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ON ALMOST r-CONTACT STRUCTURE MANIFOLDS

1. Preliminaries

Almost contact metric structure manifolds have been defined and studied by Sasaki [1], Mishra [2] and others. In the present paper, we have considered r linearly independent C^∞ vector fields $\underset{x}{U}$ and r (C^∞) I-forms $\overset{x}{u}$, r some finite integer, and studied the notion of almost r-contact structure.

Let M^{2n+r} be $(2n+r)$ dimensional differentiable manifold of class C^∞. Suppose there exists on M^{2n+r} a tensor field F of type $(1,1)$, $r(C^\infty)$ linearly independent vector fields $\underset{x}{U}$ and $r(C^\infty)$ 1-forms $\overset{x}{u}$, $x = 1,2,\dots,r$ and satisfying the following properties:

(1.1) $$F^2 = -I_n + \overset{x}{u}\otimes\underset{x}{U},\quad I_n \text{ being unit tensor field}$$

(1.2) $$\bar{X} \overset{\text{def}}{=\!=} F(X) \text{ for arbitrary vector field } X$$

and

(1.3) $$\begin{cases} \text{(i)} & F(\underset{x}{U}) = 0, \\ \text{(ii)} & \overset{x}{u}\circ F = 0, \\ \text{(iii)} & \overset{x}{u}(\underset{y}{U}) = \delta^x{}_y, \end{cases}$$

$x,y = 1,2,\dots,r$ and $\delta^x{}_y$ denotes Kronecker delta.

Thus M^{2n+r} satisfying conditions (1.1), (1.2) and (1.3) will be said to possess an almost r-contact structure.

Suppose further that M^{2n+r} is endowed with a Riemannian metric g satisfying [1]

$$(1.4) \qquad g(\bar{X},\bar{Y}) = g(X,Y) - \overset{x}{u}(X)\overset{x}{u}(Y).$$

Then in view of equations (1.1), (1.2), (1.3) and (1.4), M^{2n+r} will be said to possess an almost r-contact metric structure.

Substituting $X = \underset{l}{U}$ in (1.4) and making use of the equations (1.3)(i) and (iii), we get

$$(1.5) \qquad g(\underset{l}{U},Y) = \overset{l}{u}(Y), \quad l = 1,2,\dots,r.$$

Again, putting $\bar{Y}$ for Y in (1.4) and using (1.1) and (1.3)(ii), we get

$$(1.6) \qquad g(\bar{X},Y) + g(X,\bar{Y}) = \overset{l}{u}(Y)g(\bar{X},\underset{l}{U}).$$

Putting $X = \underset{m}{U}$ in (1.6) and making use of (1.3)(i), we obtain

$$(1.7) \qquad g(\underset{m}{U},\bar{Y}) = 0, \qquad m = 1,2,\dots,r.$$

Thus in view of (1.7), equation (1.6) takes form

$$(1.8) \qquad g(\bar{X},Y) + g(X,\bar{Y}) = 0.$$

Let us now define 2-form 'F as

$$(1.9) \qquad {}'F(X,Y) = g(\bar{X},Y).$$

2. Some results

In this section we shall prove some theorems related to almost r-contact structure.

T h e o r e m 2.1. An almost r-contact structure is not unique. If μ be a non-singular vector valued function on M^{2n+r}, let us put

$$(2.1)\quad \begin{cases} \text{(i)} & \mu \circ F' = F \circ \mu, \\ \text{(ii)} & \overset{x}{v} = \overset{x}{u} \circ \mu, \\ \text{(iii)} & \mu^{V}_{x} = \underset{x}{U}, \quad x = 1,2,\ldots,r. \end{cases}$$

Then $\left\{F', \underset{x}{V}, \overset{x}{v}\right\}$ gives an almost r-contact structure on M^{2n+r}.

P r o o f . Post multiplying (2.1)(i) by F' and making use of equations (1.1) and (2.1), we get

$$\mu \circ F'^2 = F \circ \mu \circ F' = F^2 \circ \mu = -\mu + \overset{x}{u}(\mu)\underset{x}{U}, \quad \mu \circ F'^2 = -\mu + \overset{x}{v}(\mu)\underset{x}{V},$$

or

$$(2.2)\qquad F'^2 = -I_n + \overset{x}{v} \otimes \underset{x}{V}.$$

Also from (2.1)(i) and (iii), we have $\mu \circ F'\underset{x}{V} = F \circ \mu \underset{x}{V} = F\underset{x}{U} = 0$. Thus

$$(2.3)\qquad F'\underset{x}{V} = 0, \quad x = 1,2,\ldots,r.$$

Again $\overset{x}{v} \circ F' = \overset{x}{u} \circ \mu \circ F' = \overset{x}{u} \circ F = 0$ by (1.1). Thus we have

$$(2.4)\qquad \overset{x}{v} \circ F' = 0.$$

Further $\overset{x}{v}(\underset{y}{V}) = \overset{x}{u} \circ \mu(\underset{y}{V}) = \overset{x}{u}(\underset{y}{U}) = \delta^x{}_y$ or

$$(2.5)\qquad \overset{x}{v}(\underset{y}{V}) = \delta^x{}_y, \quad x,y = 1,2,\ldots,r.$$

By virtue of equations (2.2), (2.3), (2.4) and (2.5) we conclude that $\left\{F', \underset{x}{V}, \overset{x}{v}\right\}$ gives an almost r-contact structure to M^{2n+r}.

T h e o r e m 2.2. The necessary and sufficient condition that M^{2n+r} be an almost r-contact structure manifold is that it possesses a tangent bundle π_n of dimension n tangent bundle $\tilde{\pi}_n$ conjugate to π_n and the product set $\pi_r(R^r)$ of ordered r-tuples of real numbers such that $\pi_n \cap \tilde{\pi}_n = \tilde{\pi}_n \cap \pi_r = \pi_n \cap \pi_r = \varphi$ and they span together a tangent bundle of dimension (2n+r). Projections L, M, N on π_n, $\tilde{\pi}_n$ and π_r are given by

$$(2.6)\qquad \begin{cases} \text{(i)} & 2L \overset{\text{def}}{=} -F^2 - iF, \\ \text{(ii)} & 2M \overset{\text{def}}{=} -F^2 + iF, \\ \text{(iii)} & N = F^2 + I_n = \overset{x}{u} \otimes \underset{x}{U}. \end{cases}$$

P r o o f . Suppose first that M^{2n+r} admits an almost r-contact structure. Hence corresponding to eigenvalue i[2], let $\underset{x}{P}$, $x = 1,2,\dots,n$ be n linearly independent eigenvectors. Let $\underset{x}{Q}$ be eigenvectors conjugate to $\underset{x}{P}$. Further there are r linearly independent vector fields $\underset{x}{U}$. Thus we have

$$\overset{x}{a}\underset{x}{P} = 0 \Rightarrow \overset{x}{a} = 0,$$

$$\overset{x}{b}\underset{x}{Q} = 0 \Rightarrow \overset{x}{b} = 0,$$

$$\overset{x}{c}\underset{x}{U} = 0 \Rightarrow \overset{x}{C} = 0 \Rightarrow (\overset{x}{a}, \overset{x}{b}, \overset{x}{c} \text{ are scalars}).$$

Now, if

$$(2.7)\qquad \overset{x}{a}\underset{x}{P} + \overset{x}{b}\underset{x}{Q} + \overset{x}{c}\underset{x}{U} = 0$$

then

$$(2.8)\qquad \overset{x}{a}\underset{x}{\bar{P}} + \overset{x}{b}\underset{x}{\bar{Q}} + \overset{x}{c}\underset{x}{\bar{U}} = 0.$$

In view of the equation (1.2)(i) and the fact that $\underset{x}{P}$, $\underset{x}{Q}$ are eigenvectors corresponding to eigenvalues i and -i respectively, we have

$$(2.9) \qquad \overset{x}{a}\underset{x}{P} - \overset{x}{b}\underset{x}{Q} = 0, \quad x = 1,2,\dots,n.$$

Baring (2.9) again and using the same fact that $\underset{x}{P}$, $\underset{x}{Q}$ are eigenvectors corresponding to eigenvalues i, -i, we get

$$(2.10) \qquad \overset{x}{a}\underset{x}{P} + \overset{x}{b}\underset{x}{Q} = 0.$$

Thus from (2.9) and (2.10), we have

$$\overset{x}{a} = \overset{x}{b} = 0, \quad x = 1,2,\dots,n.$$

Thus from (2.7), it follows that $\overset{x}{C} = 0$.

Thus $\left\{\underset{x}{P}, \underset{x}{Q}, \underset{x}{U}\right\}$ is a linearly independent set. From equation (2.6), we can easily show that

$$(2.11) \quad (i) \quad L\underset{x}{P} = \underset{x}{P}, \quad (ii) \quad L\underset{x}{Q} = 0, \quad (iii) \quad L\underset{x}{U} = 0,$$

$$(2.12) \quad (i) \quad M\underset{x}{P} = 0, \quad (ii) \quad M\underset{x}{Q} = \underset{x}{Q}, \quad (iii) \quad M\underset{x}{U} = 0,$$

$$(2.13) \quad (i) \quad N\underset{x}{P} = 0, \quad (ii) \quad N\underset{x}{Q} = 0, \quad (iii) \quad N\underset{x}{U} = \underset{x}{U}.$$

Thus, there exists a tangent bundle π_n of dimension n, a tangent bundle $\tilde{\pi}_n$ conjugate to π_n and the product space π_r or ordered r-tuples of real numbers such that $\pi_n \cap \tilde{\pi}_n = \pi_n \cap \pi_r = \tilde{\pi}_n \cap \pi_r = \varphi$ and $\pi_n \cup \tilde{\pi}_n \cup \pi_r$ gives a tangent bundle of dimension (2n+r), projections on π_n, $\tilde{\pi}_n$ and π_r being L, M and N respectively.

Suppose conversely that in M^{2n+r} there exists a tangent bundle π_n of dimension n, $\tilde{\pi}_n$ conjugate to π_n and product set π_r such that they are mutually disjoint and span together a tangent bundle of dimension (2n+r).

Let $\underset{x}{P}$ be n linearly independent vectors in π_n, $\underset{x}{Q}$ in $\tilde{\pi}_n$ conjugate to $\underset{x}{P}$ and $\underset{x}{U}$ be r linearly independent vectors in product set π_r. Suppose $\left\{\underset{x}{P}, \underset{x}{Q}, \underset{x}{U}\right\}$ span a tangent bundle of dimension (2n+r). Define the inverse set $\left\{\overset{x}{p}, \overset{x}{q}, \overset{x}{u}\right\}$ as

$$\overset{x}{p} \otimes \underset{x}{P} + \overset{x}{q} \otimes \underset{x}{Q} + \overset{x}{u} \otimes \underset{x}{U} = I_n. \tag{2.14}$$

Let us now put

$$F \overset{def}{=} i \left\{\overset{x}{p} \otimes \underset{x}{P} - \overset{x}{q} \otimes \underset{x}{Q}\right\}. \tag{2.15}$$

Thus we have

$$F^2 = i \left\{\overset{x}{p} \otimes \underset{x}{\bar{P}} - \overset{x}{q} \otimes \underset{x}{\bar{Q}}\right\}. \tag{2.16}$$

In view of the equation (2.15), the above equation takes form

$$F^2 = -\left\{\overset{x}{p} \otimes \underset{x}{P} + \overset{x}{q} \otimes \underset{x}{Q}\right\}$$

which by virtue of (2.14) takes the form

$$F^2 = -I_n + \overset{x}{u} \otimes \underset{x}{U}. \tag{2.17}$$

Thus M^{2n+r} admits an almost r-contact structure.

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