DEMONSTRATIO MATHEMATICA
Vol. XXII No 3 1989

Priti Bajpai, R.B. Saxena

# SOME REGULAR AND SINGULAR PAIRS OF BIRKHOFF INTERPOLATION

Let E be the interpolation matrix of order $(n+2)\times(m+1)(n+2)$ with rows $(\underbrace{11\ldots1}_{m}0100\ldots0)$ and let X be the set of knots which are the zeros of $(1-x^2)\,P_n(x)$, where $P_n(x)$ is the $n^{th}$ Legendre'a polynomial. It has been proved that if m is odd, the pair (E,X) is regular if n is even and singular if n is odd. But if m is even, the pair (E,X) is singular regardless of n. This generalizes some of known results and offers a result of negative character in the theory of lacunary interpolation.

Let $E = [e_{i,j}]$ be an interpolation matrix of order $m \times (n+1)$ of 1's and 0's with exactly N+1 1's and let $X = \{x_i\}_{i=1}^{m}$ be the interpolation knots. The pair (E,X) describes the Birkhoff interpolation problem of finding a polynomial p(x) of degree $\le$ N such that

$$(1)\qquad p^{(j)}(x_i) = c_{i,j}\,, \qquad (i,j) \in e$$

for arbitrary given reals $c_{i,j}$ where $e = \left\{(i,j) \mid e_{i,j} = 1\right\}$.

The pair (E,X) is said to be regular if the problem (1) has a unique solution for arbitrary reals $c_{i,j}$ , otherwise it is said to be singular.

If $p^{(j)}(x_i) = 0$, $(i,j) \in e$, then $p(x)$ is said to be annihilated by $(E,X)$. If $E$ is the Turan's matrix $E_{(0,2)}$ of order $n \times 2n$ with rows $(10100\ldots0)$ and $X$ is the set of knots

$$-1 = x_n < x_{n-1} < \ldots < x_2 < x_1 = 1$$

which are zeros of $(1-x^2)\ P'_{n-1}(x)$ ($P_n(x)$ being the $n^{th}$ Legendre polynomial with normalization $P_n(1) = 1$), then the pair $(E,X)$ is regular if $n$ is even and singular if $n$ is odd, see [4].

It was observed by R.B. Saxena in 1964 and recently proved by Prasad [2] that if $E$ is the interpolation matrix $E_{(0,1,3)}$ of $(0,1,3)$ interpolation i.e., $E$ is of order $(n+2)$ $\times$ $3(n+2)$ with rows $(110100\ldots0)$ and $X$ is the set $X_1$ of the zeros of $(1-x^2)P_n(x)$, then the pair $(E,X_1)$ is singular. Since he was interested only in the regular pair, he modified the matrix $E_{(0,1,3)}$ at the extreme knots of second column and considered the matrix

$$E_1 = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & . & . & . \\ 1 & 1 & 0 & 1 & 0 & . & . & . \\ . & . & . & . & . & . & . & . \\ 1 & 1 & 0 & 1 & 0 & . & . & . \\ 1 & 0 & 0 & 1 & 0 & . & . & . \end{bmatrix}$$

of order $(n+2) \times 3(n+1)$ and proved [3] that the pair $(E_1,X_1)$ is regular if $n$ is even and singular if $n$ is odd. This idea was extended by A.K. Varma [5] to the matrix $E_{(0,1,2,4)}$ of $(0,1,2,4)$ interpolation which is of order $(n+2)$ $\times$ $4(n+2)$

with rows (1110100...0). He modified $E_{(0,1,2,4)}$ at the extreme knots of $3^{rd}$ and $5^{th}$ columns and considered the matrix

$$E_2 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & . & . & . \\ 1 & 1 & 1 & 0 & 1 & 0 & . & . & . \\ . & . & . & . & . & . & . & . & . \\ 1 & 1 & 1 & 0 & 1 & 0 & . & . & . \\ 1 & 1 & 0 & 0 & 0 & 0 & . & . & . \end{bmatrix}$$

of order $(n+2) \times 4(n+1)$ and proved that $(E_2, X_1)$ is regular if n is even and singular if n is odd.

It is then natural to ask how the pair $(E, X_1)$ will behave if E is the matrix $E_{(0,1,2,4)}$ modified at the extreme knots of the $5^{th}$ column only or if E is the matrix $E_{(0,1,2,4)}$ itself. It will follow that in both cases the pair $(E, X_1)$ is regular if n is even and singular if n is odd.

Our main aim here is to consider the regularity and singularity of the pair (E,X), where E is a matrix with identical rows and X is the set of zeros of $(1-x^2)\, P_n(x)$. We shall prove the following.

T h e o r e m. If E is the matrix of order $(n+2) \times (n+2)(m+1)$, $m \geq 2$, with rows $(\underbrace{11\ldots1}_{m}0100\ldots0)$ and X is the set of zeros of $(1-x^2)\, P_n(x)$, then

(i) if m is even, the pair (E,X) is singular, and

(ii) if m is odd, the pair (E,X) is regular, if n is even and singular if n is odd.

The theorem says that the problem of $(0,1,2,\dots,2r-1, 2r+1)$, $r \ge 1$, on the zero's of $(1-x^2)\, P_n(x)$ is not uniquely solvable whereas the problem of $(0,1,2,\dots,2r,2r+2)$, $r \ge 1$, is uniquely solvable only when n is even.

In the proof of our theorem the following lemmas will play an essential role.

L e m m a 1. If

$$K_r(x) = [\ (1-x^2)\ P_n^2(x)\ ]^r\ , \quad r = 1,2,\dots,$$

then

(i) $$K_r^{i}\ (x_\nu) = 0, \qquad i = 0,1,\dots,2r-1\ ,$$

(ii) $$K_r^{2r+1}\ (x_\nu) = 0\ , \quad \nu = 1,2,\dots,n\ .$$

P r o o f. (i) is obvious. We prove (ii) using induction on r. To start the induction we first see that

(2) $$K_1'''\ (x_\nu) = 0\ , \quad \nu = 1,\dots,n.$$

This follows from the equation

$$\left[(1-x^2)\ P_n(x)^2\right]_{x=x_\nu}''' =$$

$$= 6\ P_n'(x_\nu)\ \left[(1-x_\nu^2)P_n(x_\nu)\ -\ 2x_\nu P_n'(x_\nu)\right]$$

whose right hand side vanishes owing to the differential equation

(3) $$(1-x^2)P_n''(x)\ -\ 2xP_n'(x)\ +\ n(n+1)P_n(x)\ =\ 0.$$

Now we assume that

(4) $$K_{r-1}^{(2r-1)}(x_\nu)\ =\ 0$$

and prove that

(5) $$K_r^{(2r+1)}(x_\nu) = 0.$$

By Leibnitz formula we have

$$K_r^{(2r+1)}(x) = [K_{r-1}(x)\ K_1(x)]^{(2r+1)} =$$

$$= K_{r-1}^{(2r+1)}(x)\ K_1(x) + \binom{2r+1}{1} K_{r-1}^{(2r)}(x)\ K_1'(x) +$$

$$+ \binom{2r+1}{2} K_{r-1}^{(2r-1)}(x)\ K_1''(x) + \binom{2r+1}{3} K_{r-1}^{(2r-2)}\ K_1'''(x) +$$

$$+ \sum_{i=4}^{2r+1} \binom{2r+1}{i} K_{r-1}^{(2r-i+1)}(x)\ K_1^{(i)}(x).$$

Now using (i) and the assumption (4) we immediately obtain (5).

L e m m a 2. Let $Q(x) = [(1-x^2)P_n(x)]^m\, q(x)$ be a polynomial of degree $\le (n+2)(m+1)-1$, where $q(x)$ is a polynomial of degree $\le n+1$ and let

$$Q^{m+1}(x_\nu) = 0\ , \qquad \nu = 0,1,\dots,n+1.$$

Then $q(x)$ satisfies the conditions

(i) $(1 - x_\nu{}^2)\ q'(x_\nu) - m\ x_\nu\ q(x_\nu) = 0, \quad \nu = 1,\dots,n$ ,

(ii) $2q'(1) + m(n^2+n+1)q(1) = 0,$

(iii) $2q'(-1) - m(n^2+n+1)q(-1) = 0.$

P r o o f. First we assume that $m = 2r$. Then

$$Q(x) = [(1-x^2)P_n(x)^2]^r\ [(1-x^2)^r q(x)] =$$

$$= K_r(x)\ [(1-x^2)^r q(x)].$$

By Leibnitz formula and Lemma 1, we have for $\nu = 1,\dots,n$,

$$Q^{2r+1}(x_\nu) = (2r+1)\ (1-x_\nu^2)^{r-1}\ K_r^{(2r)}(x_\nu)\ \cdot \tag{6}$$

$$\cdot\ [(1-x_\nu^{\ 2})q'(x_\nu) - 2r\ x_\nu\ q(x_\nu)].$$

To compute $Q^{2r+1}(\pm 1)$, we write

$$Q(x) = \varphi_{2r}(x)[P_n(x)^{2r}\ q(x)],$$

where

$$\varphi_{2r}(x) = (x^2-1)^{2r}.$$

Then

$$Q^{2r+1}(1) = \varphi_{2r}^{(2r+1)}(1)\ P_n(1)^{2r} q(1) + (2r+1)\varphi_{2r}^{(2r)}(1)\ \cdot$$

$$\cdot\ [P_n(1)^{2r} q'(1) + 2r\ P_n(1)^{2r-1} P_n'(1) q(1)]$$

and

$$Q^{2r+1}(-1) = \varphi_{2r}^{(2r+1)}(-1)\ P_n(-1)^{2r} q(-1) + (2r+1)\varphi_{2r}^{(2r)}(-1)\ \cdot$$

$$\cdot\ [P_n(-1)^{2r} q'(-1) + 2r\ P_n^{2r-1}(-1)\cdot P_n'(-1) q(-1)].$$

Now we using the facts that

(7) $$\begin{cases} P_n(1) = 1 = (-1)^n \, P_n(-1) \\ P'_n(1) = \dfrac{n(n+1)}{2} = (-1)^{n-1} \, P'_n(-1) \end{cases}$$

and

(8) $$\begin{cases} \varphi_{2r}^{(2r)}(1) = 2^{2r}(2r)! = \varphi_{2r}^{(2r)}(-1) \\ \varphi_{2r}^{(2r+1)}(1) = 2^{2r}(2r+1)! \, r = -\varphi_{2r}^{2r+1}(-1) \ , \end{cases}$$

we obtain

(9) $$Q^{(2r+1)}(1) = 2^{2r}(2r+1)! \ [q'(1)+r(n^2+n+1)q(1)]$$

and

(10) $$Q^{(2r+1)}(-1) = 2^{2r}(2r+1)! \ [q'(-1)-r(n^2+n+1)q(-1)].$$

Thus the conditions

$$Q^{2r+1}(x_\nu) = 0 \ , \qquad \nu = 0,1,\dots,n+1$$

imply (i), (ii) and (iii) of Lemma 2 for $m = 2r$.

Now let $m = 2r+1$. Then

$$Q(x) = [(1-x^2) \ P \ (x)]^{2r+1} \ q(x) =$$

$$= K_r(x) \ [(1-x^2)^{r+1} P_n(x) \ q(x)].$$

By Leibnitz formula and Lemma 1 we have for $\nu = 1,2,\dots,n$,

$$(11)\quad Q^{(2r+2)}(x_\nu) = (2r+2)(2r+1)(1-x_\nu^2)^r\, K_r^{(2r)}(x_\nu)P_n'(x_\nu)\cdot$$
$$\cdot\,[(1-x_\nu^2)q'(x_\nu)-(2r+1)x_\nu q(x_\nu)].$$

To compute $Q^{(2r+2)}(\pm 1)$, we write

$$Q(x) = \varphi_{2r}(x)\ [(1-x^2)P_n(x)^{2r+1}q(x)],$$

and use Leibnitz formula

$$Q^{(2r+2)}(1) = -2(2r+2)\ \varphi_{2r}^{(2r+1)}(1)\ P_n(1)^{2r+1}q(1) +$$
$$+ (2r+2)(2r+1)\ \varphi_{2r}^{(2r)}(1)\Big[P_n^{2r+1}(1)\ q(1) -$$
$$- 2\ \Big\{P_n^{2r+1}(1)q'(1)+(2r+1)P_n(1)^{2r}P_n'(1)q(1)\Big\}\Big],$$

$$Q^{(2r+2)}(-1) = -2(2r+2)\ \varphi_{2r}^{(2r+1)}(-1)\ P_n(-1)^{2r+1}q(-1) +$$
$$+ (2r+2)(2r+1)\ \varphi_{2r}^{(2r)}(-1)\Big[-P_n^{2r+1}(-1)\ q(-1) +$$
$$+\ 2\ \Big\{P_n^{2r+1}(-1)q'(-1)+(2r+1)P_n(-1)^{2r}P_n'(-1)q(-1)\Big\}\Big].$$

Now using (7) and (8) we get

$$(12)\quad Q^{(2r+2)}(1) = -2^{2r}(2r+2)!\,[2q'(1)+(2r+1)(n^2+n+1)q(1)],$$

$$(13)\quad Q^{(2r+2)}(-1)=(-1)^n\, 2^{2r}(2r+2)!\,[2q'(-1)-(2r+1)(n^2+n+1)q(-1)].$$

The conditions $Q^{(2r+2)}(x_\nu) = 0$, $\nu = 0,1,\ldots,n+1$, owing to (11), (12), (13), prove the lemma for odd m.

L e m m a 3. Let q(x) be a polynomial of degree $\le n+1$ which satisfies the following n+2 conditions :

(i) $(1-x_\nu^2)q'(x_\nu) - mx_\nu q(x_\nu) = 0, \quad \nu = 1,2,\ldots,n,$

(ii) $2q'(1) + m(n^2 + n + 1)\, q(1) = 0,$

(iii) $2q'(-1) - m(n^2 + n + 1)\, q(-1) = 0.$

Then q(x) satisfies the relation

$$(1-x^2)q'(x) - m\,x\,q(x) = a\left[x^2 - \frac{(m+3)(n^2+n+1)+1}{(m+3)(n^2+n+1)-3}\right]P_n(x)$$

with arbitrary constant a.

P r o o f. Owing to (i), we can write

(14) $(1-x^2)\,q'(x) - m\,x\,q(x) = (ax^2 + bx + c)P_n(x),$

where a, b, c are constants. From (14), we have

$$-\,m\,q(1) = (a + b + c)P_n(1)$$

and

$$m\,q(-1) = (a - b + c)P_n(-1).$$

Differentiating (14), we get

(15) $(1-x^2)\,q''(x) - (m+2)\,x\,q'(x) - m\,q(x) =$

$$= (ax^2 + bx + c)P_n'(x) + (2ax+b)P_n(x).$$

From (15), we have

$$-(m+2)\ q'(1) - m\ q(1) = (a+b+c)P'_n(1) + (2a+b)P_n(1)$$

and

$$(m+2)\ q'(-1) - m\ q(-1) = (a-b+c)P'_n(-1) + (-2a+b)P_n(-1).$$

Now using the fact that

$$(16)\qquad \begin{cases} P_n(1) = 1 = (-1)^n\ P_n(-1)\ , \\ P'_n(1) = \dfrac{n(n+1)}{2} = (-1)^{n-1}\ P'_n(-1)\ , \end{cases}$$

we obtain

$$-mq(1) = (a + b + c)\ ,$$

$$mq(-1) = (a - b + c)\ (-1)^{n-1}\ \frac{n(n+1)}{2}\ ,$$

$$-(m+2)\ q'(1) = (a - c) + (a + b + c)\ \frac{n(n+1)}{2}\ ,$$

$$(m+2)\ q'(-1) = (-1)^{n-1}\ [(a-c) + (a-b+c)\ \frac{n(n+1)}{2}\ ].$$

Substituting $q(1)$, $q'(1)$, $q(-1)$, $q'(-1)$ in (ii) and (iii), we get

$$(a-b-3c) + (a+b+c)(n^2+n+1)(m+3) = 0$$

and

$$(a+b-3c) + (a-b+c)(n^2+n+1)(m+3) = 0.$$

From these equations we have $b = 0$ and

$$(a-3c) + (a+c)(n^2+n+1)(m+3) = 0$$

i.e.,

$$C = \frac{(m+3)(n^2+n+1)+1}{(m+3)(n^2+n+1)-3} a .$$

This proves the lemma.

P r o o f of the theorem. Let E be the interpolation matrix of order $(n+2) \times (n+2)(m+1)$, $m > 2$ with rows $(\underline{11....1}10100...0)$ and $X_1$ be the set of zeros of $(1-x^2)P_n^m(x)$. Let $Q(x)$ be a polynomial of degree $\le$ $(n+2)(m+1)-1$ annihilated by $(E,X)$. We have to verify if $Q(x)$ is identically zero.

Since $Q(x_\nu) = Q'(x_\nu) =...= Q^{m-1}(x_\nu) = 0$, $\nu = 0,1,...,n+1$, we can write

$$Q(x) = [(1-x^2)P_n(x)]^m q(x),$$

where $q(x)$ is a polynomial of degree $\le n+1$. We apply the conditions

$$Q^{m+1}(x_\nu) = 0, \quad \nu = 0,1,...,n+1.$$

According to Lemma 3, the polynomial q satisfies the relation

$$(17) \qquad (1-x^2)q'(x)-m\, x\, q(x) = a\left[x^2\ \frac{(m+3)(n^2+n+1)+1}{(m+3)(n^2+n+1)-3}\right] P_n(x)$$

with constant a. The question is whether $q(x) \equiv 0$.

Since $P_k$ are linearly independent, for some constants $a_k$ we have

$$(18) \qquad q(x) = \sum_{k=0}^{n+1} a_k P_k(x).$$

Thus the left hand side of (17) takes the form

$$(19)\qquad \sum_{k=1}^{n+1} [(1-x^2)P'_k(x) - m\, x\, P_k(x)]a_k - m\, x\, P_0(x)\, a_0.$$

Using the recurrence relations

$$(20)\qquad \begin{cases} (n+1)\, P_{n+1} = (n+1)\, x\, P_n - n\, P_{n-1}\,, \\ (1-x^2)\, P_n = n\, P_{n-1} - n\, x\, P_n\,, \end{cases}$$

we get

$$(1-x^2)\, P'_k(x) - m\, x\, P_k(x) =$$

$$= \frac{1}{2k+1}\left[k(k+1-m)\, P_{k-1}(x) - (k+1)(k+m)\, P_k(x)\right]$$

and

$$x^2\, P_n(x) = \frac{n(n-1)}{4n^2-1}\, P_{n-2}(x) + \left[\frac{(n+1)^2}{(2n+1)(2n+3)} + \frac{n^2}{4n^2-1}\right] P_n(x) +$$

$$+ \frac{(n+1)(n+2)}{(2n+1)(2n+3)}\, P_{n+2}(x).$$

Since

$$P_0(x) = 1 \quad \text{and} \quad P_1(x) = x\,,$$

the equation (17) takes the form

$$\sum_{k=0}^{n} \frac{(k+1)(k+2-m)}{2k+3}\, a_{k+1}\, P_k(x) - \sum_{k=1}^{n+2} \frac{k(k-1+m)}{2k-1}\, a_{k-1}\, P_k(x) =$$

$$= A\, P_{n-2}(x) + B\, P_n(x) + C\, P_{n+2}(x)\,,$$

where

$$A = \frac{n(n-1)}{4n^2-1}\, a \ ,$$

$$B = \left[\frac{(n+1)^2}{(2n+1)(2n+3)} + \frac{n^2}{4n^2-1} - \frac{(m+3)(n^2+n+1)+1}{(m+3)(n^2+n+1)-3}\right] a \ ,$$

$$C = \frac{(n+1)(n+2)}{(2n+1)(2n+3)}\, a \ .$$

Consequently

$$\frac{2-m}{3}\, a_1 = 0 \ ,$$

$$\frac{(k+1)(k+2-m)}{2k+3}\, a_{k+1} - \frac{k(k-1+n)}{2k-1}\, a_{k-1} = 0 \ ,$$

$$k = 1,\dots,n-3,$$

$$\frac{(n-1)(n-m)}{2n-1}\, a_{n-1} - \frac{(n-2)(n-3+m)}{2n-5}\, a_{n-3} = A,$$

$$\frac{n(n+1-m)}{2n+1}\, a_m - \frac{(n-1)(n-2+m)}{2n-3}\, a_{n-2} = 0,$$

$$\frac{(n+1)(n+2-m)}{2n+3}\, a_{n+1} - \frac{n(n-1+m)}{2n-1}\, a_{n-1} = B,$$

$$-\frac{(n+1)(n+m)}{2n+1}\, a_n = 0,$$

$$-\frac{(n+2)(n+1+m)}{2n+3}\, a_{n-1} = C.$$

Let m be an odd number, then we have

(i) if n is odd ,

$$a_1 = a_3 = \dots = a_{n-2} = a_n = 0$$

and

$$a_{m-3} = a_{m-5} = \dots = a_2 = a_0 = 0$$

but $a_{m-1}$, $a_{m+1}, \dots, a_{n+1}$ are not necessarilly zero. Hence q(x) is not identically zero and in this case (E,X) is singular,

(ii) If n is even,

$$a_n = a_{n-2} = \dots = a_2 = a_0 = 0$$

and

$$a_1 = a_3 = \dots = a_{n-3} = 0.$$

For the coefficients $a_{n-1}$ and $a_{n+1}$, the equations

$$\frac{(n-1)(n-m)}{2n-1} a_{n-1} = A,$$

$$\frac{(n+1)(n+2-m)}{2n+3} a_{n+1} - \frac{n(n-1+m)}{2n-1} a_{n-1} = B,$$

$$-\frac{(n+2)(n+1+m)}{2n+3} a_{n+1} = C.$$

must be satisfied. Putting the values of A, B and C, we see that a must be zero i.e. $a_{n-1}$ and $a_{n+1}$ are also zero. Thus q(x) is identically zero and therefore, in this case (E,X) is regular.

Now let, m be even. We see that :

If n is even, then

$$a_n = a_{n-2} = \dots = a_2 = a_0 = 0$$

and

$$a_{m-3} = a_{m-5} = \dots = a_3 = a_1 = 0$$

but $a_{m-1}$, $a_{m+1}, \dots, a_{n+1}$ are not zero.

If n is odd, then

$$a_n = a_{n-2} = \dots = a_3 = a_1 = 0,$$

but not all $a_0$, $a_2, \dots, a_{n+1}$ are not zero. Thus, in both cases (n even or n odd) $q(x)$ is not identically zero and so $(E,X)$ is singular.

## REFERENCES


[1] G. G. Lorentz, K. Jetter, S. D. Riemenschneider: Birkhoff Interpolation, Addison - Wesley Publishing Company (1983)

[2] J. Prasad, C. H. Anderson: Some remarks on non-existance of interpolation polynomials, Math. Not. 29 (1981/82), 67-71.

[3] R. B. Saxena: Some interpolatory properties of Legendre polynomials III, Bulgarian Akad. Nauk. Otd. Mat. Fiz. Nauk. Izv. Math. Inst. 8 (1964), 63-94, MR 30 # 4005.

[4] J. Surányi, P. Turán: Notes of interpolation I, Acta Math. Acad. Sci. Hungar. 6(1955), 56-79.

[5] A. K. Varma: An analogue of a problem of J. Balàzs and P. Turàn III Trans. Amer. Math. Soc. 146(1969), 107-120.


DEPARTAMENT OF MATHEMATICS AND ASTRONOMY LUCKNOW UNIVERSITY
LUCKNOW 226007, INDIA

Received October 16, 1986.