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ON SOME GENERALIZATIONS OF THE POISSON DISTRIBUTION

1. Introduction

This paper deals with two extensions of the Poisson distribution. They are generalized Poisson (GP) distribution and the linear function Poisson (LPF) distribution. These two distributions have received some attention recently. For example, Consul and Jain (1973) obtain GP from the Lagrange distribution and deal with moments and other properties. Lingappaiah (1977) takes up inflation in GP. Jain (1975) works with LPF, and relates variables to waiting times in recurrent events. Lingappaiah (1986) shows that some constants of LPF and GP can be obtained from analyzing any one of these two distributions and a separate analysis for each is not needed. Consul (1986) gives the distribution of the difference of two GP variables and also gives cumulants. Janardan (1987) calls the LPF as the weighted distribution of GP and again obtains both via Lagrange distribution. Consul and Shoukri (1986) deal with the inverse moments of GP. What is being done in this paper is to deal with the sum of two LPF and GP variables under both conditions, such as, when these variables are just independent or they are i.i.d. Under both these conditions, certain recurrence relations are obtained for the moments. Next, the estimation is taken up. Bayes estimates of the parameters θ and λ in LPF are put in closed forms under different priors. Finally, LPF of order-k is dealt with. In Lingappaiah (1986), parameter θ in LPF is successively replaced and after k such replacements, a distribution is obtained. This may be termed as LPF of order-k (Type I). Here, instead of θ, parameter λ is replaced successively and the resulting distribution after k such replacements is called as LPF or order-k (Type II) distribution.

2. Recurrence relations

Generalized Poisson (GP) distribution is,

$$f_g(x,\theta,\lambda) = \theta e^{-(\theta+\lambda x)}(\theta+\lambda x)^{x-1}/x! \tag{1}$$

and the linear function Poisson (LPF) distribution is,

$$f_l(x,\theta,\lambda) = (1-\lambda)e^{-(\theta+\lambda x)}(\theta+\lambda x)^{x}/x! \tag{2}$$

In both (1) and (2), $x = 0,1,2,\ldots,\ \theta>0,\ |\lambda|<1$.

2a: Consider two independent LPF variables x and y with parameters θ_i, λ_i, $i = 1,2$, respectively. First, some recurrence relations are given below which express $\mu(\theta',\lambda')$ and $\mu(\theta_1+\lambda_1,\theta_2,\lambda')$ in terms of the derivatives, where $\theta' = (\theta_1,\theta_2)$ and $\lambda' = (\lambda_1,\lambda_2)$. These relations tell how by increasing θ_i to $\theta_i+\lambda_i$, $i = 1,2$, resulting forms can be put in terms of derivatives. Derivation of the results are given in Appendix.

$$\frac{\partial^s \mu'_{rl}(\theta',\lambda')}{\partial\theta_1^s} = \tag{3}$$

$$= (-1)^s\left[\mu'_{rl}(\theta',\lambda') + \sum_{j=1}^{s}\binom{s}{j}(-1)^j \prod_{t=1}^{j}\sum_{i_t=0}^{i_{t-1}}\binom{i_{t-1}}{i_t}\mu'_{i_j l}(\theta_1+j\lambda_1,\theta_2,\lambda')\right]$$

where $i_0 = r$, $\theta_1 + j\lambda_1 > 0$, $j = 0,1,\ldots,s$.

(4) $$\frac{\partial^2 \mu'_{21}(\theta',\lambda')}{\partial\theta_1^2} =$$

$$= \Big[\mu'_{21}(\theta',\lambda') - 2\mu'_{21}(\theta_1+\lambda_1,\theta_2,\lambda') + \mu'_{21}(\theta_1+2\lambda_1,\theta_2,\lambda') -$$

$$- 4\mu_1(\theta_1+\lambda_1,\theta_2,\lambda') + 4\mu_1(\theta_1+2\lambda_1,\theta_2,\lambda') + 2\Big].$$

Similar to (4′) in Appendix, by differentiating w.r.t. θ_2, (5) follows

(5) $$\frac{\partial \mu'_{r1}(\theta',\lambda')}{\partial\theta_2} = -\mu'_{r1}(\theta',\lambda') + \sum_{i=0}^{r} \binom{r}{i} \mu'_{i1}(\theta_1,\theta_2+\lambda_2,\lambda').$$

From (4′) and (5), one gets

(6) $$\frac{\partial^2 \mu'_{r1}(\theta',\lambda')}{\partial\theta_1\partial\theta_2} = \mu'_{r1}(\theta',\lambda') + \sum_{i=0}^{r} \binom{r}{i} \mu'_{i1}(\theta_1+\lambda_1,\theta_2+\lambda_2,\lambda') -$$

$$- \sum_{i=0}^{r} \binom{r}{i} \Big\{\mu'_{i1}(\theta_1+\lambda_1,\theta_2,\lambda') + \mu'_{i1}(\theta_1,\theta_2+\lambda_2,\lambda')\Big\}.$$

2b: Now consider two i.i.d. LPF variables x and y, each with parameters θ and λ. Define (with $z = x+y$)

(7) $$\psi(r,s,t,\theta,\lambda) = a^2 \sum_{z=0}^{\infty} z^r h_0(z) \sum_{x=0}^{z} \Big[A^x B^{(z-x)}/x!(z-x)!A^s B^t\Big],$$

where $h_0(z) = e^{-(2\theta+\lambda z)}$, $A = (\theta+\lambda x)$, $B = \theta+\lambda(z-x)$, $a = 1-\lambda$. And also

(8) $$\phi(r,s,t,\theta,\lambda) = \theta^2 \sum_{z=0}^{\infty} z^r h_0(z) \sum_{x=0}^{z} \Big[A^{x-1} B^{z-x-1}/x!(z-x)!A^s B^t\Big].$$

Below, certain relations are given involving $\psi(r,s,t,\theta,\lambda)$ and $\phi(r,s,t,\theta,\lambda)$. Their derivations are given in Appendix.

(9) $$\psi(r,s,t,\theta,\lambda) = a^2\phi(r,s,t,\theta,\lambda)+(a^2\lambda/\theta)\phi(r+1,s,t,\theta,\lambda) + \lambda^2\sum_{i=0}^{r}\binom{r}{i}2^{r-i}\psi(i,s,t,\theta+\lambda,\lambda),$$

(10) $$\mu'_{rl}(\theta,\lambda) = a^2\mu'_{rg}(\theta,\lambda) + (a^2\lambda/\theta)\mu'_{(r+1)g}(\theta,\lambda) + \lambda^2\sum_{i=0}^{r}\binom{r}{i}2^{r-i}\mu'_{il}(\theta+\lambda,\lambda),$$

(11) $$\frac{\partial\mu'_{rl}(\theta,l)}{\partial\theta} = -2\mu'_{rl}(\theta,\lambda) + (a^2/\theta)\mu'_{(r+1)g}(\theta,\lambda) + 2\lambda^2a^2\sum_{z} z^r h_0(z)\sum_{x} A^{x-1}B^{z-x-1}/(x-1)!(z-x-1)!.$$

3. Bayes estimates (LFP)

In this section, we give the Bayes estimate of θ and λ. Bayes estimates are easier to obtain than maximum likelihood estimates, which have to be gotten through iteration procedure. Simple priors are taken for θ and λ as exponential and uniform. Estimates involve repeated summations, which can be handled on computers.

3a: From (2), we get the likelihood function for the LPF case as

(12) $$L(\underset{\sim}{x},\theta,\lambda) = (1-\lambda)^n e^{-(n\theta+n\lambda\bar{x})}\prod_{i=1}^{n}\left[\sum_{j_i=0}^{x_i}\binom{x_i}{j_i}\theta^{j}\lambda^{x-j}x_i^{x_i-x_j}/x_i!\right],$$

where $\underset{\sim}{x} = (x_1,\ldots,x_n)$, $x = x_1 + \ldots + x_n$, $j = j_1 + \ldots + j_n$. Now (12) can be put as,

$$(13)\quad L(\underset{\sim}{x},\theta,\lambda) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\binom{n}{r}(-1)^{r+s}\frac{(n\bar{x})^s}{s!}\prod_{i=1}^{n}\left[H(j_i)e^{-n\theta}\lambda^c\theta^j\right],$$

where $c = x+r+s-j$, $H(j_i) = \sum_{j_i}\binom{x_i}{j_i}x_i^{x_i-j_i}/x_i!$.

Let the priors for λ and θ be,

$$(14)\quad \begin{aligned} f(\lambda) &= \tfrac{1}{2}, \quad -1<\lambda<1, \\ g(\theta) &= e^{-\theta}, \quad \theta>0. \end{aligned}$$

From (13) and (14), we get

$$(15)\quad L(\underset{\sim}{x}) = \iint L(\underset{\sim}{x},\theta,\lambda)f(\lambda)g(\theta)d\lambda d\theta =$$

$$= \left(\tfrac{1}{2}\right)\sum_r\sum_s\prod_{i=1}^{n} H(j_i)M(r,s,x,j)\phi(c),$$

where

$$(15a)\quad \phi(c) = \frac{1+(-1)^c}{c+1}, \quad M(r,s,x,j) = \binom{n}{r}\left[\frac{(n\bar{x})^s(-1)^{r+s}j!}{s!\,(n+1)^{j+1}}\right].$$

From (13) and (15), we get the Bayes estimate of θ as,

$$(16)\quad \hat{\theta} = \iint \theta L(\underset{\sim}{x},\theta,\lambda)f(\lambda)g(\theta)d\lambda d\theta/L(\underset{\sim}{x}) =$$

$$= \frac{\sum_r\sum_s\prod_{i=1}^{n} H(j_i)M(r,s,x,j)\phi(c)\left[(j+1)/(n+1)\right]}{\sum_r\sum_s\prod_{i=1}^{n} H(j_i)M(r,s,x,j)\phi(c)}.$$

Similarly, Bayes estimate of λ is,

(17)
$$\hat{\lambda} = \iint \lambda L(\underset{\sim}{x},\theta,\lambda) f(\lambda) g(\theta) d\lambda d\theta / L(\underset{\sim}{x}) =$$

$$= \frac{\sum_r \sum_s \prod_{i=1}^{n} H(j_i) M(r,s,x,j) \phi(c+1)}{\sum_r \sum_s \prod_{i=1}^{n} H(j_i) M(r,s,x,j) \phi(c)} .$$

3b: Sometimes, variation of λ in (2) is taken as $0 < \lambda < 1$, instead $|\lambda| < 1$. Now, one can have the prior for λ as

(18)
$$f(\lambda) = \lambda^{a-1}(1-\lambda)^{b-1}/B(a,b),$$

where $B(a,b)$ the beta function. Now we have

(19)
$$L(\underset{\sim}{x},\theta,\lambda) = \sum_{s=0}^{\infty} \frac{(-n\bar{x})^x}{s!} \prod_{i=1}^{n} H(j_i)\theta^j \lambda^{c'} e^{-n\theta},$$

where $c' = x+s-j$. Now we get from (19)

(19a)
$$L(\underset{\sim}{x}) = \iint L(\underset{\sim}{x},\theta,\lambda) f(\lambda) g(\theta) d\lambda d\theta =$$

$$= \sum_{s=0}^{\infty} \prod_{i=1}^{n} H(j_i) M_0(x,s,j) B(a+c',b+n),$$

where $c' = x+s-j$, $M_0(x,s,j) = \left[\frac{(-n\bar{x})^x}{s!} \frac{j!}{(n+1)^{j+1}}\right]$ and hence we get (16) and (17) as

(20)
$$\hat{\theta} = \frac{\sum_s \prod_{i=1}^{n} H(j_i) M_0(x,s,j) B(a+c',b+n) \left[(j+1)/(n+1)\right]}{\sum_s \prod_{i=1}^{n} H(j_i) M_0(x,s,j) B(a+c',b+n)}$$

and

$$(21) \qquad \hat{\lambda} = \frac{\sum_{s} \prod_{i=1}^{n} H(j_i) M_0(s,x,j) B(a+c'+1, b+n)}{\sum_{s} \prod_{i=1}^{n} H(j_i) M_0(s,x,j) B(a+c', b+n)} .$$

Though (16), (17), (20) and (21) look complex, since n sums have to be evaluated, on j_i's, $i = 1,2,\ldots,n$, on the computers, $\hat{\theta}$ and $\hat{\lambda}$ can be easily evaluated for different n.

4. LPF of order-k (Type II)

Define

$$(22) \qquad \psi(\theta,k\lambda) = (1-k\lambda)e^{-(\theta+k\lambda x)}(\theta+k\lambda x)^x/x!.$$

Now

$$(23) \qquad \psi(\theta,2\lambda) = \left(\frac{1-2\lambda}{1-\lambda}\right)(2e^{-\lambda})^x e^{-\theta/2}\left[ae^{-(\theta/2+\lambda x)}(\theta/2+\lambda x)^x/x!\right],$$

where $a = 1-\lambda$. Now (23) is

$$(24) \qquad \left(\frac{e^{\lambda}}{2}\right)^x \left(\frac{1-\lambda}{1-2\lambda}\right)\psi(\theta,2\lambda) = e^{-\theta/2}\psi(\theta/2,\lambda).$$

Similarly, we get

$$(25) \qquad \left(\frac{e^{2\lambda}}{3}\right)^x \left(\frac{1-\lambda}{1-3\lambda}\right)\psi(\theta,3\lambda) = e^{-2\theta/3}\psi(\theta/3,\lambda).$$

Continuing this way, one gets,

$$(26) \qquad \left[\frac{e^{(k-1)\lambda}}{k}\right]^x \left(\frac{1-\lambda}{1-k\lambda}\right)\psi(\theta,k\lambda) = e^{-(k-1)\theta/k}\psi(\theta/k,\lambda).$$

Eq. (26) can also be proved by induction.

Now, replacing θ in (2) several times by $\theta+\lambda x$, resulting distribution after such k replacements, is obtained in Lingappaiah (1986), which can be termed as LPF of order-k (Type I). Since λ is replaced here several times, $\psi(\theta,k\lambda)$ can be termed as LPF of order-k (Type II) and (25) expresses LPF or order-k (Type II)-in terms of LPF of order-1 (Type II) which is the regular LPF. Summing (26), we get

$$(27) \qquad \sum_{x=0}^{\infty} d^x[(1-\lambda)/(1-k\lambda)]\psi(\theta,k\lambda) = e^{-b\theta/k},$$

where

$$(27a) \qquad d = \left[\frac{e^{(k-1)\lambda}}{k}\right]^x, \quad b = (k-1).$$

Eq. (27) can also be proved by induction. Differentiating (27) w.r.t. θ and using (27) again, we get, using (1),

$$(28) \qquad e^{-b\theta/k} + \sum_x xd^x(a/\theta)f_g(\theta,k\lambda) = -(b/k)e^{-b\theta/k},$$

and (28) gives

$$(29) \qquad \sum_x xd^xf_g(\theta,\lambda) = (\theta/ka)e^{-b\theta/k}.$$

Eq. (29) can also be obtained from (26) by writing $\theta+k\lambda x = k(\theta/k+\lambda x)$ in the LHS of (26) and summing over, one gets

$$(30) \qquad \sum_x kxd^x(a/\theta)f_g(\theta,k\lambda) = e^{-b\theta/k}(ak/\theta)\mu_g(\theta/k,\lambda)$$

and (30) is (29).

Appendix

Let $z = x+y$, then from (2), we get the distribution of z as

$$(1') \qquad f_1(z,\theta',\lambda') = a_1a_2e^{-(\theta'+\lambda_2 z)}\sum_{x=0}^{z} e^{-(\lambda_1-\lambda_2)x}A_1^xB_2^{(z-x)}\Big/x!(z-x)!,$$

where $a_i = 1-\lambda_i$, $A_i = \theta_i+\lambda x_i$, $B_i = \theta_i+\lambda_i(z-x)$, $i = 1,2$, $\theta' = (\theta_1,\theta_2)$, $\lambda' = (\lambda_1,\lambda_2)$. From $(1')$, the r-th moment of z is

$$(2') \qquad \mu'_{r1}(\theta',\lambda') = \sum_{z=0}^{\infty} z^r f_1(z,\theta',\lambda').$$

Differentiating $(2')$ w.r.t. θ_1, one gets,

$$(3') \qquad \frac{\partial \mu'_r(\theta',\lambda')}{\partial\theta_1} = -\mu'_{r1}(\theta',\lambda') +$$

$$+ a_1a_2\sum_z z^r h(z)\sum_x e^{-(\lambda_1-\lambda_2)x}\left[\frac{A_1^{x-1}B_2^{z-x}}{(x-1)!(z-x)!}\right],$$

where $h(z) = e^{-(\theta'+\lambda_2 z)}$. Setting $x = 1 = u$ and writing $z = (z-1+1)$, $(3')$ reduces to

$$(4') \qquad \frac{\partial\mu'_{r1}(\theta',\lambda')}{\partial\theta_1} = -\mu'_{r1}(\theta',\lambda') + \sum_{i=0}^{r}\binom{r}{i}\mu'_{i1}(\theta_1+\lambda_1,\theta_2,\lambda').$$

For $r = 0$, $(4')$ is obvious since,

$$(4'a) \qquad \mu'_{o1}(\theta',\lambda') = 1.$$

For $r = 1$, LHS of $(4')$ gives $1/a_1$, noting from Lingappaiah (1986),

$$(4') \qquad \mu'_{11}(\theta',\lambda') = \mu_1(\theta',\lambda') = \sum_{i=1}^{2}\left(\frac{\theta_i}{a_i}+\frac{\lambda_i}{a_i^2}\right).$$

RHS of $(4')$ is $-\mu_1(\theta',\lambda')+[1+\mu_1(\theta_1+\lambda_1,\theta_2,\lambda')]$ and again, it is easy to check that LHS = RHS in $(4')$ using $(4'b)$. From $(4')$, by successive differentiating w.r.t. θ_1, we get the s-th derivative $D^s_{\theta_1}\mu'_{r1}(\theta',\lambda')$ w.r.t. θ_1 as (3). For $s = 2$, if $r = 0$, LHS of $(4')$ is zero from $(4'b)$. RHS of (3) is

(1-1) + (1-1) = 0. For s = 2, if r = 1, LHS of (4′) is again zero from (4′b) and RHS is

$$(4'c) \qquad +\mu_1(\theta',\lambda') - 2\sum_{i=0}^{1}\mu_{i1}(\theta_1+\lambda_1,\theta_2,\lambda') +$$

$$+\sum_{i_1=0}^{1}\sum_{i_2=0}^{i_1}\binom{1}{i_1}\binom{i_1}{i_2}\mu_{i_2 1}(\theta_1+2\lambda_1,\theta_2,\lambda') =$$

$$= \mu_1(\theta',\lambda') - 2[1+\mu_1(\theta_1+\lambda_1,\theta_2,\lambda')] + [2+\mu_1(\theta_1+2\lambda_1,\theta_2,\lambda']$$

and (4′c) is zero using (4′b) again.

For s = 2, r = 2, (3) is,

$$(4'd) \quad \frac{\partial^2\mu'_{21}(\theta',\lambda')}{\partial\theta_1^2} = \mu'_{21}(\theta',\lambda') - 2\sum_{i_1=0}^{2}\binom{2}{i_1}\mu'_{i_1 1}(\theta_1+\lambda_1,\theta_2,\lambda') +$$

$$+\sum_{i_1=0}^{2}\binom{2}{i_1}\sum_{i_2=0}^{i_1}\binom{i_1}{i_2}\mu'_{i_2 1}(\theta_1+2\lambda_1,\theta_2,\lambda') =$$

$$(4'e) \qquad = \mu'_{21}(\theta',\lambda') - 2[1+2\mu_1(\theta_1+\lambda_1,\theta_2,\lambda')+\mu'_{21}(\theta_1+\lambda_1,\theta_2,\lambda')] +$$

$$+ [1+2\{1+\mu_1(\theta_1+2\lambda_1,\theta_2,\lambda')\} + \{1+2\mu_1(\theta_1+2\lambda_1,\theta_2,\lambda') +$$

$$+ \mu'_{21}(\theta_1+2\lambda_1,\theta_2,\lambda')\}] .$$

(4′e) gives (4).

Since $\sigma^2(z) = \sigma^2(x) + \sigma^2(y)$, we have

$$(4'f) \qquad \mu'_{21}(z,\theta',\lambda') =$$

$$= \mu'_{21}(x,\theta',\lambda') + \mu'_{21}(y,\theta',\lambda') + 2\mu_1(x,\theta',\lambda')\mu(y,\theta',\lambda').$$

From (4′b) it is clear that LHS of (3) is $2/a_1^2$. Now using (4′b) and $\mu'_{21}(x)$ from Lingappaiah (1986), along with (4′f), it is seen from (4) that RHS of (4′d) is also $2/a_1^2$. For r = 0, using (4′a) and (4′b) gives,

$$0 = 1+1 - (1+1).$$

If r = 1, LHS of (6) is zero from (4′b). RHS of (6) is also zero which can be seen using (4′b) to each term.

From (7) and (8), it follows,

(5′) $$\psi(r,s,t,\theta,\lambda) = (a^2/\theta^2)\phi(r,s-1,t-1,\theta,\lambda),$$

(5′a) $$\psi(0,0,0,\theta,\lambda) = (a^2/\theta^2)\phi(0,-1,-1,\theta,\lambda).$$

Noting

(5′b) $$AB = (\theta+\lambda x)[\theta+\lambda(z-x)] = (\theta^2+\theta\lambda z) + \lambda^2 x(z-x)$$

we get RHS of (7) as,

(5′c) $$a^2\phi(r,s,t,\theta,\lambda) + (a^2\lambda/\theta)\phi(r+1,s,t,\theta,\lambda) + + \lambda^2 \sum_z z^r h_o(z)\theta^2 \sum_x \left[\frac{A^{x-1}B^{z-x-1}}{(x-1)!(z-x-1)!A^s B^t}\right].$$

Setting x-1 = u and z = (z-2)+2 in (5′c), one gets (9).

From (7) and (8), it follows

(6′a) $$\psi(0,0,0,\theta,\lambda) = \phi(0,0,0,\theta,\lambda) = 1,$$

(6′b) $$\phi(r,0,0,\theta,\lambda) = \mu'_{rg}(\theta,\lambda),$$

$$\psi(r,0,0,\theta,\lambda) = \mu'_{rl}(\theta,\lambda),$$

where $\mu'_{rg}(\theta,\lambda)$, $\mu'_{rl}(\theta,\lambda)$ are the r-th moment of GP and LPF respectively when x,y in z are i.i.d.

Now from (9), it follows

(6′c) $\psi(0,0,0,\theta,\lambda) = a^2\phi(0,0,0,\theta,\lambda) + (a^2\lambda/\theta)\phi(1,0,0,\theta,\lambda) +$

$$+\lambda^2\psi(0,0,0,\theta+\lambda,\lambda).$$

Now with $\mu'_{1g}(\theta,\lambda) = \mu_g(\theta,\lambda)$, from (6′a) and (6′b) one sees that both sides of (6′c) are equal to 1 since (6′c) now is

$$1 = a^2+(a^2\lambda/\theta)\mu_g(\theta,\lambda)+\lambda^2.$$

Now, from Lingappaiah (1986),

(6′e) $$\mu_g(\theta,\lambda) = \frac{2\theta}{a}, \quad \mu'_{2g}(\theta,\lambda) = \frac{2\theta}{a^3} + \frac{4\theta^2}{a^2}.$$

For s = t = 0, (9) can be put as (10).

For r = 0, we get (6′d). For r = 1, (10) is

(7′a) $\mu_1(\theta,\lambda) = a^2\mu_g(\theta,\lambda) + (a^2\lambda/\theta)\mu'_{2g}(\theta,\lambda) + \lambda^2[2+\mu_1(\theta+\lambda,\lambda)]$

which is

(7′b) $\mu_1(\theta,\lambda) - \lambda^2\mu_1(\theta+\lambda,\lambda) = a^2\mu_g(\theta,\lambda) + (a^2\lambda/\theta)\mu'_{2g}(\theta,\lambda)+2\lambda^2.$

Using (6′e) and (4′b) with $\theta_1 = \theta_2 = \theta$, $\lambda_1 = \lambda_2 = \lambda$, it is easy to check that (7′a) is true. Now,

(8′) $$\mu'_{r1}(\theta,\lambda) = a^2 \sum_z h_o(z)z^r \sum_x A^x B_{z-x}/x!(z-x)!.$$

Differentiating (8′) w.r.t. θ, one gets

(8′a) $$\frac{\partial\mu'_r(\theta,\lambda)}{\partial\theta} =$$

$$= -2\mu'_{r1}(\theta,\lambda) + \sum_z a^2 z^r h_o(z) \sum_x A^{x-1}B^{z-x-1}[Bx+A(z-x)].$$

But

(8′b) $$Bx + A(z-x) = 2\lambda x(z-x) + \theta z.$$

From (8′b), (8′) reduces to

$$(8'c) \qquad \frac{\partial \mu'_{r1}(\theta,1)}{\partial\theta} = -2\mu'_{r1}(\theta,\lambda) + (a^2/\theta)\mu'_{(r+1)g}(\theta,\lambda) +$$

$$+ 2\lambda^2 a^2 \sum_z z^r h_0(z) \sum_x A^{x-1}B^{z-x-1}/(x-1)!(z-x-1)!.$$

Setting $x-1 = u$ and $z = (z-2)+2$, we get (8′c) as (11).

For $r = 0$, LHS of (11) is zero from (4′a). RHS is $-2+(a^2/\theta)\mu_g(\theta,\lambda)+2\lambda$ which is zero using (6′e). For $r = 1$, LHS of (11) is $2/a$ from (6′e) and RHS is $-2\mu_1(\theta,\lambda) + (a^2/\theta)\mu'_{2g}(\theta,\lambda) + 2\lambda[2+\mu_1(\theta+\lambda,\lambda)]$ which is also $2/a$ what can be seen using (6′e) and (4′b) again.

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