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COMPLEX-TYPE DESCRIPTION OF 3-DIMENSIONAL HYPERBOLIC GEOMETRY

Dedicated to the memory of Professor Edward Otto

Introduction

The aim of this paper is to construct a field on the horizon of 3-dimensional hyperbolic space and coordinatize the space over the field we obtain. Our construction is a natural analogue of Hilbert's "End-Calculus". Let's recall the classical construction (compare [1], [5]). For given ends α, β we denote $\alpha\sqcup\beta$ the line joining α and β. Given lines L, M we denote $L \oplus M$ the bisector line; it is unique whenever K, M are parallel or hyperparallel. Denote by σ_k the symmetry with axis K. Let $0, 1, \infty$ be three distinct ends. Then we define for ends $\alpha, \beta \neq \infty$

$$\alpha + \beta = \gamma \quad \text{iff} \quad \gamma\sqcup\infty = \sigma_{(\infty\sqcup\alpha)\oplus(\infty\sqcup\beta)}(\infty\sqcup 0).$$

We obtain $-\alpha = \gamma$ iff $\gamma\sqcup\infty = \sigma_{0\sqcup\infty}(\alpha\sqcup\infty)$, i.e.

$-\alpha = \gamma$ iff $\gamma = \alpha = 0$ or $\alpha\sqcup\gamma \perp 0\sqcup\infty$.

$\alpha > 0$ iff $0, \infty$ does not separate $1, \alpha'$ (the horizon can be projectively ordered).

For $\alpha, \beta > 0$ we put

$$\alpha\cdot\beta = \gamma \quad \text{iff} \quad \gamma > 0 \text{ and}$$

$$\gamma u - \gamma = \sigma_{(\alpha u-\alpha)\oplus(\beta u-\beta)}(1u-1)$$

and extend this by $\alpha\cdot(-\beta) = (-\alpha)\cdot\beta = -\alpha\cdot\beta$, $(-\alpha)\cdot(-\beta) = \alpha\cdot\beta$. Drawing the figures we see that the construction involves two pencils of lines - lines parallel in the direction ∞, and lines orthogonal to $0u\infty$. Even subgroup of the first one is invariant under the even subgroup of the second - this gives distributivity. Operations $+$, $\cdot$ form groups because of simple properties of pencils (see [3] or [4]).

The above construction can be generalized to 3-dimensional hyperbolic geometry; the field we obtain is in fact a complex-type field (see [2]).

Basic notions and notations

The geometry we shall be concerned with is the 3-dimensional hyperbolic geometry. It is considered as the theory of all 3-dimensional Kleins Models, i.e. structures with universe K consisting of points inside non ruled quadric V in an ordered projective space coordinatized by some ordered Euclidean field. Lines and planes in the model are interpreted as intersections of K with projective lines and planes. Let us fix one such model; all the constructions will be presented in this arbitrary but fixed structure. One can visualise V as a sphere (in some coordinate system), this will be helpful to understand the construction. The horizon of our model is simply the set V; its elements, ends, will be usually denoted α, β, γ, δ, ... To every two ends α, β there corresponds uniquely a line $\alpha u\beta$ having α, β as ends. For given (hyperbolic) plane Q we denote by ∂Q the set of all ends of lines contained in Q, clearly Q determines Q uniquely. Moreover ∂Q is a circle on V. For any three ends α, β, γ we denote by $c(\alpha,\beta,\gamma)$ the circle on V determined by α, β, γ. Then $P(\alpha,\beta,\gamma)$ denotes the plane Q such that $\partial Q = c(\alpha,\beta,\gamma)$. Just from the construction we obtain

Fact 1. $\langle V; \{\partial Q: Q\text{-hyp. plane}\}, \epsilon \rangle$ is a Möbius plane.

Using an order we can define a half-line $\overrightarrow{a\alpha}$ with origin a and direction (end) α, and a (hyperbolic) half-plane. If Q' is a half-plane, then $\partial Q'$ denotes its boundary, the set of ends of halflines contained in Q'. Then $\partial Q'$ forms a "half-circle". For three ends α, β, γ we denote by $hc(\alpha,\beta;\gamma)$ this half-circle of $c(\alpha,\beta,\gamma)$ with ends α, β, which contains γ; analogously $HP(\alpha,\beta;\gamma)$ denotes this halfplane of $P(\alpha,\beta,\gamma)$ with boundary line $\alpha\cup\beta$ which contains on the horizon.

Let us choose $O,E,\infty \in V$-three distinct ends. They will be used as parameters in constructing field. In the surrounding projective space we consider planes Z_o, Z'_o tangent to V in O, ∞. Next we consider the stereographical projection λ with pole ∞, from $V \setminus \{\infty\}$ onto $Z = Z_o \setminus Z'_o$. Clearly $Z_o \setminus Z'_o$ can be considered as an affine plane. But we know even more.

Fact 2. $\mathbf{Z} = \langle Z, \lambda(\{c : \infty \in c,\ c\text{-circle on } V\}), \lambda(\{c : \infty \notin c,\ c\text{-circle on } V\})\rangle$ is an Euclidean plane with (euclidean) circles.

This fact will be crucial in justifying further investigations.

The construction of (complex) field on ends

For hyperbolic line L we denote by σ_L the line symmetry with axis L, analogously if Q is a plane then σ_Q denote the reflection on Q. Its known for L = $\alpha\cup\beta$ that $\lambda(\sigma_L)$ is a inversion with centers $\lambda(\alpha)$, $\lambda(\beta)$. Analogously $\lambda(\sigma_Q)$ is a inversion on $\lambda(\partial Q)$.

Let $M = \alpha\cup\infty$, $L = \beta\cup\infty$. Then there exists unique line N with end ∞, such that $\sigma_N(L) = M$; we denote $N = L\oplus M$.

Now we define addition on $V \setminus \{\infty\}$.

D e f i n i t i o n 3. $\alpha, \beta, \gamma \neq \infty \longrightarrow \alpha + \beta = \gamma :\longleftrightarrow \gamma\cup\infty = = \sigma_{(\alpha\cup\infty)\oplus(\beta\cup\infty)}(0\cup\infty)$.

One can notice that for every $\delta \neq \infty$ the function $\lambda(\sigma_{\delta\cup\infty})$ is a central symmetry on Z, with center $\lambda(\delta)$. Therefore $\lambda(+)$ is a usual (affine) addition in $\mathbb{Z}$. This proves

L e m m a 4. $\langle V \setminus \{\infty\}, 0, + \rangle$ is an abelian group.

If Q, R are planes, $Q,R \perp M$, M is a line then there is exactly one plane S such that $\sigma_s(Q) = R$; we denote $S = Q\oplus R$. If Q, R are planes and $Q,R \supseteq M$ then there are two, such bisecting planes. But if we consider two half-planes Q', R' with common boundary line M then there is exactly one plane S with $\sigma_s(Q') = R'$; we denote also $S = Q'\oplus R'$.

Given any end δ we denote

Q_δ : Q_δ a plane, $Q_s \perp 0\cup\infty$, $\delta \in \partial(Q)$,

Q'_δ : $HP(0,\infty\ ;\ \delta)$.

Now we define multiplication

D e f i n i t i o n 5. $\alpha,\beta \neq 0, \infty \longrightarrow$

$$\alpha\cdot\beta := \partial(\sigma_{Q_\alpha\oplus Q_\beta}(Q_E)) \cap \partial(\sigma_{Q'_\alpha\oplus Q'_\beta}(Q'_E))$$

$$\alpha \cdot 0 = 0 \cdot \alpha = 0.$$

One can notice that $\lambda(Q_\delta)$ is a circle in $\mathbb{Z}$, with center $0 = \lambda(0)$, and $\lambda(Q_\delta^{\cdot})$ is a half line with origin o. Then we see that $\cdot$ is a multiplication in a complex field, as a $\sqrt{-E}$ on V one can take I such that $I \in \partial(Q_E)$, $Eu{-}E \perp Iu{-}I$. The point $\lambda(I)$ will serve as a "imaginary unit" with respect to $\lambda(\cdot)$ on $\mathbb{Z}$. Finally we put

D e f i n i t i o n 6. $\alpha \neq 0 \longrightarrow$

$$\bar{\alpha} = \gamma \ :\longleftrightarrow \alpha \in \partial P(0,E,\infty) \ \& \ \alpha = \gamma \quad \text{or} \quad \alpha u \gamma \perp P(0,E,\infty).$$

More easily we can write $\bar{\alpha} = \sigma_{P(0,E,\infty)}(\alpha)$,

All the above considerations give us

T h e o r e m 7. $\mathbb{Z} = \langle V \setminus \{\infty\}, 0, E, +, \cdot, {}^{-} \rangle$ is a complex field satisfying $(\exists x)[\ x^2 = -1 \wedge \bar{x} = -x\ ]$.

R e m a r k 8. It is seen that Definition 3 is exactly the same as this used in Hilbert's construction. But we also can define multiplication in a way which will be more similar to those of Hilbert. Let $\delta \neq 0, \infty$. Then $\delta u{-}\delta \perp 0u\infty$. Denote by $L_\delta^{\cdot}$ the halfline $L_\delta^{\cdot} = \overrightarrow{(\delta u{-}\delta) \cap (0u\delta), \delta}$. For any two ends $\alpha, \beta \neq 0, \infty$ there is exactly one line M such that $\sigma_M(L_\alpha^{\cdot}) = L_\beta^{\cdot}$, $M \perp 0u\infty$. We denote $M = L_\alpha^{\cdot} \oplus L_\beta^{\cdot}$. Then we can put

$\alpha, \beta \neq 0, \infty \longrightarrow$

$$\alpha \cdot \beta = \gamma \ :\longleftrightarrow L_\gamma^{\cdot} = \sigma_{L_\alpha^{\cdot} \oplus L_\beta^{\cdot}}(L_E^{\cdot}).$$

Analytical interpretation of hyperbolic geometrical notions

To describe hyperbolic geometry in terms of the field we have constructed above it suffices to give analytical interpretation of points, lines, incidence and orthogonality. To do so we shall carefully analyze λ on $K \cup V$ and fix the field on $\mathbb{Z}$. Denote $V_o = V \setminus \{\infty\}$. Lines correspond to pairs of ends, therefore lines are interpreted as elements of $\Big\{\{\alpha,\beta\} : \alpha \neq \beta,\quad \alpha,\beta \in V_o\Big\} \cup V_o$. Then we have natural interpretation $\{\alpha,\beta\} \longmapsto \alpha\cup\beta$, $\alpha \longmapsto \infty\cup\alpha$. Let $a \in K$, $a \notin 0\cup\infty$. There is unique line L_a such that $a \in L_a \perp 0\cup\infty$. Let $L_a = \alpha\cup\beta$. There is unique line M_a such that $a \in M_a$, M_a has ∞ as an end. Let $M_a = \infty\cup\gamma$. Then a determines a triple α, β, γ. It's seen that $\alpha = -\beta$ and $\gamma \neq 0$, γ lies between α, β. Therefore every point a corresponds to $\langle \{\alpha,\beta\},\gamma \rangle$ such that $\alpha = -\beta$, $|\alpha - \beta| = |\alpha - \gamma| + |\gamma - \beta|$. The point a determined by $\{\alpha,\beta\},\gamma$ will be denoted by $p_{\alpha,\beta;\gamma}$. We have $p_{\alpha,\beta;\gamma} \in \alpha'\cup\beta'$ iff α, β, α', β' are cocincular in $\mathbb{Z}$ and γ lies between α', β'. This means $p_{\alpha,\beta;\gamma} \in \alpha'\cup\beta' \longleftrightarrow |\alpha' - \beta'| = |\alpha' - \gamma| + |\gamma - \beta'|$ &

$$\begin{vmatrix} |\alpha|^2 & \alpha & \bar{\alpha} & 1 \\ |\beta|^2 & \beta & \bar{\beta} & 1 \\ |\alpha'|^2 & \alpha' & \bar{\alpha}' & 1 \\ |\beta'| & \beta' & \bar{\beta}' & 1 \end{vmatrix} = 0.$$

Analogously $p_{\alpha,\beta;\gamma} \in \alpha'\cup\infty \longleftrightarrow \gamma = \alpha'$.

Obviously circles and lines in $\mathbb{Z}$ are interpreted as hyperbolic planes - orthogonality of such planes correspond to usual (euclidean) orthogonality. Therefore we have interpreted all primitive notions of hyperbolic geometry in terms of complex field.

Practically this means that one can investigate 3-dimensional hyperbolic geometry analytically with the help of complex fields.

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