can be shown.

For verifying (5') we have to prove that for each τ, $\mu \in \mathbb{R}^+$ and for each $x \in C$

(17) $$\sup_{s\leq\tau} |f_n(s)| \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

where $f_n = (K - K_n)x$.

Since x is continuous for fixed s we have $|f_n(s)| \longrightarrow 0$. The assertion (17) follows from the fact that the family $\{f_n\}$ is equicontinuous at each point $s \in \mathbb{R}^+$ which is a simple consequence of Lemma 1, (iii), of [2] again, and the proof is complete.

Further, notice that C is an invariant subspace for $(I-K)^{\pm 1}$ and $(I-K_n)^{\pm 1}$ if the inverse operators exist.

Thus, Proposition 4' combined with Proposition 10 gives the convergence of the Nystrom method.

P r o p o s i t i o n 11. Let (16a-c) be fulfilled. If K is invertible from PC to PC (and from C to C) then the meshes $\{z_i^{(n)}\}$ and the bounds $\beta(n)$ can be chosen so that the Nystrom method $\Pi_{\mathcal{R}}\{K_n\}$ converges on the space C of all bounded continuous functions on $\mathbb{R}^+$.

R e m a r k. See [2] for the concrete choice of the meshes and bounds.

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