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A REMARK ON A CHARACTERIZATION OF THE POISSON PROCESS

A characterization of the Poisson process by properties of conditional moments obtained by Bryc (1987) may be simplified in the following way.

Theorem. Let $X = (X_t)_{t\geq 0}$ be a square integrable stochastic process such that for any $0 \leq s_1 \leq \ldots \leq s_n < s < t$, $n = 1,2,\ldots$

$$EX_t = t, \tag{1}$$

$$\operatorname{cov}(X_s, X_t) = s, \tag{2}$$

$$E(X_s | X_{s_1},\ldots,X_{s_n}) = A_1 X_{s_n} + A_0, \tag{3}$$

$$E(X_s | X_{s_1},\ldots,X_{s_n}, X_t) = B_1 X_{s_n} + B_2 X_t + B_0, \tag{4}$$

$$\operatorname{Var}(X_s | X_{s_1},\ldots,X_{s_n}, X_t) = C(X_t - X_{s_n}), \tag{5}$$

where A_0, A_1, B_0, B_1, B_2, C are some constants depending on $s_1,\ldots,s_n$, s, t only. Then X is a Poisson process.

In Bryc (1987) it was additionally assumed that

$$\operatorname{Var}(X_s | X_{s_1},\ldots,X_{s_n}) = \text{const}. \tag{6}$$

Proof. We will prove that (6) follows from (1)-(5) and then our Theorem will be a consequence of Bryc characterization. From (1), (2) and properties of conditional expectations we easily compute

$$A_1 = 1, \quad A_0 = s - s_n, \quad B_1 + B_2 = 1, \quad B_0 = 0.$$

Let us denote $Y = (X_{s_1},\dots,X_{s_n})$. Then from (5) we obtain

$$(7) \qquad E(X_s^2|Y) = E(E(X_s^2|Y, X_t)|Y) = B_1^2X_{s_n}^2 - CX_{s_n} + (2B_1B_2X_{s_n} + C)E(X_t|Y) + B_2^2E(X_t^2|Y).$$

We observe that (3) implies

$$(8) \qquad E(X_tX_s|Y) = E(E(X_t|Y, X_s)X_s|Y) = E(X_s^2|Y) + (t - s)E(X_s|Y).$$

On the other hand from (4) we get

$$(9) \qquad E(X_tX_s|Y) = E(X_tE(X_s|Y, X_t)|Y) = B_1X_{s_n}E(X_t|Y) + B_2E(X_t^2|Y).$$

The formulas (7)-(9) imply

$$B_1E(X_s^2|Y) = B_1^2X_{s_n}^2 - CX_{s_n} + (B_1B_2\, X_{s_n} + C)E(X_t|Y) + B_2(t - s)E(X_s|Y).$$

Now we apply once again (3). Hence

$$B_1E(X_s^2|Y) = X_{s_n}^2(B_1^2 + B_1B_2) + X_{s_n}(B_1B_2(t - s_n) + B_2(t - s)) + C(t - s_n) + B_2(t - s)(s - s_n).$$

Consequently

$$E(X_s^2|Y) = X_{s_n}^2 + 2A_0X_{s_n} + \text{const},$$

since $B_2(t - s_n) = s - s_n$. Applying (3) to the above equation we get (6). □

Similar simplification may be also introduced to Proposition 3.1 in Bryc (1987). (The assumption (13) of that Proposition follows from the remained ones and thus may be deleted).

REFERENCE

W. B r y c : A characterization of the Poisson process by conditional moments, Stochastics 20 (1987), 17-26.

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