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THE ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF THE SINGULAR INTEGRO-DIFFERENTIAL EQUATION

1. Introduction and basic notions

This paper deals with the existence and asymptotic behaviour of solutions of the integro-differential equation

$$(1)\qquad g(x)\cdot y'(x) = y(x) + \int_{0+}^{x}\left[\sum_{I(4)=2}^{M} u_{i(4)}(x)\cdot v_{i(4)}(t)\cdot y^{i_1}(x)\cdot y^{i_2}(t)\cdot (y'(x))^{i_3}\cdot (y'(t))^{i_4}\right]dt,$$

where g, $u_{i(4)}$, $v_{i(4)}$ are given functions; $M \geq 2$ is the natural number, i_1, i_2, i_3, i_4 are natural numbers including zero, $i(4) = (i_1,i_2,i_3,i_4)$, $I(4) = i_1 + i_2 + i_3 + i_4$. The existence of solution of (1) is proved by use of Schauder's fixed point theorem. Then there is applied the implicit function theorem and Ważewski's topological method in the forms known for ordinary differential equations. The special cases of ordinary singular differential equations were similarly studied by Diblik [1]. Some equations of the form (1) occur in the theory of electrical systems, in the mechanics of fluids and, in the last time, in biology too (see [4]). Asymptotic methods are basic tools for the investigation of (1) in view of the fact that this equation has the complicated form. Unfortunately, the disadvantage of these methods are the complicated assumptions (e.g. [1], [5], [7]) being necessary to acquire the basic properties of a solution of such an equation.

Notations:

(i) $f(x) = O(g(x))$ for $x \to x_0^+$ denotes that there exists $K>0$ such that $\left|\frac{f(x)}{g(x)}\right| \leq K$ on some right-hand neighbourhood of the point x_0;

(ii) $f(x) = o(g(x))$ for $x \to x_0^+$ denotes that $\lim_{x \to x_0^+} \frac{f(x)}{g(x)} = 0$;

(iii) $f(x) \sim g(x)$ for $x \to x_0^+$ denotes that $\lim_{x \to x_0^+} \frac{f(x)}{g(x)} = 1$;

(iv) $f(x) \approx \sum_{i=1}^{n} a_i \cdot \varphi_i(x)$ for $x \to x_0^+$ denotes that $\varphi_{i+1}(x) = o(\varphi_i(x))$ for $x \to x_0^+$, $i = 1,\dots,n-1$, and

$$\left[f(x) - \sum_{i=1}^{n} a_i \cdot \varphi_i(x)\right] = o(\varphi_{n+1}(x)) \text{ for } x \to x_0^+;$$

(v) $i(n) = i_1,\dots,i_n$, $i_k \in N \cup \{0\}$, $1 \leq k \leq n$, $n \in N$, and $ij(n) = (i_1,\dots,i_n,j_1,\dots,j_n)$, $I(n) = i_1 + \dots + i_n$, $IJ(n) = (i_1+j_1) + \dots + (i_n+j_n)$.

Definitions:

1) Every function $\phi \in C^1(0,x_0)]$, $x_0 > 0$, satisfying (1) for each $x \in (0,x_0]$ will be called a solution of the equation (1).

2) A function $g \in C^1(0,x_0]$ such that $g(x) > 0$, $\lim_{x \to 0^+} g(x) = 0$, $g'(x) \sim \psi_1(x) \cdot g^{\lambda_1}(x)$ for $x \to 0^+$, $\lambda_1 > 0$, with $\lim_{x \to 0^+} \psi_1(x) \cdot g^{\tau}(x) = 0$ for each $\tau > 0$, will be called a singular function with respect to the equation (1). The point $(0^+, 0)$ will be called a singular point of the equation (1)

2. The construction of a formal series satisfying the equation (1)

We shall seek the solution of (1) in the form of a one-parametric series

$$(2) \qquad y(x,C) = \sum_{h=1}^{\infty} f_h(x)\cdot[\varphi(x,C)]^h,$$

where $\varphi(x,C) = C\cdot\exp\left[\int_{x_0}^{x} \frac{dt}{g(t)}\right]$, $C \neq 0$, is the general solution of the equation $g(x)\cdot y' = y$; the functions $f_h(x)$, $h\geq 2$, are unknown and $f_1(x) \equiv 1$.

Denote

$$y_n(x,C) = \sum_{h=1}^{n} f_h(x)\,[\varphi(x,C)]^h, \quad F_h(x) = f_h'(x) + \frac{h\cdot f_h(x)}{g(x)}, \; h\geq 1,$$

$$(3) \; K_h(x,t) = \tilde{K}_h\Big[f_1(x),\ldots,f_{h-1}(x),F_1(x),\ldots,F_{h-1}(x),f_1(t),\ldots$$

$$\ldots,f_{h-1}(t),\; F_1(t),\ldots,F_{h-1}(t)\Big] \equiv$$

$$\equiv \sum_{I(4)=2}^{M} u_{i(4)}(x)\cdot v_{i(5)}(t)\cdot \sum_{\alpha(2)+\beta(2)=h} A_1(i_1,\alpha_1)\,\cdot$$

$$\cdot\; A_2(i_2,\alpha_2)\cdot A_3(i_3,\beta_1)\cdot A_4(i_4,\beta_2)$$

with

$$A_1(p,\alpha_1) = \begin{cases} \displaystyle\sum_{w_1+\ldots+w_p=\alpha_1} \prod_{s=1}^{p} f_{w_s}(x)\cdot\varphi^{w_s}(x,C) & \text{for } p\in N, \\ 0 \quad \text{for} \quad p=0,\ \alpha_1\neq 0,\ I(4)<h, \\ 1 \quad \text{for} \quad p=0, \text{ in the other cases,} \end{cases}$$

$$A_2(p,\alpha_2) = \begin{cases} \displaystyle\sum_{w_1+\dots+w_p=\alpha_2} \prod_{s=1}^{p} F_{w_s}(x)\cdot\varphi^{w_s}(x,C) & \text{for } p\in N, \\ 0 \text{ for } p=0,\ \alpha_2\neq 0,\ I(4)<h, \\ 1 \text{ for } p=0, \text{ in the other cases,} \end{cases}$$

$$A_3(p,\beta_1) = \begin{cases} \displaystyle\sum_{v_1+\dots+v_p=\beta_1} \prod_{r=1}^{p} f_{v_r}(t)\cdot\varphi^{v_r}(t,C) & \text{for } p\in N, \\ 0 \text{ for } p=0,\ \beta_1\neq 0,\ I(4)<h, \\ 1 \text{ for } p=0, \text{ in the other cases,} \end{cases}$$

$$A_4(p,\beta_2) = \begin{cases} \displaystyle\sum_{v_1+\dots+v_p=\beta_2} \prod_{r=1}^{p} F_{v_r}(t)\cdot\varphi^{v_r}(t,C) & \text{for } p\in N, \\ 0 \text{ for } p=0,\ \beta_2\neq 0,\ I(4)<h, \\ 1 \text{ for } p=0, \text{ in the other cases,} \end{cases}$$

where $\alpha_k \geqslant i_k$, $\beta_k \geqslant i_{k+2}$, $k = 1,2$, and $\alpha(2) = \alpha_1 + \alpha_2$, $\beta(2) = \beta_1 + \beta_2$, $\alpha_i, \beta_i \in N \cup \{0\}$, $i = 1, 2$, $h \geqslant 2$; the functions f_{w_s}, f_{v_r} are coefficients of (2).

Put $\sum_h(x) = \varphi^{-h}(x,C)\cdot\int_{0+}^{x} K_h(x,t)dt$, $h>2$, $\sum_2(x) = T_2(x)$,

$$(3_h)\quad T_h(x) = \frac{1}{h-1}\cdot\varphi^{-h}(x,C)\cdot\int_{0^+}^{x} \tilde{K}_h\Big[f_1(x), T_2(x), \dots, T_{h-1}(x),$$

$$F_1(x), \frac{2}{g(x)}\cdot T_2(x), \dots, \frac{h-1}{(h-2)\cdot g(x)}\cdot T_{h-1}(x), f_1(t), T_2(t), \dots, T_{h-1}(t),$$

$$F_1(t), \frac{2}{g(t)}\cdot T_2(t), \dots, \frac{h-1}{(h-2)\cdot g(t)}\cdot T_{h-1}(t)\Big]dt, \quad h\geqslant 3.$$

In the sequel we shall use the following Lemma (see [1]).

Lemma 2.1. Let $g(x)$ be a singular function with respect to the equation

$$g(x)\cdot y' = q\cdot y + p(x). \tag{4}$$

Suppose that $p\in C^0(0,x_0]$, $p(x) = b_0(x)\cdot g^{\lambda}(x)+O[b_1(x)\cdot g^{\lambda+\varepsilon}(x)]$, $\varepsilon>0$, $\lim\limits_{x\to 0+} b_i(x)\cdot g^{\tau}(x) = 0$, $i = 0, 1$, $b_0\in C^1(0,x_0]$, $b_0(x)\neq 0$, $b_0'(x)\sim\psi_2(x)\cdot g^{\lambda_2}(x)$ for $x\to 0^+$, $\lambda_2+1>0$, $\lim\limits_{x\to 0+}\psi_2(x)g^{\tau}(x) = 0$, $\lim\limits_{x\to 0+} g^{\tau}(x)\cdot[b_0(x)]^{-1} = 0$, $q<0$. Then the equation (4) has a particular solution $y_0(x)$ unique in $(0,x_0]$ satisfying the relations

$$y_0(x) = -\frac{1}{q}\cdot b_0(x)\cdot g^{\lambda}(x) + O[g^{\nu}(x)],\quad y_0'(x) = O[g^{\nu-1}(x)]$$

for $x\in(0,x_0]$, $\nu\in(\lambda,\lambda+\min\{\lambda_1,\lambda_2+1,\varepsilon\})$.

We formally differentiate the series (2) and substitute into (1). Comparing the coefficients of equal powers of $\varphi(x,C)$, we obtain for unknown functions $f_h(x)$ the system of the recurrence differential equations

$$(5_h)\quad g(x)\cdot f_h'(x) = (1-h)\cdot f_h(x) + \varphi^{-h}(x,C)\cdot\int_{0+}^{x} K_h(x,t)dt,\quad h\geq 2.$$

Consider the following assumptions:

(B_1) $T_h\in C^0(0,x_0]$, $T_h(x) = b_{0h}(x)\cdot g^{\lambda_h}(x)+O\left[b_{1h}(x)\cdot g^{\lambda_h+\varepsilon_h}(x)\right]$, $\varepsilon_h>0$, $\lim\limits_{x\to 0+} b_{ih}(x)\cdot g^{\tau}(x) = 0$, $i = 0, 1$, $b_{0h}(x)\neq 0$, $b_{0h}\in C^1(0,x_0]$, $b_{0h}'(x)\sim\psi_{2h}(x)\cdot g^{\lambda_{2h}}(x)$ for $x\to 0^+$, $\lim\limits_{x\to 0+} g^{\tau}(x)\cdot[b_{0h}(x)]^{-1} = 0$, $h\geq 2$.

(B_2) There exist constants $\nu_h\in(\lambda_h,\lambda_h+\min\{\lambda_1,\lambda_{2h}+1,\varepsilon_h - \Delta^{*}_{h-1}\})$, where $\Delta^{*}_{h-1} = \max\{\Delta_1,\ldots,\Delta_{h-1}\}$, $\Delta_j = \lambda_j + \varepsilon_j - \nu_j$, $j = 2,\ldots,h-1$; $\Delta_1 = 0$, $h\geq 2$.

(B_3) $u_{i(4)}, v_{i(4)} \in C^0(0,x_0]$, $\lim\limits_{x\to 0+} u_{i(4)}(x)\cdot\varphi(x,C) = 0$,

$\lim\limits_{x\to 0+} \int\limits_x^{x_0} |v_{i(4)}(t)|\,dt < \infty$, $I(4) = 2,\dots,M$.

Theorem 2.1. If assumptions (B_1), (B_2) hold and $g(x)$ is the singular function with respect to (5_h), then the coefficients $f_h(x)$ of the series (2) are univocally defined on $(0,x_0]$ as particular solutions of the system (5_h) and the relation

$$(6_h)\quad f_h(x) = \int\limits_{0+}^{x} \left\{ \left[\exp \int\limits_x^t \frac{h-1}{g(s)}\, ds \right] \cdot \frac{\varphi^{-h}(t,C)}{g(t)} \cdot \int\limits_{0+}^{t} K_h(t,u)\,du \right\} dt$$

hold. Moreover, $f_h(x)$ and $f_h'(x)$ possess the asymptotic form

$$(7_h)\quad f_h(x) = b_{oh}(x)\cdot g^{\lambda_h}(x) + O\left[g^{\nu_h}(x)\right],\ f_h'(x) = O\left[g^{\nu_h - 1}(x)\right]$$

for $x \in (0,x_0]$, $h \geq 2$.

Proof. The relation (6_h) can be obtained by the method of variation of constants. The equation (5_2) is, by (B_1), of the form

$$g(x)\cdot f_2'(x) = -f_2(x) + b_{o2}(x)\cdot g^{\lambda_2}(x) + O\left[b_{12}(x) g^{\lambda_2+\varepsilon_2}(x)\right],$$

since $T_2 = \Sigma_2$. Lemma 2.1 gives (7_2). Substituting f_2 instead of T_2 into (3_3), we get

$$T_3(x) = b_{o3}(x)\cdot g^{\lambda_3}(x) + O\left[b_{13}^{o}(x)\cdot g^{\lambda_3+\varepsilon_3-\Delta_2^*}(x)\right],$$

where the function b_{13}^{o} has the same properties as b_{13}, because f_2 and T_2 possess the same asymptotic form. This substitution can lead to a deteriotation of the asymptotic form of T_3, see Δ_2^*, because $\lambda_2 < \nu_2 < \lambda_2 + \varepsilon_2$. Now the equation (5_3) has the form

$$g(x)\cdot f_3'(x) = -2\cdot f_3(x) + 2\cdot T_3(x).$$

The assumptions of Lemma 2.1 are fulfilled. Hence there exists the unique solution $f_3(x)$ of (5_3) and (7_3) holds on $(0,x_o]$.

Generally, substituting f_k instead of T_k into (3_h), $k = 2,\dots,h-1$, we obtain

$$g(x)\cdot f_h'(x) = (1-h)\cdot f_h(x) + b_{oh}(x)g^{\lambda_h}(x) + O\left[b_{1h}^o(x)\cdot h^{\lambda_h+\varepsilon_h-\Delta_{h-1}^*(x)}\right],$$

where $b_{1h}^o(x)$ has the same properties as $b_{1h}(x)$ and Δ_{h-1}^* possesses the same meaning as above. Lemma 2.1 gives the assertion (7_h).

The proof of Theorem 2.1 is complete.

3. The existence and asymptotic behaviour of solutions of the equation (1)

The technique used for prooving the existence and asymptotic behaviour of solutions of (1) is based on well-known Schauder's fixed point theorem and Ważewski's topological method.

Schauder's theorem. Let E be a Banach space and S its nonempty convex and closed subset. If P is a continuous mapping of S into itself and PS is relatively compact, then the mapping P has at least one fixed point.

T h e o r e m 3.1. If the assumptions (B_1), (B_2), (B_3) hold and $g(x)$ is the singular function with respect to (1), then for each value of a parameter $C \neq 0$ there exists a solution $y(x,C)$ of the equation (1) such that

$$(8)\qquad \left|y^{(i)}(x,C) - y_{n-1}^{(i)}(x,C)\right| \leq \delta \left|(f_n(x)\cdot\varphi^n(x,C))^{(i)}\right|, \quad i=0,1,$$

for $x \in (0,x_o]$, where $\delta > 1$ is a constant, x_o depends on δ, C, n and $f_n(x)$ is the solution (6_n) of the equation (5_n).

P r o o f . 1^o) Let the Banach space E be the set $C^1(0,x_o]$ of all functions h continuously differentiable on the interval $[0,x_o]$ with the norm

$$\|h(x)\| = \max_{x\in[0,x_0]} |h(x)| + \max_{x\in[0,x_0]} |h'(x)|$$

and S the set of all functions $h \in C^1[0,x_0]$ satisfying the inequalities

$$(9)\qquad \begin{cases} \left|h(x) - y_{n-1}(x,C)\right| \leq \delta\cdot\left|f_n(x)\cdot\varphi^n(x,C)\right|, \\ \left|h'(x) - y'_{n-1}(x,C)\right| \leq \delta\cdot\left|(f_n(x)\cdot\varphi^n(x,C))'\right| . \end{cases}$$

The set S is obviously nonempty and, as it is easy to see, convex and closed.

2^0) Construct the mapping P of S into itself. Let $h_0 \in S$ be an arbitrary function. Substituting $h_0(t)$, $h_0'(t)$ instead of $y(t)$, $y'(t)$ into (1) we obtain the differential equation

$$(10)\qquad g(x)\cdot y'(x) =$$

$$= y(x) + \int_{0+}^{x}\left[\sum_{I(4)=2}^{M} u_{i(4)}(x)\cdot v_{i(4)}(t)\cdot y^{i_1}(x)\cdot h_0^{i_2}(t)\cdot \cdot(y'(x))^{i_3}\cdot(h_0'(t))^{i_4}\right] dt.$$

Set

$$(11_1)\qquad y(x) = y_{n-1}(x,C) + \varphi^{n-1}(x,C)\cdot Y_0(x),$$

$$(11_2)\qquad y'(x) = y'_{n-1}(x,C) + \frac{1}{n-1}\left(\varphi^{n-1}(x,C)\right)'\cdot Y_1(x),$$

where the new functions Y_0, Y_1 satisfy the differential equation

$$(12)\qquad g(x)\cdot Y_0'(x) = (1-n)\cdot Y_0(x) + Y_1(x).$$

From (9) it follows that

$$(13)\quad \begin{cases} h_o(x) = y_{n-1}(x,C) + |H_o(x)|, & |H_o(x)| \le \delta \cdot |f_n(x) \cdot \varphi^n(x,C)|, \\ h_o'(x) = y_{n-1}'(x,C) + H_1(x), & |H_1(x)| \le \delta \cdot |(f_n(x) \cdot \varphi^n(x,C))'|. \end{cases}$$

Substituting (11_1), (11_2), (13) into (10), by (5_h), we get

$$(14)\qquad Y_1(x) = Y_o(x) + \varphi(x,C)\sum\nolimits_n +$$

$$+ \varphi^{1-n}(x,C) \cdot \int_{0+}^{x} Q_n(x,t,Y_o(x),Y_1(x),H_o(t),H_1(t))dt,$$

where $Q_n = \sum_{JK(2)=0}^{M} P_{j(2)}(x) \cdot Y_o^{j_1}(x) \cdot Y_1^{j_2}(x) \cdot R_{k(2)}(t) \cdot H_o^{k_1}(t) \cdot H_1^{k_2}(t)$, $P_{j(2)}(x)$ is the polynomial with respect to arguments $u_{i(4)}(x)$, $f_m(x)$, $F_m(x)$, $\varphi(x,C)$ and $R_{k(2)}(t)$ is the polynomial with respect to arguments $v_{i(4)}(t)$, $f_m(t)$, $F_m(t)$, $\varphi(t,C)$, $m=1,\dots,n-1$, $I(4) = 2,\dots,M$. Denote $\varphi_n(x,C) = \varphi(x,C) \cdot \sum_n$,

$$R^o_{k(2)}(x) = \varphi^{1-n}(x,C) \cdot \int_{0+}^{x} R_{k(2)}(t) \cdot H_o^{k_1}(t) \cdot H_1^{k_2}(t)dt,$$

$$V_{jk(2)}(x) = P_{j(2)}(x) \cdot R^o_{k(2)}(x), \quad JK(2) = 2,\dots,M.$$

Then the equation (14) has the form

$$Y_1(x) - Y_o(x) - \varphi_n(x,C) - \sum_{JK(2)=0}^{M} V_{jk(2)}(x) \cdot Y_o^{j_1}(x) \cdot Y_1^{j_2}(x) = 0.$$

Let $D_o = \{(x, Y_o, Y_1): 0 \le x \le x_o, |Y_o| < \delta_o, |Y_1| < \delta_1;\ \delta_o, \delta_1$ are constants$\}$. Consider the function

$$(15) \qquad W(x) = \widetilde{W}(V_{jk(2)}, \varphi_n, Y_o, Y_1) =$$

$$= Y_1(x) - Y_o(x) - \varphi_n(x, C) - \sum_{JK(2)=0}^{M} V_{jk(2)}(x) \cdot Y_o^{j_1}(x) \cdot Y_1^{j_2}(x).$$

If x_o is sufficiently small, then there exists a constant $\delta_2 = \delta_2(x_o, C, n)$ such that

$$|V_{jk(2)}(x)| < \delta_2, \ |\varphi_n(x, C)| < \delta_2, \ JK(2) = 0, \dots, M.$$

Now we can apply to (15) the implicit function theorem (see [2]) at the point $G = (V_{jk(2)} = 0,\ \varphi_n = 0,\ Y_o = 0,\ Y_1 = 0)$.

It is obvious that, after a suitable extension of W for $x = 0$, $\widetilde{W} \in C^2(D_o)$ and $\widetilde{W}(G) = \widetilde{W}'_{V_{jk(2)}}(G) = 0$, $j_1 + j_2 \neq 0$, $JK(2) = 0, \dots, M$, $W'_{\varphi_n}(G) = \widetilde{W}'_{V_{jk(2)}}(G) = W'_{Y_o}(G) = -W_{Y_1}(G) = -1$, $j_1 = j_2 = 0$, $K(2) = 0, \dots, M$. Hence the equation $W(x) = 0$ univocally defines the implicit function $Y_1 = W_1(V_{jk(2)}, \varphi_n, Y_o)$ in a region $D_1 = \{(x, Y_o):\ 0 \le x \le x_o,\ |Y_o| < \delta_{oo} \le \delta_o\}$ for sufficiently small x_o.

Denote $V_{ok(2)}(x) = V_{jk(2)}(x)$ for $j_1 = j_2 = 0$. The first approximation of the function W_1 has the form

$$(16) \quad W_1(V_{jk(2)}, \varphi_n, Y_o) = Y_o(x) + \varphi_n(x, C) + V_{ok(2)}(x) + R_2(V_{jk(2)}, Y_o),$$

R_2 being a continuous function of the second order with respect to arguments Y_o, $V_{jk(2)}$.

Substituting (16) into (12) we obtain the differential equation

(17) $$g(x)\cdot Y_o'(x) = (2-n)\cdot Y_o(x) + \varphi_n(x,C) + V_{ok(2)}(x) + R_2(V_{jk(2)}, Y_o).$$

The equation (17) satisfies in the region $D_2 = D_1 - \{0\}$ the conditions of existence and uniqueness of the solutions. It follows from condition (B_3) and definitions of the functions φ, $V_{jk(2)}$, $V_{ok(2)}$. In view of (11_1), (11_2), it is obvious that the solution of (17) determines the solution of (10).

In the sequel we shall use Ważewski's topological method (see [3]). Investigate the behaviour of the integral curves of (17) with respect to the boundary of the set

$$\Omega_o = \left\{(x,Y_o): 0<x<x_o,\ u_o(x,Y_o)<0,\ u_o(x,Y_o)=Y_o^2-[\delta\cdot f_n(x)\cdot\varphi(x,C)]^2\right\}.$$

Calculating the derivative $\dot u_o(x,Y_o)$ along the trajectories of (17) on the set $U_o = \left\{(x,Y_o): 0<x<x_o,\ u_o(x,Y_o) = 0\right\}$, we obtain

(18) $$\dot u_o(x,Y_o) = \frac{2}{g(x)}\Big[-\delta^2\cdot f_n(x)\cdot\varphi^2(x,C)\textstyle\sum_n \pm \pm\,\delta\cdot f_n(x)\cdot\varphi^2(x,C)\cdot\sum_n + Y_o(x)\cdot V_{ok(2)}(x) + Y_o(x)\cdot R_2(V_{jk(2)},Y_o)\Big].$$

The assumptions of Theorem 3.1 and the relation $\lim\limits_{x\to 0+}\dfrac{\varphi^{\tau}(x,C)}{g^{\sigma}(x)} = 0$ (σ is arbitrary real number) imply that the powers of $\varphi(x,C)$ influence, in decisive way, the convergence to zero of the terms in (18). The first two of them are of second order with respect to $\varphi(x,C)$. The polynomials $P_{j(2)}(x)\cdot R^o_{k(2)}(x)$ are at least of the second order with respect to $\varphi(x,C)$. Hence, the terms $Y_o\cdot V_{ok(2)}(x)$, $R_2(Y_o,V_{jk(2)})\cdot Y_o$ are at least of the third order with respect to $\varphi(x,C)$. Since

$f_n(x)\cdot\sum_n \sim (n-1)\cdot b^2_{on}(x)\cdot g^{\lambda_n}(x)$ for $x \to 0^+$, we have

$$\operatorname{sgn} \dot{u}_0(x,Y_0) = \operatorname{sgn}\left(-f_n(x)\cdot \textstyle\sum_m\right) = -1 \tag{19}$$

for sufficiently small x_0 depending on C, δ, n. The relation (19) implies that each point of the set U_0 is a strict ingress point with respect to the equation (17). Change the orientation of the axis x into opposite. Now each point of the set U_0 is a strict egress point with respect to the new system of coordinates. By Ważewski's topological method, we state that there exists at least one integral curve of (17) lying in Ω_0. It is obvious that this assertion remains true for arbitrary function $h_0 \in S$.

Now we shall prove the uniqueness of the solution of (17). Let $\overline{Y}_0$ be also a solution of (17). Putting $Z_0 = Y_0 - \overline{Y}_0$ and substituting it into (17), we obtain

$$\begin{aligned} g(x)\cdot Z_0'(x) &= (2-n)\cdot Z_0(x) + \\ &+ \varphi^{1-n}(x,C)\cdot\left[R_2(V_{jk(2)}(x),\ Z_0(x) + \overline{Y}_0(x)) - \right.\\ &\left. - R_2(V_{jk(2)}(x),\ \overline{Y}_0(x))\right]. \end{aligned} \tag{20}$$

Let

$$\Omega_1 = \left\{ (x,Z_0):\ 0<x<x_0,\ u_1(x,Z_0) = Z_0^2 - \left[\delta\cdot f_n(x)\cdot\ {}^{1-\alpha}(x,C)\right]^2 < 0\right\}$$

with sufficiently small constant $\alpha > 0$. Investigate the behaviour of the integral curves of (20) with respect to the boundary of Ω_1. Using the same method as above, we have

$$\operatorname{sgn} \dot{u}_1(x,Z_0) = -1 \tag{21}$$

for sufficiently small x_0. It is obvious that $\Omega_0 \subset \Omega_1$. Let $\overline{Z}_0(x)$ be any nonzero solution of (20) such that $[x_1,\overline{Z}_0(x_1)] \in \Omega_1$ for $0 < x_1 < x_0$. Let $\overline{\delta} \in (0,\delta)$ be such a constant that $[x_1,\overline{Z}_0(x_1)] \in \partial\Omega_1(\overline{\delta})$. If the curve $\overline{Z}_0 = \overline{Z}_0(x)$ lays in $\Omega_1(\overline{\delta})$

for $0<x<x_1$, it would have to be valid that $[x_1,\bar{Z}_0(x)]$ is a strict egress point of $\partial\Omega_1(\bar{\delta})$. This contradicts the relation (21). Hence in $\Omega_0\subset\Omega_1$ there is only the trivial solution $Z_0(x)\equiv 0$ of (20), so $\bar{Y}_0(x)$ is unique solution of (17).

From (11_1) we obtain

$$(22)\quad |y_0(x)-y_{n-1}(x,C)| = |\varphi^{n-1}(x,C)\ \bar{Y}_0|\leqslant\delta\cdot|f_n(x)\cdot\varphi^n(x,C)|,$$

where $y_0(x)$ is a solution of (10) for $x\in(0,x_0]$. Similarly, from (11_2), (14) we have

$$(23)\quad |y_0'(x)-y_{n-1}'(x,C)| = \frac{\varphi^{n-1}(x,C)}{g(x)}\cdot|\bar{Y}_1|\leqslant\delta|(f_n(x)\cdot\varphi^n(x,C))'|\ .$$

It is obvious (after a suitable extension of $y_0(x)$ for $x = 0$) that $P: h_0 \longrightarrow y_0$ maps S into itself and $PS\subseteq S$.

3^0) We shall prove that PS is relatively compact and P is a continuous mapping. It is easy to see, by (22), (23), that PS is the set of uniformly bounded and equicontinuous functions for $x\in[0,x_0]$. By Ascoli's theorem (see [6]), PS is relatively compact. Let $\{h_r(x)\}_1^\infty$ be an arbitrary sequence in S such that

$$\|h_r(x) - h_0(x)\| = \varepsilon_r,\quad \lim_{r\to\infty}\varepsilon_r = 0,\quad h_0\in S.$$

Denote

$$r_{V_{jk(2)}}(x) = P_{j(2)}(x)\cdot\varphi^{1-n}(x,C)\cdot\int_{0^+}^{x} R_{k(2)}(t)\cdot H_{r0}^{k_1}(t)\cdot H_{r1}^{k_2}(t)dt,$$

where $H_{r0}(t)$, $H_{r1}(t)$, $r\geqslant 1$, have the same properties as $H_0(t)$, $H_1(t)$. It is obvious that the solution $\bar{Y}_r(x)$ of the equation

$$(24)\qquad g\cdot Y_0' = (2-n)Y_0 + \varphi_n + r_{V_{ok(2)}} + R_2\left(r_{V_{jk(2)}},\ Y_0\right)$$

corresponds to the function $h_r(x)$ and $\overline{Y}_r \in \Omega_0$. Similarly, the solution $\overline{Y}_0(x)$ of (17) corresponds to the function $h_0(x)$. We shall show that $|\overline{Y}_r(x) - \overline{Y}_0(x)| \longrightarrow 0$ uniformly on $[0,x_0]$. Consider the region

$$\Omega_{or} = \Big\{(x,Y_0):\ 0<x<x_0,\ u_{or}(x,Y_0) = \\ = (Y_0-\overline{Y}_0)^2 - [\varepsilon_r f_n(x)\varphi^{1-\beta}(x,C)]^2 < 0;\ r>1\Big\}$$

with sufficiently small constant $\beta > 0$. Evidently $\Omega_0 \subset \Omega_{or}$ for any $r \geqslant 1$ and sufficiently small x_0. Investigate the behaviour of integral curves of (24) with respect to the boundary of Ω_{or}. Using the same method as above we obtain for trajectory derivatives

$$\operatorname{sgn} \dot{u}_{or}(x,Y_0) = -1$$

for sufficiently small x_0 and $r \geqslant 1$. By Ważewski's topological method we state that there exists at least one integral curve of (24) lying in Ω_{or}. Therefore

$$|\overline{Y}_r(x) - \overline{Y}_0(x)| \leqslant \varepsilon_r\ |f_n(x)\ \varphi^{1-\beta}(x,C)| \leqslant L\cdot\varepsilon_r\ ,$$

$L>0$ is a constant depending on n, x_0 and hence, by (11_1), we obtain

$$|y_r(x) - y_0(x)| \leqslant \varphi^{n-1}(x,C)\cdot|\overline{Y}_r(x) - \overline{Y}_0(x)| \leqslant$$

$$\leqslant \varepsilon_r\cdot L\cdot\varphi^{n-1}(x,C) \leqslant \varepsilon_r\cdot m\ ,$$

$m>0$ being a constant depending on n, x_0, L; $x \in [0,x_0]$. This estimate implies that P is continuous.

We have thus proved that the mapping P satisfies the assumptions of Schauder's fixed point theorem and hence there exists a function $h \in S$ with $h(x) = P(h(x))$. The proof is complete.

T h e o r e m 3.2. If the assumptions (B_1), (B_2), (B_3) are fulfilled and $g(x)$ is the singular function with respect to (1), then the solution $y(x,C)$ of (1) and its derivative have the following asymptotic expansions

$$y^{(i)}(x,C) \approx \sum_{k=1}^{n-1} \left(f_k(x)\cdot\varphi^k(x,C)\right)^{(i)} \quad \text{for} \quad x \to 0^+,\ i=0,1,\ n \geqslant 2,$$

where $f_k(x)$ are the functions (6_k).

P r o o f . By Theorems 2.1, 3.1, it is sufficient to show that

$$\lim_{x\to 0^+} \frac{\left(f_h(x)\cdot\varphi^h(x,C)\right)^{(i)}}{\left(f_{h-1}(x)\cdot\varphi^{h-1}(x,C)\right)^{(i)}} = 0 \quad \text{for} \quad i=0,1;\quad h=2,\dots,n-1.$$

But it follows from (7_h) and from the relation

$$\lim_{x\to 0^+} \varphi(x,C)\cdot g^{\sigma}(x) = 0 \text{ for any } \sigma\in R.$$

E x a m p l e . Consider the equation

$$x^2\cdot y'(x) = y(x) + \int_{0^+}^{x} \left[t\cdot y(x)\cdot y'(t) + [y'(t)]^2\right]dt.$$

The recurrence equations (5_h) for $h = 2, 3$ are, respectively,

$$x^2\cdot f_2'(x) =$$

$$= -f_2(x) + \varphi^{-2}(x,C)\cdot\int_{0^+}^{x}\left[\frac{1}{t}\,\varphi(x,C)\cdot\varphi(t,C) + \frac{1}{t^4}\cdot\varphi^2(t,C)\right]dt,$$

$$x^2\cdot f_3'(x) = -2\,f_3(x) + \varphi^{-3}(x,C)\int_{0^+}^{x}\left[\frac{1}{t}\cdot f_2(x)\cdot\varphi^2(x,C)\cdot\varphi(t,C) + \right.$$

$$\left. + \left(t\,f_2'(t) + \frac{2f_2(t)}{t}\right)\varphi^2(t,C)\cdot\varphi(x,C) + \left(\frac{2f_2'(t)}{t^2} + \frac{4f_2(t)}{t^4}\right)\cdot\varphi^3(t,C)\right]dt.$$

By Theorem 2.1 and Lemma 2.1, we obtain

$$f_2(x) = \frac{1}{2x^2} + O\left(x^{2\nu_2}\right), \quad f_2'(x) = O\left(x^{2\nu_2-1}\right), \quad \nu_2 \in \left(-1, -\frac{1}{2}\right),$$

$$f_3(x) = \frac{1}{3x^4} + O\left(x^{2\nu_3}\right), \quad f_3'(x) = O\left(x^{2\nu_3-1}\right), \quad \nu_3 \in \left(-2, -\frac{3}{2}\right).$$

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