DEMONSTRATIO MATHEMATICA
Vol. XXII No 2 1989

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VIBRATION CONTROL OF THE SYSTEM OF STRINGS CONNECTED IN A NODE

Formulation of the problem

In this work we shall consider the problem of optimal control for a system of N strings connected in a node. These strings, lying at rest in a plane, have the same lengths S and the same masses per unit length ρ, and also the tension T_0 is the same for each of them.

We introduce the following assumptions:

1) the node is massless;

2) both the points of the strings at the node can move only in the direction perpendicular to the plane in which whole the system lays in its equilibrium state (the node moves in this direction without any drag);

3) the node does not transmit any forces acting in the plane of strings.

For simplifying calculations we take $\sqrt{T_0/\rho} = 1$. For each string we introduce its axis Ox directed along the string at rest, to the node where x = S. Let $U^i(x,t)$ denotes the displacement of the point x from its equilibrium state position at the time t for i-th string. The oscillations of each string are described by the wave equation:

$$(1) \quad U^i_{tt}(x,t) = U^i_{xx}(x,t) \quad \text{for} \quad x \in (0,S),\ t>0,\ i = 1,2,\dots,N$$

with the initial conditions

(2) $$U^i(x,0) = f^i(x), \quad U^i_t(x,0) = F^i(x) \text{ for } x \in \langle 0,S \rangle \qquad i = 1,2,\dots,N$$

and the boundary conditions

(3) $$U^i(0,t) = u^i(t) \quad \text{for} \quad t \geqslant 0, \quad i = 1,2,\dots,N.$$

The geometry of node gives the conditions of the equality of displacements in the node

(4) $$U^1(S,t) = U^2(S,t) = \dots = U^N(S,t) \quad \text{for} \quad t \geqslant 0.$$

Moreover from the conditions of the equilibrium of vertical forces in the node, treated as a point, we obtain the following condition

(5) $$\sum_{i=1}^{N} U^i_x(S,t) = 0.$$

The problem of looking for the solutions of the equations (1) under conditions (2)-(5) is described in the paper [5]. The problem of the optimal control may be formulated as follows: one should find the time interval T and control functions $u^i(t) \in L_2[0,T]$ with the norms limited by a given number $l > 0$ i.e.

$$\|u^i(t)\|_{L_2[0,T]} = \left\{ \int_0^T |u(t)|^2 dt \right\}^{1/2} \leqslant l$$

which will set a rest during the time interval T the vibrations of the system (1) caused by nonzero initial conditions (2) i.e. the equilibrium conditions will be satisfied at a time T

(6) $$U^i(x,T) = 0, \quad \frac{\partial U^i}{\partial t}(x,T) = 0 \quad \text{for} \quad x \in \langle 0,S \rangle.$$

We shall assume that $f^i(x) \in C^2(0,S)$ and they have their third derivatives piecewise continuous, and moreover $f^i(0) = \frac{\partial^2 f^i}{\partial x^2}(0) = 0$, $f^i(S) = 0$, functions $F^i(x) \in C^1(0,S)$ and they have their second derivatives piecewise continuous, and also $F^i(0) = 0 = F^i(S)$.

We shall mean the solutions $U^i(x,t)$ of the system (1) under conditions (2)-(5) in the generalized sense, as a limit for $n \to \infty$ of the solutions $U^i_n(x,t)$ of the system (1)-(5) corresponding to the control functions to the control functions $u^i_n(t) \in C^2[0,T]$ satisfying the conditions $u^i_n(0) = \frac{\partial u^i_n}{\partial t}(0) = 0$, where $\{\dot{u}^i_n(t)\}$ converges on $[0,T]$ to the functions $u^i(t)$ in the norm $L_2[0,T]$.

If for any functions $f^i(x)$, $F^i(x)$ satisfying the above assumptions the control functions $u^i(t)$ exist, we shall say that the system is controllable.

The problem of control for a single string was investigated by A.G. Butkowski [1] by applying the control functions to the both ends of the string, or controlling the one end only, the same time the second one beeing built-in. For both the above cases A.G. Butkowski gives an effective method of the calculation of the control functions. S. Rolewicz in his papers [2] and [3] investigated the system of the strings connected in the form of "net", with the control functions applied in its nodes.

Solution of the problem

Passing on to the solution of the problem formulated in this paper, we shall assume that the system contains three strings i.e. $N = 3$.

Let us investigate first the following case: we assume that the vibrations of such a system are caused by non-zero initial conditions for the first string. The question arises if such a system can be set at rest by single control function acting at a point $x = 0$ of the first string (Fig.1) i.e. does such a control function $u(t) = U^1(0,t)$ exist, for which the condition (6) is satisfied.

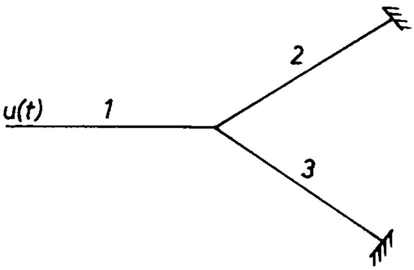

Fig. 1

In this case the initial conditions (2) have the form

$$(2') \quad \begin{matrix} U^1(x,0) = f(x) & U^2(x,0) = U^3(x,0) = 0 \\ U^1_t(x,0) = F(x) & U^2_t(x,0) = U^3_t(x,0) = 0 \end{matrix} \quad \text{for } x \in \langle 0,S \rangle$$

and the boundary conditions (3) become the conditions

$$(3') \quad U^1(0,t) = u(t) \quad U^2(0,t) = U^3(0,t) = 0 \quad \text{for} \quad t \geqslant 0.$$

We find the solutions of the equations (1) under conditions (2′), (3′), (4), (5) by the method given in the work [5].

They are of the form

$$U^1(x,t) = \frac{1}{3} Q(x,t) + \frac{2}{3} z(x,t),$$

$$U^2(x,t) = U^3(x,t) = \frac{1}{3} Q(x,t) - \frac{1}{3} z(x,t),$$

where $Q(x,t)$ is the solution of the equation

$$Q_{tt}(x,t) = Q_{xx}(x,t) \quad \text{for} \quad x \quad (0,S), \quad t > 0$$

under conditions

$$Q(x,0) = f(x) \quad Q(0,t) = u(t)$$
$$Q_t(x,0) = F(x) \quad Q_x(S,t) = 0 \quad \text{for} \quad x \in \langle 0,S \rangle, \quad t \geqslant 0$$

and z(x,t) is the solution of the equation

$$z_{tt}(x,t) = z_{xx}(x,t) \quad \text{for} \quad x \in (0,S), \quad t > 0$$

under conditions

$$z(x,0) = f(x) \qquad z(0,t) = u(t)$$

$$z_t(x,0) = F(x) \qquad z(S,t) = 0 \quad \text{for} \quad x \in \langle 0,S \rangle, \quad t \geq 0.$$

Using the method of separation of the variables to the above problems we obtain the solutions in the form

$$(7) \qquad Q(x,t) = \sum_{n=1}^{\infty} \left[C_n \cos \frac{(2n-1)\pi}{2S} t + \frac{2S}{(2n-1)\pi} D_n \sin \frac{(2n-1)\pi}{2S} t \right] \sin \frac{(2n-1)\pi}{2S} x + \sum_{n=1}^{\infty} \frac{2}{S} \sin \frac{(2n-1)\pi}{2S} x \int_0^t u(\tau) \sin \frac{(2n-1)\pi}{2S} (t-\tau) d\tau$$

where

$$C_n = \frac{2}{S} \int_0^S f(\xi) \sin \frac{(2n-1)\pi}{2S} \xi \, d\xi ,$$

$$D_n = \frac{2}{S} \int_0^S F(\xi) \sin \frac{(2n-1)\pi}{2S} \xi \, d\xi$$

and

$$(7') \quad z(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos \frac{n\pi}{S} t + \frac{S}{\pi n} B_n \sin \frac{n\pi}{S} t \right] \sin \frac{n\pi}{S} x + \sum_{n=1}^{\infty} \frac{2}{S} \sin \frac{n\pi}{S} x \int_0^t u(\tau) \sin \frac{n\pi}{S} (t-\tau) d\tau ,$$

where

$$A_n = \frac{2}{S}\int_0^S f(\xi) \sin \frac{n\pi}{S} \xi \, d\xi ,$$

$$B_n = \frac{2}{S}\int_0^S F(\xi) \sin \frac{n\pi}{S} \xi \, d\xi .$$

For simplicity of writing we denote $\frac{(2n - 1)\pi}{2S} = \alpha_n$, $\frac{n\pi}{S} = \beta_n$. Let us come back to our main problem, i.e. finding such a control function $u(t)$, which satisfied the equalities (6) at the time T. From the condition $U^1(x,T) = 0$ we obtain

$$Q(x,T) = z(x,T) = 0 \quad \text{for} \quad x \in \langle 0,S \rangle.$$

Similarly, from the condition $U^1_t(x,T) = 0$ we have

$$Q_t(x,T) = z_t(x,T) = 0 \text{ for } x \in \langle 0,S \rangle.$$

Let us consider the conditions imposed on the function Q

$$Q(x,T) = \sum_{n=1}^{\infty} \left[C_n \cos \alpha_n T + \frac{1}{\alpha_n} D_n \sin \alpha_n T + \int_0^T \frac{2}{S} \sin \alpha_n (T - \tau) u(\tau) d\tau \right] \sin \alpha_n x = 0$$

(8)

$$Q_t(x,T) = \sum_{n=1}^{\infty} \left[-\alpha_n C_n \sin \alpha_n T + D_n \cos \alpha_n T + \int_0^T \frac{2}{S} \alpha_n \cos \alpha_n (T - \tau) u(\tau) d\tau \right] \sin \alpha_n x = 0$$

for $x \in (0,S)$.

The system of functions $\{\sin \alpha_n x\}$ is complete in $L_2[0,S]$, thus for satisfaction of the condition (8) it is necessary and sufficient that all coefficients at $\sin \alpha_n x$ should equal to zero. From here we obtain the infinite system of equations

$$(9)\quad \int_0^T \frac{2}{S} \sin\alpha_n(T-\tau)u(\tau)d\tau + C_n \cos\alpha_n T + \frac{1}{\alpha_n} D_n \sin\alpha_n T = 0,$$

$$(10)\quad \int_0^T \frac{2}{S} \cos\alpha_n(T-\tau)u(\tau)d\tau - C_n \sin\alpha_n T + \frac{1}{\alpha_n} D_n \cos\alpha_n T = 0$$

for $n = 1,2,\ldots$

Multiplying (for fixed n) the equation (9) by i ($i \in C$) an adding to the equation (10), and then dividing the resulting equation by exp(inT) we obtain the new system of equations, which can be written, after the separation of the real and imaginary parts, as follows

$$(11)\quad \int_0^T u(t) \sin\alpha_n t\, dt = \frac{S}{2} C_n, \quad \int_0^T u(t) \cos\alpha_n t\, dt = \frac{-S}{2\alpha_n} D_n$$

for $n = 1,2,\ldots$

Considering the conditions imposed on function z

$$z(x,T) = z_t(x,T) = 0$$

taking into account the fact that the system $\{\sin\beta_n x\}$ is complete in $L_2[0,S]$ we obtain the system of equations

$$(12)\quad \int_0^T u(t) \sin\beta_n t\, dt = \frac{S}{2} A_n, \quad \int_0^T u(t) \cos\beta_n t\, dt = -\frac{S}{2\beta_n} B_n$$

for $n = 1,2,\ldots$

The equations (11) and (12) can be rewritten in the form

$$(13)\quad \int_0^T u(t) \sin\gamma_n t\, dt = \frac{S}{2} a_n, \quad \int_0^T u(t) \cos\gamma_n t\, dt = -\frac{S}{2\gamma_n} b_n$$

where $\gamma_n = \frac{n\pi}{2S}$, and

$$a_n = \begin{cases} A_{\frac{n}{2}} & \text{for even } n \\ C_{\frac{n+1}{2}} & \text{for odd } n \end{cases} \qquad b_n = \begin{cases} B_{\frac{n}{2}} & \text{for even } n \\ D_{\frac{n+1}{2}} & \text{for odd } n \end{cases}$$

for $n = 1,2,\ldots$

Let us assume the length of the strings $S = \pi/2$. Then the equations (13) take the form ($\gamma_n = n$)

$$(14) \quad \int_0^T u(t)\sin nt\,dt = \frac{\pi}{4}a_n, \quad \int_0^T u(t)\cos nt\,dt = -\frac{\pi}{4n}b_n.$$

This way, the problem of finding the optimal control was reduced to the problem of finding such a function $u(t)\in L_2[0,T]$ which norm is limited by a number $l>0$, $\|u\|\leq l$ and for which the infinite system of equations (14) is satisfied i.e. the moments of which are given by system (14) in relation to some system of functions. In our case it is a system of the functions sin nt and cos nt for $n = 1,2,\ldots$.

Basic results of the theory concerning the problem of moments were obtained by M.G. Krejn [6], [7], but H.H. Krassowski [8,9] was the first who applyied the results obtained from this theory to the problem of optimal control.

In our problem the beeing restricted for function u(t) will belong to the space $\text{lin}\{\sin nt, \cos nt\}\subset L_2[0,T]$ so it will be a periodic function with the period 2π.

Let us express T in a form of $T = 2\pi k + \varepsilon$, where $k = 0,1,\ldots$ and $0\leq\varepsilon<2\pi$. Then the equations (14) under condition $u(t+2\pi) = u(t)$ can be expressed in the form

$$(15) \quad \int_0^{2\pi k+\varepsilon} u(t)\sin nt\,dt = (k+1)\int_0^{\varepsilon} u(t)\sin nt\,dt +$$

$$+ k\int_{\varepsilon}^{2\pi} u(t)\sin nt\,dt = \int_0^{2\pi}\psi(t)\sin nt\,dt = \frac{\pi}{4}a_n$$

$n = 1,2,\ldots$

$$(15)\quad \int_0^{2\pi k+\varepsilon} u(t)\cos nt\,dt = (k+1)\int_0^{\varepsilon} u(t)\cos nt\,dt + $$

$$+ k\int_{\varepsilon}^{2\pi} u(t)\cos nt\,dt = \int_0^{2\pi}\psi(t)\cos nt\,dt = -\frac{\pi}{4n} b_n$$

$$n = 1,2,\dots$$

where

$$(16)\qquad \psi(t) = \begin{cases} (k+1)u(t) & \text{for } t \in \langle 0,\varepsilon\rangle \\ k\,u(t) & \text{for } t \in (\varepsilon, 2\pi\rangle. \end{cases}$$

The function $\psi(t)$ is calculated from the conditions (15) exact to some constant C, as it is seen that $\frac{1}{4} a_n$ and $-\frac{1}{4n} b_n$ for $n = 1,2,\dots$ are the Fourier coefficients of the function $\psi(t)$ on the interval $\langle 0,2\pi\rangle$. From the above

$$(17)\quad \psi(t) = \frac{1}{4}\sum_{n=1}^{\infty}\left(a_n \sin nt - \frac{1}{n} b_n \cos nt\right) + C = G(t) + C.$$

From the conditions imposed on f(x) and F(x) it appears that the above series is uniformly convergent. From the formulae (16) it is seen that if $k \neq 0$ then the beeing looked for control function u(t) is on the interval $\langle 0,2\pi\rangle$ of the form

$$(18)\qquad u(t) = \begin{cases} \frac{1}{k+1}\left[G(t) + C\right] & \text{for } t \in \langle 0,\varepsilon\rangle \\ \frac{1}{k}\left[G(t) + C\right] & \text{for } t \in (\varepsilon, 2\pi\rangle \end{cases}$$

where for $t > 2\pi$ (but $t < T$) the function u(t) is periodically extended with the period 2π. For $k = 0$ and $0 \leqslant \varepsilon < 2\pi$ the equations (15) can be rewritten in the form

$$(19)\quad \int_0^{2\pi} v(t)\sin nt\,dt = \frac{\pi}{4} a_n, \quad \int_0^{2\pi} v(t)\cos nt\,dt = -\frac{\pi}{4n} b_n$$

where

$$v(t) = \begin{cases} u(t) & \text{for} \quad t \in \langle 0, \varepsilon \rangle \\ 0 & \text{for} \quad t \in (\varepsilon, 2\pi \rangle \end{cases}$$

and v(t) is also calculated exact to some constant C

(20) $$v(t) = G(t) + C \quad \text{for} \quad 0 \leqslant t \leqslant 2\pi.$$

But, for $t \in (\varepsilon, 2\pi\rangle$ it is seen from the formula (19) that the function v(t) should be equal to zero. Thus, in this case i.e. for k = 0 that is to say for $0 \leqslant T < 2\pi$, we obtain from (20) that the constant $C = -G(t^*)$, where t^* - any element from $(\varepsilon, 2\pi\rangle$ and the being looked for control function is calculated explicitly

(21) $$u(t) = G(t) - G(t^*), \quad t \in \langle 0, \varepsilon \rangle, \quad t^* \in (\varepsilon, 2\pi).$$

It is seen from (20) that in the case $0 \leqslant T < 2\pi$ the control function u(t) exists if and only if G(t) is constant on the interval $(\varepsilon, 2\pi)$. If the initial functions f(x) and F(x) are such that G(t) is not constant on the interval $(\varepsilon, 2\pi)$, then the control function u(t) giving the setting the system at rest during the time interval $T < 2\pi$ does not exist. In such a case the choice of the constant C for $T \geqslant 2\pi$ in the formula (18) remains. The constant C can be choosen e.g. so as for a given $T \geqslant 2\pi$ to minimize $\|u(t)\|_{L_2[0,T]}$ where u(t) is described by the formula (18), or minimal time of setting at rest T_{min} ($T_{min} \geqslant 2\pi$) can by looked for choosing the constant C so as $\|u(t)\|_{L_2[0,T_{min}]} \leqslant 1$. It is easy to show that the function G(t) is of the form

(22) $$G(t) = \frac{1}{2}\bar{f}(t) + \frac{1}{2}\int_0^t \bar{F}(\tau)d\tau \quad \text{for} \quad t \in \langle 0, 2\pi \rangle$$

where

$$\bar{f}(t) = \begin{cases} f(t) & \text{for } t \in \langle 0, \frac{\pi}{2} \rangle \\ 0 & \text{for } t \in (\frac{\pi}{2}, \frac{3\pi}{2} \rangle \\ -f(2\pi - t) & \text{for } t \in (\frac{3\pi}{2}, 2\pi \rangle \end{cases} \qquad \bar{F}(t) \text{ - similarly.}$$

Finally, for $T = 2\pi k + \varepsilon$, $k = 1,2,\ldots$ the optimal control function $u(t)$ under the initial conditions $f(x)$ and $F(x)$ can be written in the form

$$(23) \qquad u(t) = \begin{cases} \frac{1}{k+1}\left[\frac{1}{2}\bar{f}(t) + \frac{1}{2}\int_0^t \bar{F}(\tau)d\tau + C\right] & \text{for } t \in \langle 0, \varepsilon \rangle \\ \frac{1}{k}\left[\frac{1}{2}\bar{f}(t) + \frac{1}{2}\int_0^t \bar{F}(\tau)d\tau + C\right] & \text{for } t \in (\varepsilon, 2\pi \rangle \end{cases}$$

and for $t > 2\pi$ function $u(t)$ is periodically extended with the period 2π.

Let us consider once more the vibrations of the system caused by non-zero initial conditions on the first string i.e. by conditions (2′), but let us ask the questions concerning the setting the system at rest in an other form: it is possible to set at rest such a system by the control function applied at zero point of the second and third string (Fig.2).

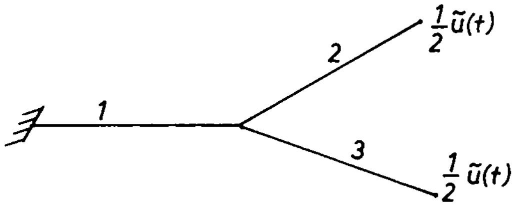

Fig.2

In this case the initial conditions (3) become the form

$$(3'') \qquad U^1(0,t) = 0, \quad U^2(0,t) = U^3(0,t) = \frac{1}{2}\tilde{u}(t) \quad \text{for } t \geq 0.$$

The solutions of the problem (1) under conditions (2'), (3''), (4), (5) are as follows

$$U^1(x,t) = \frac{1}{3} Q(x,t) + \frac{2}{3} w(x,t),$$

$$U^2(x,t) = U^3(x,t) = \frac{1}{3} Q(x,t) - \frac{1}{3} w(x,t)$$

where the function $Q(x,t)$ is designated by the formula (7) but $w(x,t)$ differs from $z(x,t)$ in boundary condition at a point $x = 0$, namely $w(0,t) = -\frac{1}{2} \tilde{u}(t)$ for $t \geqslant 0$.

From here

$$w(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos \beta_n t + \frac{1}{\beta_n} B_n \sin \beta_n t\right] \sin \beta_n x +$$

$$- \frac{1}{2} \sum_{n=1}^{\infty} \frac{2}{S} \sin \beta_n x \int_0^t \tilde{u}(\tau) \sin \beta_n (t - \tau) d\tau.$$

From the conditions (6), which we put on the control function $u(t)$, we obtain

$$Q(x,T) = w(x,T) = 0, \quad Q_t(x,T) = w_t(x,T) = 0.$$

The above system leads us (similarly as before) to the infinite system of equations

$$(24) \qquad \int_0^T \tilde{u}(t) \sin nt \, dt = \frac{\pi}{4} c_n, \quad \int_0^T \tilde{u}(t) \cos nt \, dt = -\frac{\pi}{4n} d_n$$

where

$$c_n = \begin{cases} -2A_{\frac{n}{2}} & \text{for even } n \\ C_{\frac{n+1}{2}} & \text{for odd } n \end{cases}, \qquad d_n = \begin{cases} -2B_{\frac{n}{2}} & \text{for even } n \\ D_{\frac{n+1}{2}} & \text{for odd } n. \end{cases}$$

Further treatment leading to calculation of the control function $\tilde{u}(t)$ will be similar to that described before. In this case the function G(t) take the form

$$G(t) = \frac{1}{4}\sum_{n=1}^{\infty} (c_n \sin nt - \frac{1}{n} d_n \cos nt). \tag{25}$$

Taking into account the formula for c_n and d_n it is easy do calculate the sum of the above series. Thus

$$G(t) = \frac{1}{2}\tilde{f}(t) + \frac{1}{2}\int_0^t \tilde{F}(\tau)d\tau \quad \text{for} \quad t \in \langle 0, 2\pi\rangle \tag{26}$$

where

$$\tilde{f}(t) = \begin{cases} 0 & \text{for} \quad t \in \langle 0, \frac{\pi}{2}\rangle \\ f(\pi - t) & \text{for} \quad t \in (\frac{\pi}{2}, \pi\rangle \\ -f(t - \pi) & \text{for} \quad t \in (\pi, \frac{3}{2}\pi\rangle \\ 0 & \text{for} \quad t \in (\frac{3}{2}\pi, 2\pi\rangle \end{cases} \qquad \tilde{F}(t) \text{ - similarly}.$$

The form of functions $\tilde{f}(t)$ and $\tilde{F}(t)$ result in fact that G(t) is constant on the interval $(\frac{3}{2}\pi, 2\pi\rangle$ and equals to zero. It is the necessary and sufficient condition for existing control function $\tilde{u}(t)$ which will set at rest the system in the time period $T < 2\pi$. Thus for $T = \frac{3}{2}\pi$ exists explicitly calculated control function

$$\tilde{u}(t) = \frac{1}{2}\tilde{f}(t) + \frac{1}{2}\int_0^t \tilde{F}(\tau)d\tau \quad \text{for} \quad t \in \langle 0, \frac{3}{2}\pi\rangle. \tag{27}$$

Conclusions

The above considerations lead to the following conclusions: it is seen that the vibrations of the system caused by the non-zero initial conditions put on the one string only can by set at rest in two manners:

- by the control function applied at the end of this string (Fig.1), in this case the shortest possible time of setting the system at rest equals $T_{min} = 2\pi$
or

- by two the identical control functions applied at the ends of the remaining strings (Fig.2), in this case the shortest possible time of setting the system at rest equals $T_{min} = \frac{3}{2}\pi$.

It is easy to show that the system under consideration can not be set at rest by the control function applied at the end of only one of the remaining strings i.e. in our case the second or the third one.

On the superposition principle we can conclude that the vibrations of the system with any initial conditions can by also set at rest in two manners:

- applying the control functions at the end of all three strings, in this case the shortest possible time of setting the system at rest equals to $\frac{3}{2}\pi$
or

- applying the control functions at the ends of two strings only, in this case the time of setting the system at rest will be elongated to 2π.

In the case of a greater number of strings, the optimal control problem may be solved similarly.

Generally, the system consisting of N strings, can be set at rest by N control functions or (N-1) ones.

BIBLIOGRAPHY

[1] А.Г. Бутковский: Методы управления системами с распределёнными параметрами. Москва 1975.

[2] S. Rolewicz: On a problems of moments, Studia Math. 30 (1968) 183-191.

[3] S. Rolewicz: On controllability of systems of strings, Studia Math. 36 (1970) 105-110.

[4] S. R o l e w i c z : Analiza funkcjonalna i teoria sterowania. Warszawa 1977.

[5] A. D y m i t r u k , J. M u s z y ń s k i : Damped vibrations of strings connected in a node, Demonstratio Math. 12 (1979) 501-508.

[6] Н.И. А х е з е р , М.Г. К р е й н : О некоторых вопросах теории моментов. Харков 1938.

[7] М.Г. К р е й н , А.А. Н у д е л ь м а н : Проблема моментов Маркова и экстремальные задачи. Москва 1973.

[8] Н.Н. К р а с о в с к и й : Теория управления движением. Москва 1968.

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Received December 7, 1987.