DEMONSTRATIO MATHEMATICA
Vol. XXII No 3 1989

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# THEOREMS ON MAPPINGS INTO HAUSDORFF SPACE

In the present paper some properties of H.a.c. functions and weakly continuous functions into Hausdorff space are investigated.

Let X be a topological space. A subset S is said to be semi-open [4] if there exists an open set U of X such that $U \subset S \subset Cl\, U$. A subset S is said to be preopen (resp. $\alpha$-set) [6] (resp. [9]) if $S \subset Int\, Cl\, S$ (resp. $S \subset Int\, Cl(Int\, S)$). The family of all preopen set in X will be denoted by PO(X). The complement of semi-open set (resp. preopen set, resp. $\alpha$-set) is called semi-closed (resp. preclosed, resp. $\alpha$-closed). The set $\bigcap \{ F \subset X ;\ F$ is preclosed and $A \subset F \}$ is called the preclosure of A and denoted by Pcl A [1].

Let X and Y be two topological spaces. The function $f: X \longrightarrow Y$ is H-almost continuous (H. a. c.) at $x \in X$ if for each open set $V \subset Y$ containing $f(x)$, $x \in Int\, Cl(f^{-1}(V))$ [2].

The function $f: X \longrightarrow Y$ is H. a. c. if it has this property in any point $x \in X$. The function $f: X \longrightarrow Y$ is weakly continuous (w. c.) [5] at $x \in X$ if for any open set $G \subset Y$ with $f(x) \in G$, there exists an open set $U \subset X$ containing x such that $f(U) \subset Cl\, G$.

The function $f: X \longrightarrow Y$ is w.c. if has this property in any point $x \in X$. The function $f: X \longrightarrow Y$ is semi-continuous

[4] (resp. precontinuous [6], resp. $\alpha$-continuous [8]) if inverse image of each open set is semi-open (resp. preopen, resp. $\alpha$-set) in X.

The concepts of H-almost continuity and precontinuity are equivalent [6].

Some properties on mappings into Hausdorff space are studied in [7], [10] - [13], [15], [16].

The purpose of the present note is to investigate some properties of H. a. c. and w. c. functions into Hausdorff space.

The following theorem was proved in [15].

T h e o r e m 1. Let X be an arbitrary topological space and let Y be a Hausdorff space. Let f and g be functions from X into Y. If f is continuous and g is H-almost continuous, then the set $\{x \in X : f(x) = g(x)\}$ is preclosed in X (Theorem 3 ; [15]).

The purpose of Theorem 2 is to generalize and improve Theorem 1. The method used to prove Theorem 2 is similar with the method used in the demonstration of Theorem 1 from [3].

T h e o r e m 2. Let X and Y be two topological spaces. If $g : X \longrightarrow Y$ is H. a.c. and S is a closed subset of $X \times Y$ then $p_1(S \cap G(g))$ is preclosed in X, where G(g) denotes the graph of a function g and $p_1 : X \times Y \longrightarrow X$ represents the projection.

P r o o f. Let $x \in \mathrm{Pcl}\ [p_1(S \cap G(g))]$, where S is a closed subset of $X \times Y$, let U be an arbitrary open set containing x, and let V be an arbitrary open set containing

$g(x)$ in Y. Since g is H. a. c., $g^{-1}(V) \in PO(X)$ by Theorem 1 of [6]. It is shown in Lemma G of [7] that an intersection of an open set with a preopen set is preopen. Therefore $U \cap g^{-1}(V)$ is preopen. Since $x \in Pcl\ [p_1(S \cap G(g))]$, by Lemma 2.2 of [1], $[U \cap g^{-1}(V)] \cap p_1(S \cap G(g)) \neq \emptyset$.

Let $u \in [U \cap g^{-1}(V)] \cap [p_1(S \cap G(g))]$ be. This implies that $(u,g(u)) \in S$ and $g(u) \in V$. Therefore $(U\times V) \cap S \neq \emptyset$ and, consequently, $(x,g(x)) \in Cl\ S$, thus $(x,g(x)) \in S \cap G(g)$. Hence $x \in p_1(S \cap G(g))$ is preclosed.

C o r o l l a r y 1. If $f: X \longrightarrow Y$ has a closed graph and $g : X \longrightarrow Y$ is H. a. c. then the set $\{x\in X : f(x) = g(x)\}$ is preclosed.

P r o o f. Since $\{x\in X : f(x) = g(x)\} = p_1[G(f) \cap G(g)]$ and since $G(f)$ is a closed subset of $X \times Y$ the result follows from Theorem 2.

C o r o l l a r y 2. If $f: X \longrightarrow Y$ is weakly continuous, $g : X \longrightarrow Y$ is H. a. c. and Y is Hausdorff, then the set $\{x\in X : f(x) = g(x)\}$ is preclosed.

P r o o f. It is shown in Theorem 10 of [12] that a weakly continuous function into a Hausdorff space has a closed graph, hence the result follows from Corollary 1.

C o r o l l a r y 3. Theorem 1.

P r o o f. It is known from [5] that continuity implies weak continuity.

C o r o l l a r y 4. If $f,g: X \longrightarrow Y$ are $\alpha$-continuous and Y is a Hausdorff then the set $\{ x\in X : f(x) = g(x)\}$ is $\alpha$-closed. (Theorem 4.9 [13]).

P r o o f. If f and g are $\alpha$-continuous then by Theorem 3.2 of [13] f and g are semi-continuous and H.a.c. and by

Theorem 1 of [14], f and g are w. c. Then by Corollary 2 of [3] the set $\{ x \in X : f(x) = g(x)\}$ is semi-closed and by the set $\{ x \in X : f(x) = g(x)\}$ is preclosed. By Lemma 3.1 of [13] A is $\alpha$-closed.

The following theorem was proved in [15].

T h e o r e m 3. Let $f_1 : X_1 \longrightarrow Y$ and $f_2 : X_2 \longrightarrow Y$ be two H. a. c. mappings. If Y is a Hausdorff space, then the set $\{(x_1, x_2) : f_1(x_1) = f_2(x_2)\}$ is a preclosed set in the space $X_1 \times X_2$.

The following theorem shows that "$f_1$ is H. a. c." in Theorem 3 can be replaced by "$f_1$ is weakly continuous".

T h e o r e m 4. Let $f_1 : X_1 \longrightarrow Y$ and $f_2 : X_2 \longrightarrow Y$ be two mappings where :

a) Y is a Hausdorff space,

b) $f_1$ is weakly continuous,

c) $f_2$ is H. a. c..

Then the set $\{(x_1, x_2) : f_1(x_1) = f_2(x_2)\}$ is a preclosed set in the space $X_1 \times X_2$.

P r o o f. Let $A = \{ (x_1, x_2) : f_1(x_1) = f_2(x_2) \}$. We shall show that the set $X_1 \times X_2 - A$ is preopen. Let $(x_1, x_2) \notin A$. Then $f_1(x_1) \neq f_2(x_2)$. As Y is the Hausdorff, there exists two open sets $V_i \subset Y$, $i=1,2$ such that $f_i(x_i) \subset V_i$ and $V_1 \cap V_2 = \emptyset$. This implies $\mathrm{Cl}\, V_1 \cap V_2 = \emptyset$. As $f_1$ is w. c. by Theorem 2 of [5], $x \in f_1^{-1}(V_1) \subset \mathrm{Int}\, f_1^{-1}(\mathrm{Cl}\, V_1)$. As $f_2$ is H. a. c., $x \in f_2^{-1}(V_2)$, were by Theorem 1 of [6], $f_2^{-1}(V) \in PO(X_2)$. Let $U = \mathrm{Int}\, f_1^{-1}(\mathrm{Cl}\, V_1) \times f_2^{-1}(V_2)$ be, then by Lemma 2 of [15], $U \in PO(X_1 \times X_2)$. If $(y_1, y_2) \in U$ then $y_1 \in \mathrm{Int}\, f_1^{-1}(\mathrm{Cl}\, V_1)$ and

$y_2 \in f_2^{-1}(V_2)$ so $f_1(y_1) \in \mathrm{Cl}\ V_1$ and $f_2(y_2) \in V_2$ or $\mathrm{Cl}\ V_1 \cap V_2 = \emptyset$. So $(y_1, y_2) \in X_1 \times X_2 - A$. Then $(x_1, x_2) \in U \subset X_1 \times X_2 - A$ and by [6] $X_1 \times X_2 - A$ is a preopen set in $X_1 \times X_2$ and so A is a preclosed set in $X_1 \times X_2$.

C o r o l l a r y 5. If $f : X \longrightarrow Y$ is $\alpha$-continuous and Y is Hausdorff, then the set $\{(x_1, x_2) : f(x_1) = f(x_2)\}$ is $\alpha$-closed in the product space.

P r o o f. By Theorem 3.2 of [13] f is semi-continuous and H. a. c.. By Corollary 1 of [15] H is a preclosed set and by Theorem 4 of [11] A is semi-closed set. By Lemma 3.1 of [13] A is $\alpha$-closed.

D e f i n i t i o n. A space X is said to be pre-$T_2$ if for each pair of distinct points x and y in X there exist disjoint pre-open sets U and V such that $x \in U$ and $y \in V$.

T h e o r e m 5. If X is a topological space and for each pair of different points $x_1$ and $x_2$ from X, there is a H. a. c. function $f: X \longrightarrow Y$ with Y a Hausdorff space so that $f(x_1) \neq f(x_2)$, then X is a pre-$T_2$ space.

P r o o f. Let $x_1 \neq x_2$ be. Then as Y is Hausdorff and $f(x_1) \neq f(x_2)$, there are two open sets $V_1$ and $V_2$ with $f(x_i) \subset V_i$, $i = 1,2$ so that $V_1 \cap V_2 = \emptyset$. As f is H. a. c. by Theorem 1 of [6] there are two preopen sets $U_i$, $i = 1,2$ such that $x_i \in U_i$ and $f(U_i) \subset V_i$, $i = 1,2$. From $V_1 \cap V_2 = \emptyset$ it follows that $f(U_1) \cap f(U_2) = \emptyset$, which implies that $U_1 \cap U_2 = \emptyset$, i.e. X is a pre-$T_2$ space.

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Received July 23, 1987.