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ON VARIETIES OF ALGEBRAS DEFINED BY FIRST REGULAR IDENTITIES

0. Let K be a variety of algebras of type τ, $\tau : T \longrightarrow N \cup \{0\}$. We denote by E(K) the set of all identities of type τ satisfied in K.
In the sét E(K) one can distinguish subsets of identities of some special form e.g. regular identities (see [5]), non-trivializing identities or defined below first regular identities.

All these sets of identities are closed under consequences and it is interesting to study the equational classes of algebras defined by these sets.

In this paper we show that the equational class K_F defined by all first regular identities satisfied in K is the smallest class containing K and the class F defined by all first regular identities of type τ.

We also show here how to construct an equational base of K when we are given the equational base of K_F and we construct the equational base of K_F when K is the variety of groups, distributive lattices or Boolean algebras.

1. If ϕ is a term of type τ, then variables and nullary fundamental polynomial symbols occuring in ϕ will be called the arguments of ϕ.

Following J.Płonka an identity $\phi = \psi$ will be called first regular iff the first argument in ϕ and ψ is the same variable or the first argument in ϕ and ψ is a nullary fundamental polynomial symbol (these symbols need not be the same).

For example identity $x + x \circ y = x$ is first regular whereas identities $x \circ y + y = y$ and $x \circ y = y \circ x$ are not first regular.

An identity which is not first regular will be called first nonregular.

(i) An identity $\phi = \psi$ is first nonregular if one of the two cases holds :

(1) the first argument of ϕ is a variable x_i, and the first argument of ψ is another variable x_j ,

(2) the first argument of ϕ is a variable and the first argument of ψ is a nullary fundamental polynomial symbol, or conversely.

Let K be an arbitrary variety of type τ. We denote by F(K) the set of all first regular identities in K. Obviously the set F(K) is closed under consequences. So K_F is the variety of algebras of type τ defined by F(K).

Let F be the variety of type τ defined by all identities of the form,

(a) $f_t\,(x_1,\ldots,x_{\tau(t)}) = x_1$ if $\tau(t) \neq 0$,

(b) $f_{t_1} = f_{t_2}$ if $\tau(t_1) = \tau(t_2) = 0.$

Note that the class F is the left zero-band when considering it as a class of type <2> with

$$f_t(x_1,\dots,x_{\tau(t)}) = x_1 \cdot \ \dots \ \cdot x_{\tau(t)}$$

for each $t \in T$.

(ii) The class F is not trivial.

In fact for any $n > 1$ there exists an algebra A in F, such that $|A| = n$.

Indeed, let $A = (a_1,\dots,a_n)$,

$$f_t = a_n \quad \text{for} \quad \tau(t) = 0$$

and

$$f_t(a_{i_1},\dots,a_{i_k}) = a_{i_1} \qquad \text{for} \quad \tau(t) \neq 0.$$

Then A belongs to F and $|A| = n$.

L e m m a 1. Every identity of the form $\phi = q$, where ϕ is a term of type τ and q is the first argument of ϕ is satisfied in F.

The proof is by standard induction of the complexity of ϕ.

L e m m a 2. If an algebra A from F satisfies a first nonregular identity $\phi = \psi$ then $|A| = 1$.

P r o o f. At first consider the case (1) from (i). By Lemma 1 it follows that the identities $\phi = x_i$ and $\psi = x_j$, are satisfied in A. Thus $x_i = x_j$ holds in A, for some $i \neq j$.

If the case (2) from (i) holds then by Lemma 1 we have

$$\phi = x_i \qquad \text{and} \qquad \psi = f_t \ .$$

Hence $x_i = f_t$ and $x_i = x_j$ holds in A.

L e m m a 3. The set E(F) consists exactly of all first regular identities of type τ.

P r o o f. If $\phi = \psi$ is first regular and x_1 is the first argument in ϕ and ψ, then by Lemma 1 the identities $\phi = x_1$ and $\psi = x_1$ are satisfied in F, whence $\phi = \psi$ holds in F as well.

If f_{t_1} is the first argument in ϕ and f_{t_2} is the first argument in ψ where $\tau(t_1) = \tau(t_2) = 0$, then by (b) and Lemma 1

$$\phi = f_{t_1} = f_{t_2} = \psi .$$

If $\phi = \psi$ is first nonregular, then by Lemma 2 and (ii) it does not belong to E(F).

C o r o l l a r y. The variety F is equationally complete. The proof follows from (ii), Lemma 3 and Lemma 2.

T h e o r e m 1. For an arbitrary variety K of type τ we have

$$K_F = K \cup F.$$

P r o o f. We have

$$e \in E(K \cup F) \Leftrightarrow e \in E(K) \cap E(F) \Leftrightarrow$$

$$\Leftrightarrow e \in E(K) \wedge e \in E(F) \Leftrightarrow e \in E(K_F) .$$

Thus $K \cup F = K_F$.

For two varieties K_1, K_2 of type τ we denote by $K_1 \times K_2$, the class of all products $A_1 \times A_2$, where

$$A_1 \in K_1 , \qquad A_2 \in K_2 .$$

T h e o r e m 2. If K is a variety of type τ, $\phi(x,y)$ is a term of type τ of two variables such that x is the first argument in ϕ and the identity

$$\phi(x,y) = y \in E(K)$$

then

$$K_F = F \times K.$$

P r o o f. It was proved in [2] that :

If K_1 and K_2 are two varieties of type τ, there exists a term $\phi(x,y)$ of two variables such that the identity

$$\phi(x,y) = x$$

belongs to $E(K_1)$ and the identity

$$\phi(x,y) = y$$

belongs to $E(K_2)$, then

$$K_1 \cup K_2 = K_1 \times K_2.$$

By Lemma 1 the identity $\phi(x,y) = x$ belongs to $E(F)$.

By Theorem 1 we have

$$K_F = F \cup K = F \times K .$$

T h e o r e m 3. If K is a variety of type τ, $\phi(x,y)$ is a term of type τ of two variables such that x is the first argument in ϕ and the identity $\phi(x,y) = x \in E(K)$, then for any $e \in E(K)\setminus F(K)$ the set $F(K) \cup \{e\}$ is an equational base of K.

P r o o f. Let K^* be the variety of type τ defined by $F(K) \cup \{e\}$. We shall that $K^* = K$. Obviously $F(K) \cup \{e\} \subseteq E(K)$ whence $K \subseteq K^*$. Further $F(K) \subset F(K) \cup \{e\}$ whence $K^* \subseteq K_F$. Let $A^* \in K^*$. Then $A \in K_F$. By Theorem 2, $A = A_1 \times A_2$ where $A_1 \in F$ and $A_2 \in K$. It is enough to show that $|A_1| = 1$. Since $e \in E(K^*)$ it follows that e is satisfied in A and A_1. Now by Lemma 2 we get that $|A_1| = 1$. It follows that A is isomorphic to A_2, whence $A \in K$.

Under assumptions of Theorem 3 we get

C o r o l l a r y 2. The variety K_F covers K.

T h e o r e m 4. If K is the variety of groups considered as algebras with fundamental operations $\circ$ and $^{-1}$, then the variety K_F is defined by the following identities

(1) $(x \circ y) \circ z = x \circ (y \circ z)$,

(2) $x \circ y \circ y^{-1} = x$,

(3) $(x^{-1})^{-1} = x$,

(4) $x \circ (y \circ z)^{-1} = x \circ (z^{-1} \circ y^{-1})$,

(5) $(x \circ x^{-1} \circ y)^{-1} = x^{-1} \circ x \circ y^{-1}$,

(6) $x^{-1} \circ x \circ x = x$.

P r o o f. Let K^* be a variety defined by (1)-(6). We shall show that $K^* = K_F$. Obviously $K_F \subseteq K^*$ since (1)-(6) are first regular identities of groups.

Let $A = (A, \circ, ^{-1}) \in K^*$. We show that A is isomorphic to $A_1 \times A_2$ where $A_1 \in F$ and $A_2 \in K$ what by Theorem 2 means that $A \in K_F$.

In the set A we define two relations R_1 and R_2 by putting

$$\forall\ (a,b \in A)\quad [a\ R_1\ b \Leftrightarrow a \circ a^{-1} \circ b = b\ ]$$

$$\forall\ (a,b \in A)\quad [a\ R_2\ b \Leftrightarrow a \circ a^{-1} \circ b = a\ ] .$$

Using (1)-(6) and Theorem 3 p.120 in [1] one proves that each of R_i, $i = 1,2$ is congruence and

$$A_1 = A/R_1 \in F , \qquad A_2 = A/R_2 \in K .$$

T h e o r e m 5. If K is the variety of distributive lattices, then the variety K_F is defined by the following identities :

(1) $x+x = x$,

(2) $x \circ x = x$,

(3) $x+x \circ y = x \circ y+x = x$,

(4) $x \circ (x+y) = x$,

(5) $x+(y+z) = (x+y)+z$,

(6) $x \circ (y \circ z) = (x \circ y) \circ z$,

(7) $(x+y) \circ z = x \circ z+y \circ z$,

(8) $x \circ (y+z) = x \circ y+x \circ z$,

(9) $x+y+z = x+z+y$,

(10) $x \circ y \circ z = x \circ z \circ y$,

(11) $x+y \circ z+z = x+z$,

(12) $x \circ y \circ z+z = x \circ z+z$,

(13) $x \circ y+x \circ z = x \circ z+x \circ y$,

(14) $x \circ y+y \circ x = x \circ y$.

P r o o f. Let K^* be a variety defined by (1)-(14). We shall show that $K^* = K_F$. Obviously $K_F \subseteq K^*$ since (1)-(14) are first regular identities of distributive lattices.

Let $A = (A, \circ, +) \in K^*$. We show that A is isomorphic to $A_1 \times A_2$, where $A_1 \in F$ and $A_2 \in K$ what by Theorem 2 means that $A \in K_F$.

In the set A we define two relations R_1 and R_2 by putting :

$$\forall\ (a,b \in A)\quad [a\ R_1\ b \Longleftrightarrow a \circ b+b = b\ ],$$

$$\forall\ (a,b \in A)\quad [a\ R_2\ b \Longleftrightarrow a \circ b+b = a\ ]\ .$$

Using (1)-(14) one proves that each R_i, $i=1,2$ is a congruence and

$$A_1 = A/R_1 \in F\ , \qquad A_2 = A/R_2 \in K\ .$$

Note that, if K is the variety of distributive lattices, then all algebras in K_F are idempotent distributive semirings. Such semirings were studied e.g. in [3] and [4]. Another equational base of K_F and representation of K_F by subdirect product were considered in [4].

T h e o r e m 6. If B is the variety of Boolean algebras, then the variety B_F is defined by the following identities :

(1) - (14) and

(15) $x+x'+1 = x+x'$,

(16) $x\circ 1 = x$,

(17) $x+0 = 0$,

(18) $x'\circ x+x = x$,

(19) $(x\circ x')+0 = x\circ x'$,

(20) $1\circ 0 = 0$.

The proof is anologous to the proof of Theorem 5. The relations R and R_1 and R_2 are defined in the same way.

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