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CONNECTIONS ON TANGENT BUNDLES OF HIGHER ORDER

Introduction

As it is well known the tangent bundle TM of any manifold M carries a canonical integrable almost tangent structure J (see [Go], [Gr], [YI]). By means of J, Grifone [Gr] gave a new definition of (non-homogeneous) connection on M. In fact, a (non-homogeneous) connection on M is a vector 1-form Γ on TM such that $J\Gamma = J$ and $\Gamma J = -J$. The connection Γ is said to be homogeneous if Γ is homogeneous as a vector 1-form. Obviously, if Γ is C^{∞} on all TM, then it is a linear connection (see [V]); so, we must suppose that Γ is C^{∞} only on $\tilde{T}M$ = TM − { zero section } in order to obtain strictly non-linear homogeneous connections.

The purpose of this paper is to generalize the results of Grifone to tangent bundles of higher order. In fact, the tangent bundle T^kM of order k of any manifold M carries a canonical almost tangent structure of order k, namely J_1 (see [CSC], [DLV1], [DLV2], [DLR1], [DLR2]). Now, the variety of fibrations on T^kM permit us to consider several types of connections of order k on M. If we put $J_r = (J_1)^r$, $1 \le r \le k$, then we obtain k canonical vector 1-forms on T^kM. Thus, a connection of order k and type r, $1 \le r \le k$, on M (that is, a connection in the fibration $\rho^k_{k-r}: T^kM \longrightarrow T^{k-r}M$) is given by a vector 1-form Γ on T^kM such that $J_r\Gamma = J_r$ and $\Gamma J_{k-r+1} = -J_{k-r+1}$. The tension $H = (1/2)\ [C_1,\Gamma]$ (where C_1 is the generalized Liouville vector field on T^kM) measures the non-homogenity of Γ ; so, Γ is homogeneous if and only if H vanishes. Now, we may associate to Γ a semispray ξ on T^kM of the same type r in such a way that Γ and ξ are the same paths. The converse is also proved ; in fact, to each semispray ξ of type 1 on T^kM we associate k connections Γ_1, ..., Γ_k of order k on M and types 1,...,k, respectively. Furthermore, the Frölicher -Nijenhuis formalism permit us to obtain the curvature and torsion forms of Γ. In fact, the curvature form R of Γ is given by $R = (1/2)\ [h,h]$, where h is the horizontal projector associated to Γ ; the weak and strong torsion forms t and T are given by $t = [J_r,h]$ and $T = i_\xi t - H$, respectively. Finally, we prove the main result of this paper which generalizes the corresponding one obtained by Grifone for the tangent bundle :

Let ξ be a semispray of type 1 and T a semibasic vector 1-form of type 1 on T^kM such that $i_\xi T + \xi^* = 0$ (where $\xi^* = [C_1,\xi] - \xi$ denotes the deviation of ξ). Then there exists a unique connection Γ of order k and type 1 on M such that its associated semispray is ξ and its strong torsion is T. The connection Γ is given by

$$\Gamma = (1/(k+1))\ \{ -2\ L_\xi J_1 + (k-1)\ I + 2\ T \}.$$

Consequently, a connection of order k and type 1 on M is completely determined by its associated semispray and its strong torsion.

In a forthcoming paper [ALR], we shall apply these results to obtain a canonical connection associated to a regular Lagrangian system of higher order (see [DLR2] for the homogeneous case).

1. Tangent bundles of higher order

Let M be an n-dimensional manifold. The *tangent bundle of order k* of M is the (k+1)n-dimensional manifold T^kM of k-jets at $0 \in R$ of differentiable mappings $\sigma : R \longrightarrow M$. We denote by $\beta : T^kM \longrightarrow M$ the canonical projection defined by $\beta(j_0^k \sigma) = \sigma(0)$. Then T^kM has a bundle structure over M. If $k = 1$, then $T^1M = TM$ is the tangent bundle of M. However, if $k > 1$, $\beta^k : T^kM \longrightarrow M$ is not a vector bundle. As well as being fibred over M, T^kM is also fibred over T^rM, $0 < r < k$. A projection map $\rho_r^k : T^kM \longrightarrow T^rM$ is defined by $j_0^k \sigma \longrightarrow j_0^r \sigma$. These projection maps verify

$$\rho_s^k = \rho_s^r \rho_r^k ,$$

for any r, s, with $0 \leq s < r < k$; here ρ_0^k is interpreted as β^k.

Notice that T^kM is associated to the principal bundle F^kM of the frames of order k on M (see [Ga]). In fact, T^kM is the tangent bundle of 1^k-velocities of M introduced by Ehresmann (see [Eh], [Mo], [Tu1]).

We shall now describe the local coordinates in T^kM. Let (U, z^A) be a coordinate neighborhood of M and $\sigma : R \longrightarrow M$ a curve on M such that $\sigma(0) \in U$. Put $\sigma^A = z^A \circ \sigma$, $1 \leq A \leq n$. Then the k-jet $j_0^k \sigma$ is uniquely represented in $(\beta^k)^{-1}(U) =$

T^kU by

$$(z^A, z_1^A, \ldots, z_k^A)$$

where

$$z^A = \sigma^A(0), \quad z_i^A = (1/i!)\,(d^i\sigma^A/dt^i)\Big|_{t=0}, \quad 1 \le i \le k.$$

(in the sequel we put $z_0^A = z^A$).

Now, we can define a canonical mapping

$$T_{k,r-1} : T^kM \longrightarrow T(T^{r-1}M), \; 1 \le r \le k,$$

given by

$$j_0^k\,\sigma \longrightarrow j_0^1\,\tau,$$

where $\tau : R \longrightarrow T^{r-1}M$, $t \longrightarrow \tau(t) = j_0^{r-1}\sigma_t$, $\sigma_t(s) = \sigma(s+t)$. A simple computation shows that $T_{k,r-1}$ is locally given by

$$(z^A, z_1^A, \ldots, z_k^A) \longrightarrow (z^A, z_1^A, \ldots, z_{r-1}^A;\; z_1^A,\; 2z_2^A, \ldots, r\,z_r^A).$$

We use these maps $T_{k,r-1}$ to construct a differential operator d_T which maps each function f on T^kM to a function d_Tf on $T^{k+1}M$ defined by

$$(d_Tf)\,(j_0^{k+1}\,\sigma) = df(j_0^k\,\sigma)\,(T_{k+1,k}(j_0^{k+1}\,\sigma)).$$

Then d_Tf is locally expressed by

$$(d_Tf) = \sum_{i=0}^{k} (i+1)\, z_{i+1}^A\, (\partial f/\partial z_i^A).$$

So, we have

(1.1) $$d_T(z_i^A) = (i+1)\, z_{i+1}^A, \quad \text{and}$$

(1.2) $$d_T^i\,(z_0^A) = (i!)z_i^A,$$

with $0 \le i \le k$.

Since $d_T(fg) = d_T(f)g + d_T(g)\, f$, then d_T extends to an operator which maps a p-form α on T^kM into a p-form $d_T\alpha$ on $T^{k+1}M$ in such a way that $d_T d = dd_T$ (see [DLR2], [Tu1]).

From (1.1) and (1.2) we deduce

P r o p o s i t i o n 1.1. Let X and Y be a vector fields on T^kM. Then $X = Y$ if and only if

$$X((\rho^k_r)^* (d^r_T f)) = Y((\rho^k_r)^* (d_T f)),$$

for every function f on M, $0 \leq r \leq k$ (here, ρ^k_k is to be interpreted as the identity map and T^0M is identified to M).

Now, we shall describe a lifting operator which generalises the vertical lift in tangent bundle geometry, as described in [YI], for example.

D e f i n i t i o n 1.1. Let X be a vector field on T^rM, $0 < r < k$. The *vertical lift* of X to T^kM (with respect to $\rho^{\ k}_{k-r-1} : T^kM \longrightarrow T^{k-r-1}M$) is the unique vector field X^k on T^kM defined by

$$X^k((\rho^k_s)^*(d^s_T f)) = \begin{cases} 0\,, & \text{if } 0 \leq s \leq k-r-1 \\ (s!)/(s-(k-r))!\ X((\rho^{\ k}_{s-(k-r)})^*(d^s_T f)), & \\ \qquad \text{if } k-r \leq s \leq k & \end{cases}$$

for every function f on M.

We remark that the case $r = 0$ was considered by Crampin, Sarlet and Cantrijn [CSC].

If $X = \sum_{i=0}^{r} X^A_i\, (\partial/\partial z^A_i)$, then we deduce from (1.1) and (1.2) that

(1.3) $$X^k = \sum_{i=0}^{r} X_i^A\ (\partial/\partial z^A_{k-r+i}).$$

Using the vertical lift and the map $T_{k,k-r}: T^kM \longrightarrow T(T^{k-r}M)$ we construct a canonical vector field C_1 on T^kM as follows : $C_1(j_0^k\ \sigma) = (T_{k,k-1}(j_0^k\ \sigma))^k$.

So, C_1 is locally expressed by

(1.4) $$C_1 = \sum_{i=1}^{k} i\ z_i^A\ (\partial/\partial z_i^A).$$

The vector field C_1 is generalization of the Liouville vector field (or dilation field) on TM (see [CSC], [DLR1], [DLR2], [DLV1]).

We may also use the vertical lift construction to define k canonical tensor fields of type (1.1) (or vector 1-forms) on T^kM. In fact, for each r, $1 \leq r \leq k$, we can define a linear endomorphism $(J_r)_z$ of the tangent space $T_z(T^kM)$ of T^kM at $z \in T^kM$ as follows :

$$(J_r)_z(X) = ((\rho^k_{k-r})_*\ X)^k.$$

So, we have

P r o p o s i t i o n 1.2. J_r has constant rank $(k-r+1)n$ and verifies

$$(J_r)^s = \begin{cases} 0\ , & \text{if } rs \leq k+1, \\ J_{rs}, & \text{if } rs > k+1. \end{cases}$$

Furthermore J_1 determines an almost tangent structure of order k on T^kM.

R e m a r k. An alternative definition of J_r, $1 \le r \le k$, has been given by de Leon and Villaverde (see [DLV1], [DLR2]).

If we put $C_r = J_{r-1}C_1$, $2 \le r \le k$, we have the following identities :

$$(1.6) \qquad J_rC_s = \begin{cases} 0, & \text{if } r+s \le k+1 \\ C_{r+s}, & \text{if } r+s > k+1, \end{cases}$$

$$(1.7) \qquad L_{C_r}J_s = \begin{cases} 0, & \text{if } r+s > k+1 \\ -sJ_{r+s-1}, & \text{if } r+s \le k+1, \end{cases}$$

$$(1.8) \qquad [J_r, J_s] = 0.$$

A direct consequence of (1.8) is the vanishing of the Nijenhuis tensor $N_{J_1} = (1/2)\,[J_1, J_1]$ of J_1. Therefore the almost tangent structure of order k on T^kM defined by J_1 is always integrable (see [DLV1], [DLR2]).

2. Homogeneous and semibasic forms

Let us recall the following definitions (see [DLV2], [DLR2]).

(a) Homogeneous forms

D e f i n i t i o n 2.1. A function f on T^kM is said to be *homogenous* of degree a if $L_{C_1} f = a\, f$.

Let $h_t : R \longrightarrow R$ be the homotetic of ratio e^t and let $H_t : T^kM \longrightarrow T^kM$ denote the fibre-preserving transformation induced from h_t, that is $H_t(j^k_0\ \sigma) = j^k_0\ (\sigma \circ h_t)$. Since C_1 generates the 1-parameter group of transformations H_t then the condition in Definition 2.1 is equivalent to $f \circ H_t = e^{at}f$.

Definition 2.2. A (scalar) p-form α on T^kM is said to be *homogeneous* of degree a if $L_{C_1}\alpha = a\ \alpha$.

Definition 2.3 A vector 1-form L on T^kM is said to be *homogeneous* of degree a if $L_{C_1}L = (a-1)L$.

(b) Semibasic forms

Definition 2.4. A (scalar) p-form α on T^kM is said to be *semibasic of type r* if $\alpha \in \operatorname{Im} J^*_r$.

Since $\operatorname{Im} J_{k-r+1} = \operatorname{Ker} J_r$ we deduce that a Pfaff form α on T^kM is semibasic of type r if and only if $\alpha(J_{k-r+1}X) = 0$, for every vector field X on T^kM. Then a Pfaff form α on T^kM is semibasic of type r if and only if α is locally expressed by

$$\alpha = \sum_{i=0}^{k-r} \alpha^i_A\ dz^A_i. \tag{2.1}$$

Let α be a semibasic Pfaff form of type r on T^kM. Then we can define a map $D : T^kM \longrightarrow T^*(T^{k-r}M)$ as follows :
$D(j^k_0\ \sigma)(Y) = \alpha(j^k_0\ \sigma)(X)$,
where $Y \in T_{j^{k-r}_0\sigma}(T^{k-r}M)$ and $X \in T_{j^k_0\sigma}(T^kM)$, with $(\rho^k_{k-r})_*X = Y$ (since α is semibasic of type r then $D(j^k_0\ \sigma)$ is well-defined.

Moreover, if α is locally given by (2.1) then we have

$$D(z^A_0, \ \ldots\ , z^A_k) = \sum_{i=0}^{k-r} \alpha^i_A\ dz^A_i.$$

Therefore, the diagram

$$\begin{tikzcd} T^kM \arrow[rr, "D"] \arrow[dr, "\rho^k_{k-r}"'] & & T^*(T^{k-r}M) \arrow[dl, "q_{T^{k-r}M}"] \\ & T^{k-r}M & \end{tikzcd}$$

is commutative, where $q_{T^{k-r}M}$ is the canonical projection. Now, let λ_{k-r} be the Liouville form of $T^*(T^{k-r}M)$ (see [AM], [Go]). Then a simple computation shows that

$$D^*\lambda_{k-r} = \alpha.$$

Therefore, we can state the following.

P r o p o s i t i o n 2.1. Let α be a semibasis Pfaff form on T^kM of type r. Then α determines the map $D : T^kM \longrightarrow T^*(T^{k-r}M)$ such that

(1) $q_{T^{k-r}M} \circ D = \rho^k_{k-r}$,

(2) $D^* \lambda_{k-r} = \alpha$.

R e m a r k. An alternative proof of Proposition 2.1was given by de Lepn and Rodrigues [DLR2] following the one of Godbillon [Go] for the case k = 1.

D e f i n i t i o n 2.5. A vector 1-form L on T^kM, $1 \geq 1$ is said to be *semibasic of type r* if $J_rL = 0$ and $i_{J_{k-r+1}X} L = 0$, for any vector field X on T^kM.

3. Semisprays of higher order. Potentials

The aim of this section is to introduce a special kind of vector fields on higher tangent bundles.

D e f i n i t i o n 3.1. A vector field ξ on T^kM is said to be *semispray of type r* if $J_r\xi = C_r$.

From (1.5) we deduce that a semispray ξ on T^kM of type r is locally given by

$$\xi = z_1^A\,(\partial/\partial z_0^A) + 2\,z_2^A\,(\partial/\partial z_1^A) + \ldots + (k-r+1)\,z_{k-r+1}^A(\partial/\partial z_{kr}^A) +$$

$$+ \xi_{k-r+1}^A(\partial/\partial z_{k-r+1}^A) + \ldots + \xi_k^A\,(\partial/\partial z_k^A),$$

where $\xi_i^A = \xi_i^A(z_0^B, \ldots, z_k^B)$, $k-r+1 \le i \le k$, $1 \le A,B \le n$.

D e f i n i t i o n 3.2. Let ξ be a semispray on T^kM of type r. A curve σ in M is called *a path* (or *solution*) of ξif $j^k\sigma$ is an integral curve of ξ.

Consequently, a curve σ in M is a path of ξ if and only if it verifies the following system of differential equations :

$$(3.1)\quad (1/i!)\,(d^{i+1}\sigma^A/dt^{i+1}) = \xi_i^A(\sigma^B,\ d\sigma^B/dt,\ \ldots\ ,\ d^k\sigma^B/dt^k),$$

$$k-r+1 \le i \le k, \qquad 1 \le A,B \le n.$$

We shall express the non-homogeneity of a semispray

D e f i n i t i o n 3.3. Let ξ be a semispray on T^kM of type r. We shall call *deviation* of ξ the vector field $\xi^* = [C_1,\xi] - \xi$.

A simple computation shows that $J_r\xi^* = 0$.

D e f i n i t i o n 3.4. A semispray ξ on T^kM of type r is called *spray* of type r if ξ has zero deviation, that is $[C_1,\xi] = \xi$. If ξ is a spray then their paths are called *geodesics*.

From (3.1) and (3.2) we deduce that ξ is a spray if and only if the functions ξ_i^A, $k-r+1 \le i \le k$, are homogeneous of degree i+1.

The next proposition is an easy consequence of (1.6), (1.7) and (1.8).

P r o p o s i t i o n 3.1. Let ξ be a semispray on T^kM of type 1. Then we have

(1) $J_1[\xi, J_kX] = -k(J_kX)$,

(2) $J_1[\xi, J_1X] - J_2[\xi, X] = -J_1X$,

for every vector field X on T^kM.

Next, we shall introduce the potential of a semibasic form.

D e f i n i t i o n 3.5. Let α (resp. L) be a scalar p-form (resp. a vector l-form) on T^kM semibasic of type r. Then the *potential* α^0 of α (resp. L^0 of L) is the scalar (p-1)-form (resp. vector (l-1)-form) given by

$$\alpha^0 = i_\xi\alpha, \quad (\text{resp. } L^0 = i_\xi L)$$

where ξ is an arbitrary semispray of type r on T^kM. Obviously, α^0 (resp. L^0) does not depend on the choice of ξ since α (resp. L) is semibasic. Moreover, we have

P r o p o s i t i o n 3.2. α^0 and L^0 are semibasic of type r.

(We notice that the scalar p-form α is not necessarily skew-symmetric).

From (1.1) and (1.3) we deduce

P r o p o s i t i o n 3.3. Let ξ be a semispray on T^kM of type 1. Then

$$L_\xi((\rho^k_r)^*\alpha) = (\rho^k_{r+1})^*(d_T\alpha),$$

for every p-form α on T^kM, where r < k.

4. Connections of higher order

The variety of fibrations on T^kM permit us to consider several types of connections of order k on M.

D e f i n i t i o n 4.1. A vector 1-form Γ on T^kM such that

$$(4.1) \qquad J_r \Gamma = J_r, \quad \Gamma J_{k-r+1} = -J_{k-r+1}$$

will be called a *connection on M of order k and type r.*

R e m a r k. Obviously, we may consider connections of order k on M which are C^∞ on $\tilde{T}^kM = T^kM$-{ zero section }, not necessarily on all T^kM. Then a connection of order k and typer on M is a connection in the fibration $\rho^k_{k-r} : T^kM \longrightarrow T^{k-r}M$ (see [R]). Since

$$V^{\rho^k_{k-r}}(T^kM) = \operatorname{Im} J_{k-r+1} = \operatorname{Ker} JJ_r,$$

we deduce that Γ defines an almost product structure on T^kM, that is, $\Gamma^2 = I$. Therefore, we can consider the *horizontal* and *vertical projectors* associated to Γ :

$$h = (1/2)\ (I + \Gamma), \qquad v = (1/2)(I - \Gamma),$$

respectively. From (4.1) we have

$$(4.2) \quad \begin{cases} J_r h = J_r,\ hJ_{k-r+1} = 0,\ J_r v = 0,\ vJ_{k-r+1} = J_{k-r+1}\ , \\ \\ \operatorname{Im} v = V^{\rho^k_{k-r}}(T^kM)\ . \end{cases}$$

If we put $H = \operatorname{Im} h$, then

$$T(T^kM) = H \oplus V^{\rho^k_{k-r}}(T^kM).$$

Thus, the linear map

$$(\rho^k_{k-r})_* : H_z \longrightarrow T_{\rho^k_{k-r}(z)}(T^{k-r}M), \; z \in T^kM,$$

is an isomrphism. So, is X is a vector field on $T^{k-r}M$ then there exists a unique horizontal vector field X^H on T^kM such that

$$(\rho^k_{k-r})_* X^H = X \; ;$$

X^H will be called the *horizontal lift* of X to T^kM with respect to Γ. Also, let σ be a curve on $T^{k-r}M$, and $z \in T^kM$ such that $\sigma(0) = y$ and $(\rho^k_{k-r})(z) = y$. Then there exists a unique horizontal curve σ^H on T^kM such that $\sigma^H(0) = z$, and $\rho^k_{k-r} \circ \sigma^H = \sigma$; σ^H will be called the *horizontal lift* of σ to T^kM with respect to Γ.

From (4.1) we deduce that Γ is locally given by the matrix

$$(4.3) \qquad \Gamma = \begin{bmatrix} I_{(k-r+1)n} & 0 \\ & \\ -2\,\Gamma^{(i,j)B}{}_A & -I_{rn} \end{bmatrix}$$

where $\Gamma^{(i,j)B}{}_A = \Gamma^{(i,j)B}{}_A(z^C_0,\ldots,z^C_k)$, $0 \le i \le r-1$, $0 \le j \le k-r$.

Next, we shall consider a particular case of connections.

D e f i n i t i o n 4.2. Let Γ be a connection on M of order k and type r. The *tension* of Γ is the vector 1-form *H* on T^kM given by

$$H = (1/2)\ [C_1,\Gamma] = [C_1,h\ ].$$

The connection Γ is said to be *homogeneous* if its tension vanishes, that is Γ is an homogeneous vector 1-form of degree 1.

From Definition 4.2 we easily deduce that Γ is homogeneous if and only if the function $\Gamma^{(i,j)B}_{A}$ is homogeneous of degree $k-r+1-i-j$, $0 \le i \le r-1$, $0 \le j \le k-r$, $1 \le A,B \le n$.

The following proposition is a direct consequence of (4.3).

P r o p o s i t i o n 4.1. The tension H of a connection Γ of order k and type r on M is a semibasic vector 1-form of type r.

D e f i n i t i o n 4.3. A curve σ in M is called a *path* of a connection Γ of order k and type r on M if $j^k\sigma$ is a horizontal curve in T^kM. If Γ is homogeneous, then their paths are called *geodesics*.

From (4.3) we get that σ is a path of Γ if and only if σ verifies the following system of differential equations :

$$(4.4)\qquad (1/i!)(d^{i+1}\sigma^B/dt^{i+1}) =$$

$$= -\sum_{j=0}^{k-r} (1/j!)\ \Gamma^{(i-k+r-1,j)B}_{A}(d^{j+1}\sigma^A/dt^{j+1}),$$

$$k-r+1 \le i \le k,\ 1 \le A,B \le n.$$

R e m a r k. If Γ is a homogeneous connection of order 1 on M, then Γ defines a linear connection ∇ on M. In such a case, this system of differential equations becomes the usual one for the geodesics of ∇ (see [Gr], [DLR2]).

5. Semisprays associated to connections of higher order

In this section, we shall prove that, canonically associated to a connection of order k and type r on M, there exists a semispray on T^kM of the same type.

Let Γ be a connection of order k and type r on M. If ξ' is an arbitrary semispray on T^kM of the same type, then, from (4.2) $\xi = h\xi'$ is a semispray on T^kM of type r which not depends on the choice of ξ'. ξ will be called the *associated semispray* of Γ. Notice that $\Gamma\xi = \xi$. Moreover, if H is the tension of Γ, we have

$$H^0 = i_\xi H = H(\xi) = [C_1, h\xi] - h[C_1, \xi] = [C_1, \xi] - h[C_1, \xi].$$

Since $\xi^* = [C_1, \xi] - \xi \in \operatorname{Im} J_{k-r+1}$, we deduce

$$h([C_1, \xi] - \xi) = h[C_1, \xi] - \xi = 0.$$

Therefore, we have $H^0 = \xi^*$. So, the following proposition has been proved.

P r o p o s i t i o n 5.1. Let Γ be a connection of order k and type r on M with tension H. Then the associated semispray ξ of Γ verifies $\xi^* = H^0$.

C o r o l l a r y 5.1. If Γ is homogeneous, then ξ is a spray.

From (3.1) and (4.3) we deduce that the local expression of the semispray associated to Γ is

$$\xi = z_1^A(\partial/\partial z_0^A) + 2z_2^A(\partial/\partial z_1^A) + \ldots + (k-r+1)z_{k-r+1}^A(\partial/\partial z_{k-r}^A) +$$
$$+ \xi_{k-r+1}^A(\partial/\partial z_{k-r+1}^A) + \ldots + \xi_k^A(\partial/\partial z_k^A),$$

where

$$(5.1)\qquad \xi_i^A = -\sum_{j=0}^{k-r}(j+1)\, z_{j+1}^{B}\, \Gamma_B^{(i-k+r-1,j)A},$$

$$k-r+1 \le i \le k, \quad 1 \le A,B \le n.$$

Next, we give an alternative construction of the semispray ξ associated to Γ. Indeed, ξ is given by

$$\xi_z = (T_{k,k-r}(z))^H_z, \quad z \in T^kM.$$

Therefore, we have

P r o p o s i t i o n 5.2. Γ and ξ have the same paths.

P r o o f. Let σ be a path of Γ. Then $j^k\sigma$ is a horizontal curve in T^kM. Thus,

$$((j^k\sigma)(t)) = (T_{k,k-r}(j^k\sigma))^H_{(j^k\sigma)(t)} = \widehat{((j^{k-r}\sigma)(t))}^{\,\dot{}\,H}_{(j^k\sigma)(t)},$$

since $T_{k,k-r}\, j^k\sigma\,(t) = \dot{\widehat{(j^{k-r}\sigma)(t)}}$. On the other hand, we have $(\rho^k_{k-r})^*\, \dot{\widehat{(j^k\sigma)(t)}} = \rho^k_{k-r}\, \dot{\widehat{(j^k\sigma)(t)}} = \dot{\widehat{(j^{k-r}\sigma)(t)}}$. Consequently, $\dot{\widehat{(j^k\sigma)(t)}}$ is the horizontal lift of $T_{k,k-r}((j^k\sigma)(t))$ to T^kM at $(j^k\sigma)(t)$, and, so, σ is a path of ξ.
The converse is trivial.

R e m a r k. The reader can obtain directly Propositition 5.2. from (3.1) and (4.4).

6. Torsion and curvature of higher order connections

Let Γ be a connection of order k and type r on M.

D e f i n i t i o n 6.1. The *weak torsion* of Γ is the vector 2-form t on T^kM given by

$$t = (1/2)\,[J_r,\Gamma] = [J_r,h].$$

From the above definitions one has

P r o p o s i t i o n 6.1. The weak torsion t of Γ is a semibasic form of type r.

D e f i n i t i o n 6.2. The *strong torsion* of Γ is the vector 1-form T on T^kM given by

$$T = t^0 - H,$$

where H is the tension of Γ.

P r o p o s i t i o n 6.2. We have $T^0 + \xi^* = 0$, where ξ is the associated semispray to Γ.

P r o o f. In fact,

$$T^0 = (t^0 - H)^0 = (t^0)^0 - H^0 = -H^0 = -\xi^*,$$

since Proposition 5.1.

Now, we introduce the curvature of Γ.

D e f i n i t i o n 6.3. The *curvature* of Γ is the vector 2-form on T^kM given by

$$R = -\,(1/2)\,[h,h].$$

A straightforward computation shows that R is semibasic of type r.

We end this section proving the Bianchi identities for Γ.

P r o p o s i t i o n 6.3. We have

(1) $[J_r,R] = [h,t]$,

(2) $[h,R] = 0$.

P r o o f. (1) From the Jacobi identity, we deduce

$$[J_r,[h,h]] + [h,[h,J_r]] + [h,[J_r,h]] = 0.$$

Then

$$[J_r, R] = [h, t].$$

(2) is proved in a similar way.

P r o p o s i t i o n 6.4. We have

$$[C_1, R] = - [h, H].$$

P r o o f. As above, the result follows directly from the Jacobi identity.

C o r o l l a r y 6.1. If Γ is homogeneous, then also is R.

R e m a r k. If Γ is an homogeneous connection of order 1 on M, then the torsion and curvature forms of Γ may be related, in a natural way, with the torsion and curvature tensors óf the induced linear connection ∇ on M (see [Gr], [V]).

7. Associated connections to semisprays of higher order

In this section, we shall prove that, associated to a semispray ξ on T^kM of type 1, there exist k connections Γ_1, ..., Γ_k on M of order k and types 1, ... , k, respectively.

Before proceeding further, we shall need the following auxiliary lemma, obtained directly from Proposition 3.1.

L e m m a 7.1. We have

(1) $(L_\xi J_1)\, J_k = k\, J_k$,

(2) $J_1\, (L_\xi J_1) = - J_1$.

P r o p o s i t i o n 7.1. Let ξ be a semispray on T^kM of type 1. Then the vector 1-form Γ_1 given by

$$\Gamma_1 = (1/(k+1))\ \{-2\ L_\xi J_1 + (k-1)\ I\ \} \tag{7.1}$$

defines a connection of order k and type r on M.

P r o o f. In fact, from (2) of Lemma 7.1, we have

$$J_1\Gamma_1 = -\ (2/(k+1))\ J_1\ (L_\xi J_1) + ((k-1)/(k+1))\ J_1 =$$

$$= (2/(k+1))\ J_1 + ((k-1)/(k+1))\ J_1 = J_1.$$

On the other hand, from (1) of Lemma 7.1, we obtain

$$\Gamma_1 J_k = -\ (2/(k+1))\ (L_\xi J_1)\ J_k + ((k-1)/(k+1))\ J_k =$$

$$= -\ (2/(k+1))\ J_k + ((k-1)/(k+1))\ J_k = -\ J_k.$$

This ends the proof.

R e m a r k. We notice that, for each integer r, $1 \le r \le k$, there exists a connection Γ_r of order k and type r on M. Γ_r is given by

$$\Gamma_r = A_0\ I + \sum_{i=1}^{r} A_i\ L_\xi^i\ J_i\ , \tag{7.2}$$

where

$$A_0 = (k-2r+1)/(k+1),$$

$$A_r = (-1)^r\ (2/(k+1)k(k-1)\ \dots\ (k-r+2)),$$

$$A_{r-s} = (-1)^s\ (s!)\ \binom{r}{s}\ \binom{k-r+s}{s}\ A_r\ , \qquad 1 \le s \le r-1.$$

For k = 1, (7.2) becomes

$$\Gamma = -\ (L_\xi J) \quad \text{(see [Gr])}.$$

For k = 2, we have

$$\Gamma_1 = -\ (2/3)\ L_\xi J_1 + (1/3)\ I\ ,$$

$$\Gamma_2 = (1/3)\, L^2_\xi J_2 - (2/3)\, L_\xi J_1 - (1/3)\, I$$

(see [Ca2], [DL1]).

Next, we shall compute the tension, weak and strong torsions and the semispray associated to Γ_1.

First, let us notice that the horizontal and vertical projectors of Γ_1 are

$$h_1 = (1/(k+1))\, \{k\, I - L_\xi J_1\}, \text{ and } v_1 = (1/(k+1))\, \{I + L_\xi J_1\},$$

respectively. Then the associated semispray of Γ_1 is

$$\xi_1 = h_1\xi = (1/(k+1))\{k\xi - (L_\xi J_1)\xi\} = (1/(k+1))\{k\xi - [\xi, C_1]\} =$$

$$= \xi + (1/(k+1))\, \xi^*.$$

Now, from the Jacobi identity, we obtain that the weak torsion t_1 of Γ_1 vanishes, that is $t_1 = 0$. Moreover, the tension of Γ_1 is

$$H_1 = (1/2)\, [C_1, \Gamma_1] = -\, (1/(k+1))\, [C_1, [\xi, J_1]],$$

since $[C_1, I] = 0$. Therefore, from the Jacobi identity, we obtain

$$H_1 = -\, (1/(k+1))\, [\xi^*, J_1].$$

Finally, the strong torsion of Γ_1 is

$$T_1 = (1/(k+1))\, [\xi^*, J_1].$$

From these facts we deduce

P r o p o s i t i o n 7.2. If ξ is a spray on T^kM, then Γ_1 is an homogeneous torsionless connection which associated semispray is ξ.

8. Decomposition theorem

In this section, we shall prove that the strong torsion and the associated semispray characterize a connection of order k and type 1 on M.

We shall need the following result.

P r o p o s i t i o n 8.1. Let Γ and Γ' be two connections of order k and type 1 on M with the same strong torsion and the same associated semispray. Then $\Gamma = \Gamma'$.

P r o o f. Let ξ, t, T and H (resp. ξ', t', T' and H') be the associated semispray, weak torsion, strong torsion and tension of Γ (resp. Γ'). If we put $B = \Gamma' - \Gamma$, we have $J_1B = 0$ and $BJ_k = 0$. Therefore, B is a semibasic vector 1-form of type 1 on T^kM. Moreover, $B\xi = \Gamma'\xi - \Gamma\xi = \xi' - \xi = 0$, since $\xi = \xi'$. Now, we have

$$t' = (1/2)[J_1,\Gamma'] = (1/2)[J_1,\Gamma+B] = (1/2)[J_1,\Gamma] + (1/2)[J_1,B] =$$
$$= t + (1/2)[J_1,B],$$

$$H' = (1/2)\ [C_1,\Gamma'] = H + (1/2)\ [C_1,B],$$

$$T' = (\dot{t}')^0 - H = T + (1/2)\ ([J_1,B]^0 - [C_1,B]).$$

Since T' = T, we get

$$[J_1,B]^0 = [C_1,B].$$

Therefore, we deduce

$$BJ_1[\xi,X] - J_1[\xi,BX] - B\,[\xi,J_1X] = 0, \tag{8.1}$$

for every vector field X on T^kM. Since $BX \in \mathrm{Im}\, J_k$ and B is semibasic, we obtain

$$J_1[\xi, BX] = - k\ (BX),$$

by Proposition 3.1. On the other hand, we have

$$BJ_1[\xi,X] - B\ [\xi, J_1X] = B(J_1[\xi,X] - [\xi,J_1X])$$

$$= - B([\xi,J_1]X) = - B([\xi,J_1](h_1X)),$$

since B is semibasic. But $[\xi,J_1]h_1 = -h_1$. Then, we deduce

$$BJ_1[\xi,X] - B\ [\xi,J_1X] = B(h_1X) = BX.$$

Consequently, (8.1) becomes

$$BX + k\ (BX) = (k+1)\ (BX) = 0.$$

So, $BX = 0$, and then B vanishes. This ends the proof.

Next, we can state our main theorem, which generalizes a result due to Grifone [Gr] for the case $k = 1$.

T h e o r e m 8.1. (Decomposition theorem). Let ξ be a semispray of type 1 and T a semibasic vector 1-form of type 1 on T^kM such that $T^0 + \xi^* = 0$. Then there exists a unique connection Γ of order k and type 1 on M such that its associated semispray is ξ and its strong torsion is T. The connection Γ is given by

$$\Gamma = \Gamma_1 + (2/(k+1))\ T\ ,$$

where Γ_1 is given by (7.1).

P r o o f. <u>Existence</u> : Let $\Gamma = \Gamma_1 + (2/(k+1))\ T$. Then $J_1\Gamma = J_1$ and $\Gamma J_k = J_k$, because T is semibasic of type 1. So Γ is a connection of order k and type 1 on M. Now, if h is the horizontal projector of Γ, we have

$$h\xi = (1/2)\ (I + \Gamma)(\xi) = (1/2)\ (I + \Gamma_1)(\xi) + (1/(k+1))T(\xi)$$

$$= h_1(\xi) + (1/(k+1))T^0 = \xi + (1/(k+1))\xi^* + (1/(k+1))\ T^0 = \xi.$$

Furthermore, the weak torsion of Γ is

$$t = (1/2)\ [J_1,\Gamma] = t_1 + (1/(k+1))\ [J_1,T] = (1/(k+1))\ [J_1,T],$$

and the tension of Γ is

$$H = (1/2)\ [C_1,\Gamma] = H_1 + (1/(k+1))\ [C_1,T]$$

$$= (1/(k+1))\ ([C_1,T] - [\xi^*,T]).$$

Therefore, the strong torsion of Γ is

$$T' = t^0 - H = (1/(k+1))\ ([J_1,T]^0 + [C_1,T] - [\xi^*,J_1]).$$

But, an easy computation shows that

$$([J_1,T]^0 + [C_1,T] - [\xi^*,J_1])(X) = -\ T([\xi,J_1]X) - J_1[\xi.TX].$$

Consequently, we have

$$T'X = T'(h_1X) = -\ (1/(k+1))\ \{T([\xi,J_1](h_1X) + J_1[\xi,T(h_1X)]\} =$$

$$= -\ (1/(k+1))\ (-T(h_1X) - k\ T(h_1X)) = T(h_1X) = TX,$$

since T and T' are semibasic of type 1. Then T = T'.

Uniqueness : It is a direct consequence of Proposition 8.1.

R e m a r k. The decomposition theorem proves that a connection of order k and type 1 on M is completely determined by its associated semis pray and its strong torsion.

C o r o l l a r y 8.1. Let Γ be a connection of order k and type 1 on M. Then the strong torsion of Γ vanishes if and only if its weak torsion and tension also vanish.

Consequently, there are no non-homogeneous connections (of order k and type 1) with zero strong torsion.

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