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ASYMPTOTIC EQUIVALENCE OF DIFFERENTIAL EQUATION IN BANACH SPACE

1. The purpose of this paper is the study of the asymptotic equivalence between the solutions of the differential equations

(I) $x' = A(t)x + f(t,x,T(x))$ and (II) $x' = A(t)x$

in Banach space E. Here x, f are the elements of E, A(t) is a linear operator on E. More precisely, we give some conditions which guarantee that for each bounded solution $y: J \to E$ ($J = \langle 0,\infty)$, E-Banach space with norm $\|\cdot\|$) of (II) there exists a bounded solution $x: J \to E$ of (I) such that

$$(*) \qquad \lim_{t\to\infty} \|x(t) - y(t)\| = 0$$

and conversely. We prove the existence of a homeomorphism between bounded solutions of (I) and (II).

In this paper we use some notations, definitions and results from [4]-[6].

Let $\tilde{E}$ denote the space of continuous linear mappings $E \to E$, $C = C(J,E)$ the space of bounded continuous functions $u: J \to E$ with the norm $\|u\|_C = \sup\{\|u(t)\| : t \in J\}$, $L^1 = L^1(J,E)$ the space of Bochner integrable functions $u: J \to E$ with norm

$$\|u\|_1 = \int_0^\infty \|u(t)\|\,dt,$$

and let $L = L(J,E)$ represent the space of strongly measurable functions $u: J \longrightarrow E$, Bochner integrable in every finite subinterval J' of J with the topology of the convergence in the mean on every such J', i.e. convergence in $L^1(J',E)$ of the restrictions to J'. The symbol $B(J,R)$ denotes a Banach function space such that:

1° $B(J,R) \subset L(J,R)$ and $B(J,R)$ is stronger than $L(J,R)$,

2° $B(J,R)$ is not stronger than $L^1(J,R)$,

3° $B(J,R)$ contains all essentially bounded functions with compact support,

4° if $u \in B(J,R)$ and v is a real-valued measurable function on J such that $|v| \leq |u|$, then $v \in B(J,R)$ and $\|v\|_B \leq \|u\|_B$.

By $B = B(J,E)$ we represent the Banach space of all strongly measurable functions $u: J \longrightarrow E$ such that $\|u(t)\| \in B(J,R)$ and with $\|u\|_B = \big\| \|u(t)\| \big\|_B$. Let $A \in L(J,\tilde{E})$. Let U be the fundamental solution for (II), i.e. U is the continuously differentiable function from J to $\tilde{E}$ such that $U(t_0) = I$ and $U' = A(t)U$ whenever $t \in J$. Let E_1 be the subspace of E to which x belongs if and only if the function from J to E, described by $t \longrightarrow U(t)x$, is bounded. Let E_1 be closed and have closed complement E_2 such that $E = E_1 \oplus E_2$. Let P_1 and P_2 be supplementary projections of E onto E_1 and E_2, respectively.

Assume that for every $b \in B$ there exists at least one bounded solution of the differential equation

$$\text{(III)} \qquad x' = A(t)x + b(t).$$

Then, by Theorem 51.E of [5], there exists a constant $K > 0$ such that for every $b \in B$ the equation (III) has a unique bounded solution x with $x(0) \in E_2$, and this solution satisfies $\|x\|_C \leq K\|b\|_B$. For every $b \in B$ denote by $F(b)$ the bounded solution of (III) such that $x(0) \in E_2$. Then F is a mapping of B into C and

$$1^\circ \qquad \|F(b)\|_C \leq K\|b\|_B,$$

2° $F(k_1 b_1 + k_2 b_2) = k_1 F(b_1) + k_2 F(b_2)$ for $b_1, b_2 \in B$ and $k_1, k_2 \in R$.

Moreover, by [5] (Theorem 52.J),

$$(1)\quad F(b)(t) = \int_0^t U(t)P_1U^{-1}(s)b(s)ds - \int_t^\infty U(t)P_2U^{-1}(s)b(s)ds, \qquad t \in J,$$

for every $b \in B$ with compact support.

2. Consider the equation (I) in which $(t,x,y) \longrightarrow f(t,x,y)$ is a function from $J \times E \times E$ into E, continuous in x,y for any fixed $t \in J$, and strongly measurable in t for any fixed $x,y \in E$, and $T:\bar{C}(J,E) \longrightarrow \bar{C}(J,E)$ is continuous operator with $T(0) = 0$, the space $\bar{C}(J,E)$ contains all continuous functions $y:J \longrightarrow E$.

Theorem 1. If

1^0 $r_0:J \times J \longrightarrow J$ is continuous function such that

(i) $\sup_{\substack{a\le z_1\le u \\ a\le z_2\le v}} r_0(z_1,z_2) = r(u,v),\quad a>0,\quad a \in R,$

(ii) $\sup\left\{\frac{r(u,v)}{\max(u,v)} : a \le u,v \le b\right\} < 1$ for each a,b, $0 < a \le b$,

2^0 there exist a function $f \in B(J,R)$ and a constant $L_0 > 0$ such that $K\|h\|_B \le 1$, $0 < L_0 \le 1$ and

$$\|f(t,x,u) - f(t,y,v)\| \le h(t)\, r_0(\|x - y\|, \|u - v\|)$$

for any $x,y \in E$, $u,v \in D \subset E$, $t \in J$, and the operator T satisfies the condition

$$\|T(x) - T(y)\|_{\bar{C}} \le L_0\|x - y\|_{\bar{C}} \quad \text{for any} \quad x,y \in \bar{C}(J,E),$$

3^0 $f(\cdot,0,0) \in B$,

then for any $p \in E_1$ there exists a unique bounded solution $x(\cdot,p)$ of (I) with $P_1x(\cdot,p) = p$ and

$$(2)\qquad x = U(\cdot)P_1x(0) + F(f(\cdot,x,T(x))).$$

The method of the proof is the same as that of Theorem 1 from [6].

Consider now the integral equation

$$(3)\qquad x(t) = U(t)p + \int_0^t U(t)P_1U^{-1}(s)f(s,x(s),T(x)(s))ds - $$

$$- \int_t^\infty U(t)P_2U^{-1}(s)f(s,x(s),T(x)(s))ds.$$

Clearly, (3) defines a mapping which can be written symbolically in the form

$$(4)\qquad G_px = U(\cdot)p + F(f(\cdot,x,T(x))).$$

L e m m a 1. Let $f(\cdot,0,0) = 0$ and $m = \sup_{t\in J}\|U(t)P_1\|$. If the assumptions of Theorem 1 hold, then for each $r_0 > 0$ and $p \in S(r_1)$, where

$$S(r_1) := \left\{p : p\in E_1, \|p\| \leq r_1 = m^{-1}(r_0 - r(r_0,r_0)K\|h\|_B)\right\},$$

G_p is a mapping of $\sum(r_0) := \left\{x : x\in C, \|x\|\leq r_0\right\}$ into $\sum(r_0)$ and is a contraction on $\sum(r_0)$.

P r o o f . For any $x \in \sum(r_0)$ from (4) we have

$$\|G_px\|_C \leq \|U(\cdot)p\|_C + \|F(f(\cdot,x,T(x)))\|_C \leq m\|p\| + K\|f(\cdot,x,T(x))\|_B \leq$$

$$\leq m\|p\| + K\|h\|_B r(r_0,r_0) \leq r_0.$$

From this it follows that $G_p\sum(r_0) \subset \sum(r_0)$.

Now we verify that the operator G_p is a contraction map. Let $x_1,x_2 \in \sum(r_0)$, then

$$\|G_px_1 - G_px_2\|_C \leq K\|f(\cdot,x_1,T(x_1)) - f(\cdot,x_2,T(x_2))\|_B \leq$$

$$\leq K\|h\|_B\, r(\|x_1-x_2\|_C, \|x_1-x_2\|_C).$$

Applying Krasnosielskii's Theorem [6] we deduce that there exists $x \in \sum(r_0)$ such that $x = G_px$. This completes the proof of Lemma.

We define a mapping V as follows $Vp = x$ for $p \in S(r_1)$, where x is a unique solution of the equation $x = G_p x$ in $\sum(r_0) \subset E$. Let $VS(r_1) = A$.

L e m m a 2. If $V: S(r_1) \to \sum(r_0)$, then V is a homeomorphism of $S(r_1)$ onto $\sum(r_0)$, and V, V^{-1} satisfy Lipschitz's condition.

P r o o f . For any $p_1, p_2 \in S(r_1)$ we have

$$\|U(t)(p_1 - p_2)\|_C \leqslant m\|p_1 - p_2\|.$$

Let $x_1, x_2 \in A$ and $Vp_1 = x_1$, $Vp_2 = x_2$. From the definition of the operator V we obtain

$$x_1 = U(\cdot)p_1 + F(f(\cdot, x_1, T(x_1))), \quad x_2 = U(\cdot)p_2 + F(f(\cdot, x_2, T(x_2)))$$

and

$$\|x_1 - x_2\|_C \leqslant m\|p_1 - p_2\| + K\|h\|_B r(\|x_1 - x_2\|_C, \|x_1 - x_2\|_C) \leqslant$$
$$\leqslant m\|p_1 - p_2\| + K\|h\|_B \|x_1 - x_2\|_C.$$

Hence, V satisfies the Lipschitz condition

$$(5) \qquad \|x_1 - x_2\|_C \leqslant m(1 - K\|h\|_B)^{-1} \|p_1 - p_2\|.$$

Conversely, for any $x_1, x_2 \in \sum(R)$, we have

$$(6) \quad \|p_1 - p_2\| \leqslant \|x_1 - x_2\|_C + K\|h\|_B \|x_1 - x_2\|_C = (1 + K\|h\|_B)\|x_1 - x_2\|_C,$$

so V^{-1} exists.

T h e o r e m 2. Let the hypotheses of Theorem 1 and condition $K\|h\|_B m(1 - K\|h\|_B)^{-1} < 1$ hold. Then the set $S(r_1) \subset E$ is homeomorphic with some set $H \subset E$ and

(a) for every point $x(0) \in H$ there exists a continuable to infinity solution of (I),

(b) on the basis of a homeomorphism, the solutions y and x of (II) and (I) passing through the points $p \in S(r_1)$ and $x(0) = Zp \in H$, respectively, satisfy the inequality

$$\|x(t) - y(t)\| \leq mK\|h\|_B(1 - K\|h\|_B)^{-1}\|p\|,$$

(c) the mappings $Zp = x(0)$ and $Z^{-1}x(0) = p$ satisfy Lipschitz's condition, and furthermore

$$Zp = p + w_1(p), \qquad Z^{-1}x(0) = x(0) + w_2(x(0)),$$

where w_1 and w_2 are such that

$$\|w_1(p_1) - w_1(p_2)\| \leq mK\|h\|_B(1 - K\|h\|_B)^{-1} \|p_1 - p_2\|,$$

$$\|w_2(x_1(0)) - w_2(x_2(0))\| \leq mK\|h\|_B(1-K\|h\|_B(1+m))^{-1}\|x_1(0)-x_2(0)\|.$$

P r o o f . Let $H = \{x(0) : x(t) \in A\}$. Let $Z : S(r_1) \to H$, where $Zp = Vp|_{t=0} = x(0)$. Then Z has inverse Z^{-1} defined by $Z^{-1}x(0) = V^{-1}x(0) = p$. For every $p_1, p_2 \in S(r_1)$ and $x_1(0) = Zp_1$, $x_2(0) = Zp_2$, we have

$$\|x_1(0) - x_2(0)\| \leq \|p_1 - p_2\| + \|F(f(t,x_1(t),T(x_1)(t))) -$$

$$- F(f(t,x_2(t),T(x_2)(t)))\|\big|_{t=0} \leq \|p_1 - p_2\| + K\|h\|_B\|x_1 - x_2\|_C.$$

This implies, by (5), that

$$(7) \quad \|x_1(0) - x_2(0)\| \leq (1 + mK\|h\|_B(1 - K\|h\|_B)^{-1})\|p_1 - p_2\|.$$

Analogically

$$\|p_1 - p_2\| \leq \|x_1(0) - x_2(0)\| + K\|h\|_B\|x_1 - x_2\|_C \leq$$

$$\leq \|x_1(0) - x_2(0)\| + mK\|h\|_B(1 - K\|h\|_B)^{-1}\|p_1-p_2\|.$$

Hence

$$(8) \quad \|p_1 - p_2\| \leq (1 - mK\|h\|_B(1 - K\|h\|_B)^{-1})^{-1}\|x_1(0) - x_2(0)\|,$$

so Z and Z^{-1} satisfy Lipschitz's condition (7) and (8), respectively. Let

$$w_1(p) = Zp - p = x(0) - p = F(f(t,x(t),T(x)(t)))\big|_{t=0},$$

$$w_2(x(0)) = Z^{-1}x(0) - x(0) = p - x(0) = -F(f(t,x(t),T(x)(t)))\big|_{t=0}.$$

Then

$$(9) \qquad \|w_1(p_1) - w_1(p_2)\| \leq$$

$$\leq \sup_{t\in J} \|F(f(t,x_1(t),T(x_1)(t))) - F(f(t,x_2(t),T(x_2)(t)))\| \leq$$

$$\leq mK\|h\|_B(1 - K\|h\|_B)^{-1}\|p_1 - p_2\|$$

and, since $w_2(x(0)) = -w_1(p)$, by (9), (8), we have

$$\|w_2(x_1(0)) - w_2(x_2(0))\| = \|w_1(p_1)) - w_1(p_2)\| \leq$$

$$\leq mK\|h_B\|(1 - K\|h\|_B)^{-1}\|p_1 - p_2\| \leq$$

$$\leq mK\|h\|_B(1 - K\|h\|_B(1+m))^{-1} \|x_1(0) - x_2(0)\|$$

which proves the thesis (c). Using (5) and (8) we obtain the inequality

$$\|x_1 - x_2\|_C \leq m(1-K\|h\|_B(1+m))^{-1}\|x_1(0) - x_2(0)\|.$$

Thus, for $x \in A$, we get

$$(10) \qquad \|x\|_C \leq m(1-K\|h\|_B(1+m)^{-1}\|x(0)\|$$

which proves the thesis (a).

Let $y = U(\cdot)p$ be a bounded solution of (II). Then, for every bounded solution x of (I) with $x(0) = Zp$, we have

$$\|x(t) - y(t)\| = \|F(f(t,x(t),T(x)(t)))\| \leq$$

$$\leq K\|h\|_B\|x\|_C \leq m\, K\|h\|_B(1 - K\|h\|_B)^{-1}\|p\|.$$

The proof of Theorem 2 is complete.

Theorem 3. If

1^0 the assumptions of Theorem 1 hold,

2^0 $\lim_{d\to\infty} \|\chi_{<d,\infty)} b\|_B = 0$ for every $b \in B(J,R)$,

3^0 $\lim_{t\to\infty} \|U(t)P_1\| = 0$,

4^0 $f(\cdot,0,0) = 0$,

then (*) holds.

Proof. Let x be a bounded solution of (I). For any $\tau \in J$ put

$$u_\tau = F(\chi_{<0,\tau>} f(\cdot,x,T(x))), \quad v_\tau = F(\chi_{<\tau,\infty)} f(\cdot,x,T(x))).$$

Because

$$\|\chi_{<\tau,\infty)}(t) f(t,x(t),T(x)(t))\| \leqslant \chi_{<\tau,\infty)}(t) h(t) r(\|x\|_C, \|x\|_C)$$

for $t \in J$, so we have

$$\|v_\tau\|_C \leqslant K \|\chi_{<\tau,\infty)} f(\cdot,x,T(x))\|_B \leqslant K \|\chi_{<\tau,\infty)} h(\cdot)\|_B r(\|x\|_C, \|x\|_C).$$

By assumption 2^0, $\lim_{\tau\to\infty} \|\chi_{<\tau,\infty)} h(\cdot)\|_B = 0$, and therefore for any $\varepsilon > 0$ we can choose $\tau > 0$ such that $\|v_\tau\|_C \leqslant \frac{\varepsilon}{2}$. Moreover, by 3^0, $\lim_{t\to\infty} \|U(t)P_1\| = 0$. Hence, there exists a $t_0 > \tau$ such that

$$\|u_\tau(t)\| \leqslant \|U(t)P_1\| \, \Big\| \int_0^\tau U^{-1}(s) f(s,x(s),T(x)(s))\,ds \Big\| \leqslant \frac{\varepsilon}{2}$$

for $t_0 \leqslant t$. Let $y = U(\cdot)p$ be a bounded solution of (II). Then for every fixed bounded solution of (I) with $x(0) = Zp$ we have

$$\|x(t) - y(t)\| \leqslant \|u_\tau(t)\| + \|v_\tau(t)\| \leqslant \varepsilon$$

for $t \geqslant t_0$, which implies (*) (ε being arbitrary).

R e m a r k 1. Theorem 3 is a generalization of an analogous result of [2] for $E = R^n$, $B = L^p$, $p = 1$ and $r(u,v) = qu$, $q<1$, and of [3] for $E = R^n$, $B = L^p$, $p = 1$.

Let B' denote the space associated to B. According to Theorem 22.M [5], if $u \in B(J,R)$ and $v \in B'(J,R)$, then $|uv| \in L^1(J,R)$ and "Hölder's inequality"

$$(11) \qquad \int_J |u(s)v(s)|\,ds \leq \|u\|_B \|v\|_{B'}$$

holds. Denote by G a function from $J \times J$ to $\tilde{E}$ such that

$$G(t,s) = \begin{cases} U(t)P_1U^{-1}(s) & \text{if} \quad 0 \leq s \leq t, \\ -U(t)P_2U^{-1}(s) & \text{if} \quad s > t \geq 0. \end{cases}$$

T h e o r e m 4. If

1^0 the assumptions of Theorem 3 hold,

2^0 $G(t,\cdot) \in B'$, $\|G(t,\cdot)\|_{B'} \leq K$ for all $t \in J$,

then the equations (I) and (II) are asymptotically equivalent.

P r o o f . Let x be a bounded solution of (I). It will now be shown that $\lim_{t\to\infty} \|w(t)\| = 0$, where

$$w(t) = \int_0^t U(t)P_1U^{-1}(s)f(s,x(s),T(x)(s))ds -$$

$$- \int_t^\infty U(t)P_2U^{-1}(s)f(s,x(s),T(x)(s))ds =$$

$$= \int_0^\infty G(t,s)f(s,x(s),T(x)(s))ds.$$

Since

$$\|\chi_{<\tau,\infty)}(t)f(t,x(t),T(x)(t))\| \leq \chi_{<\tau,\infty)}(t)h(t)r(\|x\|_C,\|x\|_C)$$

for $t, \tau \in J$, we can write

$$\|\chi_{<\tau,\infty)} f(\cdot, x, T(x))\|_B \leq \|\chi_{<\tau,\infty)} h(\cdot)\|_B r(\|x\|_C, \|x\|_C).$$

By assumption 2^0 of Theorem 3, for any $\varepsilon > 0$ we can choose $\tau > 0$ such that

$$(12) \qquad \|\chi_{<0,\infty)} h(\cdot)\|_B r(\|x\|_C, \|x\|_C) \leq \frac{\varepsilon}{2K}.$$

On the other hand, from $\lim_{t\to\infty} \|U(t)P_1\| = 0$ it follows that there exists $t_0 \geq \tau$ such that

$$(13) \qquad \|U(t)P_1\| \, \|\int_0^\tau U^{-1}(s) f(s, x(s), T(x)(s)) ds\| \leq \frac{\varepsilon}{2}$$

for $t \geq t_0$. Therefore, for $t \geq \tau$ we have

$$x(t) - y(t) = U(t)P_1 \int_0^\tau U^{-1}(s) f(s, x(s), T(x)(s)) ds +$$

$$+ \int_\tau^\infty G(t,s) f(s, x(s), T(x)(s)) ds,$$

where y is a solution of (II).

By (12), (13), 2^0 and (11), we infer that

$$\|x(t) - y(t)\| \leq \|U(t)P_1\| \, \|\int_0^\tau U^{-1}(s) f(s, x(s), T(x)(s)) ds\| +$$

$$+ \|G(t,\cdot)\|_{B'} \|\chi_{<\tau,\infty)} h(\cdot)\|_B r(\|x\|_C, \|x\|_C) \leq \varepsilon$$

for $t \geq t_0$. The proof of Theorem 4 is complete.

R e m a r k 2. The results contained in Theorems 2, 4 are some extension of those of [1] for $E = R^n$, $B = M_p$, $r(u,v) = qu$, $q < 1$, where M_p is the space of measurable functions $x: J \to R^n$ with

$$\sup_{t\geq 0}\left(\int_t^{t+1}\|x(s)\|^p ds\right)^{\frac{1}{p}}$$

Remark 3. If $x:J\longrightarrow E$ is a bounded solution of (I), then routine computations show that the function y defined by

$$y(t) = x(t) - \int_0^\infty G(t,s)f(s,x(s),T(x)(s))ds$$

is a solution of (II).

Remark 4. By Theorems 1, 4, we can show the asymptotic equivalence of the equations (I) and

$$\text{(IV)} \qquad x' = A(t)x + g(t,x,T(x))$$

where $(t,x,y)\longrightarrow g(t,x,y)$ is a function from $J\times E\times E$ into E, continuous in x, y for any fixed $t\in J$, and strongly measurable in t for any fixed $x,y\in E$.

Theorem 5. Let z be an unique solution of the equation

$$z' = A(t)z + f_1(t,z,T(z))$$

defined on J and such that $z(t)\longrightarrow 0$ as $t\longrightarrow\infty$, where $(t,\bar{x},\bar{y})\longrightarrow f_1(t,\bar{x},\bar{y})$ is a function from $J\times E\times E$ into E which fulfils the hypotheses of Theorem 1. Then (I) and (IV) are asymptotically equivalent.

Proof. Let $z(t) = x(t) - u(t)$, where $x(t)$ and $u(t)$ are solutions of (I) and (IV), respectively; then, by differentiation, we obtain

$$\text{(14)} \qquad z' = A(t)z + f_1(t,z,T(z)) \quad \text{for} \quad z\in E,\ t\in J,$$

where

$$f_1(t,z,T(z)) = f(t,z+u,T(z+u)) - g(t,u,T(u)).$$

Thus, the previous problem is reduced to finding a solution z of (14) such that $\lim_{t\to\infty} \|z(t)\| = 0$. Since z is the solution of the equation (14), so $x(t) = z(t) + u(t)$ is the solution of the equation (I). The solutions u and z exist and are bounded for $t \in J$, thus x is also bounded for $t \in J$.

Hence, by $\lim_{t\to\infty} z(t) = 0$, we have (*).

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