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THE SEMIGROUP OF ENDOMORPHISMS OF THE FREE GROUP OF INFINITE RANK

1. Introduction

It is known [9] that an endomorphism of a free group F_n of a finite rank which is invertible from one side must be an automorphism. We consider the situation in the semigroup of endomorphisms of the free group F of countably infinite rank. Two endomorphisms $\underline{u}$ and $\underline{v}$ are called equivalent if there exists an automorphism $\underline{a}$ such that $\underline{u} = \underline{a} \circ \underline{v}$. The duality of automorphisms and the infinite Nielsen transformations allows us to use Theorem 6 from [7] to state that every endomorphism is equivalent to the so-called free endomorphism which maps a part of a base into 1 and another part into a Nielsen-reduced set. Applying this fact we generalize the proofs of several known theorems from F_n to F. We show also that every endomorphism of F is a product of a monomorphism and an epimorphism. The same is proved for the free abelian group F/F' which implies that every infinite matrix over Z with a finite number of non-zero components in every row is a product of a matrix with lineary independent rows and a left invertible matrix. We will be dealing with subsemigroups of endomorphisms of F which are listed below:

End - semigroup of all endomorphisms,

End_∞ - subset of endomorphisms with the infinitely generated image,

Epi - semigroup of epimorphisms,
Mono - semigroup of monomorphisms,
Sur - semigroup of surjective endomorphisms,
Inj - semigroup of injective endomorphisms,
M - semigroup of endomorphisms which map a base into a set independent mod F',
R - semigroup of endomorphisms invertible from the right side,
L - semigroup of endomorphisms invertible from the left side,
Aut - group of invertible endomorphisms that is automorphisms.

2. Definitions and preliminaries

Following [9] we consider a fixed relatively free group $G = F/V$ of countably infinite rank. As the basis of G a set of free generators $x_1, x_2, \ldots$ is chosen and will remain fixed. Its characteristic property is that every mapping of these generators into the group can be extended to an endomorphism of the group. We shall use a presentation of an endomorphism $\underline{u}$ by means of its components, namely, the images of the free generators x_i. If $\underline{u}$ maps x_i into u_i, $i=1,2,\ldots$, we interpret the ordered set of elements u_i as an infinite vector $\underline{u} = (u_1, u_2, \ldots)$. The word infinite will be omitted. If now $\underline{u}$ is arbitrarily chosen, the corresponding endomorphism $\underline{u}$ is defined to map the element $g(x_1,\ldots,x_n)$ into $g(x_1,\ldots,x_n)\underline{u} = g(x_1\underline{u},\ldots,x_n\underline{u}) = g(u_1,\ldots,u_n)$ or briefly, $g(\underline{x})\underline{u} = g(\underline{u})$, where $\underline{x} = (x_1, x_2, \ldots)$. The set of all endomorphisms $\underline{u}$ of G is represented by the set of all vectors with components $u_i = x_i\underline{u}$. Two vectors represent the same endomorphism if and only if the corresponding components interpreted as elements of the absolutely free group F on the same generators x_i differ by elements of V.

The product $\underline{u} \circ \underline{v}$ of endomorphisms is defined by $x_i(\underline{u} \circ \underline{v}) = (x_i\underline{u})\underline{v} = u_i(\underline{v})$ and is represented by the vector $\underline{u} \circ \underline{v} = (u_1(\underline{v}), u_2(\underline{v}), \ldots)$. The vector $\underline{x}$ represents the identity endomorphism. We shall use the same small bold-faced letter to signify either an endomorphism or a corresponding vector with components from G.

2.1. Definition. ([11], 2.2.1). An endomorphism $\underline{u}$ is called a monomorphism (epimorphism) if $\underline{v} \circ \underline{u} = \underline{w} \circ \underline{u}$ implies $v = w$ ($\underline{u} \circ \underline{v} = \underline{u} \circ \underline{w}$ implies $\underline{v} = \underline{w}$).

2.2. Definition. An endomorphism $\underline{u}$ is called invertible from the right side if there exists $\underline{v}$ such that $\underline{u} \circ \underline{v} = \underline{x}$. Respectively, $\underline{v}$ is called invertible from the left side. An endomorphism invertible from both sides is called invertible or an automorphism.

In the category of groups the notions of monomorphism and injective morphism (epimorphism and surjective morphism) coincide ([11], 2.2.4, 2.2.12) but, since we restricted ourselves to the subcategory with one object G and the semigroup of morphisms End G, the situation is not obvious.

2.3. The notions of monomorphism and injective endomorphism coincide. Indeed, $\underline{v} \circ \underline{u} = \underline{w} \circ \underline{u}$ implies $\underline{v} = \underline{w}$ if and only if $v_i\underline{u} = w_i\underline{u}$ implies $v_i = w_i$ for all i which gives the required result.

2.4. There exists an endomorphism $\underline{u}$ of the free group F which is a monomorphism and an epimorphism, but is not surjective.

Proof. We take $\underline{u} = (x_1^2, x_2^2, \ldots)$. If now $\underline{u} \circ \underline{v} = \underline{u} \circ \underline{w}$, then $(v_1^2, v_2^2, \ldots) = (w_1^2, w_2^2, \ldots)$ and, by [4], $v_i^2 = w_i^2$ implies $v_i = w_i$ and hence $\underline{v} = \underline{w}$ which shows that $\underline{u}$ is an epimorphism. Since $\underline{u}$ is injective, by 2.3, $\underline{u}$ is a monomorphism. The endomorphism $\underline{u}$ is not surjective, since $gp(\underline{u}) \neq F$.

2.5. $R \subseteq \mathrm{Mono}$, $L = \mathrm{Sur} \subseteq \mathrm{Epi}$ for a relatively free group G.

Proof. Let $\underline{u} \circ \underline{v} = \underline{x}$, that is $\underline{u} \in R$, $\underline{v} \in L$. If $\underline{w}_1 \circ \underline{u} = \underline{w}_2 \circ \underline{u}$, then multiplied by $\underline{v}$ we have $\underline{w}_1 = \underline{w}_2$ and hence $R \subseteq \mathrm{Mono}$. Similarily $L \subseteq \mathrm{Epi}$. Now $G = gp(\underline{x}) = gp(\underline{u} \circ \underline{v}) \subseteq gp(\underline{v})$ implies $L \subseteq \mathrm{Sur}$. If $\underline{v} \in \mathrm{Sur}$, then we denote by u_i the contra-image of x_i under $\underline{v}$, that is $u_i\underline{v} = x_i$. For $\underline{u} = (u_1, u_2, \ldots)$ we get $\underline{u} \circ \underline{v} = \underline{x}$ and hence $\underline{v} \in L$ which completes the proof.

In the case of F both inclusions in 2.5 are proper, because of 2.4.

2.6. $\mathrm{Aut} = R \cap L = \mathrm{Mono} \cap L = R \cap \mathrm{Epi} \subseteq \mathrm{Mono} \cap \mathrm{Epi}$.

P r o o f . All that is required is to ensure that $\underline{u} \in \mathrm{Mono} \cap L$ implies $\underline{u} \in R$ and $\underline{u} \in R \cap \mathrm{Epi}$ implies $\underline{u} \in L$. In both cases there exists $\underline{v}$ such that $\underline{u} \circ \underline{v} \circ \underline{u} = \underline{u}$. The alternative cancellings give the results.

2.7. D e f i n i t i o n ([5], [8]). An endomorphism $\underline{r}$ of G onto a subgroup $H = gp(\underline{r})$ is called a retraction and H is called a retract if and only if $\underline{r}^2 = \underline{r}$. The notion of the retract in [11] has a different meaning.

2.8. A monomorphism $\underline{u}$ is invertible from the right side if and only if gp(u) is a retract.

P r o o f . If $\underline{u} \in R$, then there exists $\underline{v}$ such that $\underline{u} \circ \underline{v} = \underline{x}$. We write $\underline{r} = \underline{v} \circ \underline{u}$, then $gp(\underline{u}) = gp(\underline{u} \circ \underline{v} \circ \underline{u}) = gp(\underline{u} \circ \underline{r}) \subseteq gp(\underline{r}) = gp(\underline{v} \circ \underline{u}) \subseteq gp(\underline{u})$ which gives $gp(\underline{r}) = gp(\underline{u})$. Moreover, $\underline{r}^2 = \underline{v} \circ \underline{u} \circ \underline{v} \circ \underline{u} = \underline{r}$, and hence $gp(\underline{u})$ is a retract. Conversely, if $\underline{u} \in \mathrm{Mono}$ and $gp(\underline{u})$ is a retract, then there exists a retraction $\underline{r}$ (obviously $\underline{r} \neq \underline{u}$ if $\underline{u} \neq \underline{x}$), such that $gp(\underline{u}) = gp(\underline{r})$. Since every u_i can be written as a word in generators from $\underline{r}$ and every r_i can be written in generators from $\underline{u}$, we have $\underline{r} = \underline{v} \circ \underline{u}$, $\underline{u} = \underline{w} \circ \underline{r} = \underline{w} \circ (\underline{v} \circ \underline{u}) = (\underline{w} \circ \underline{v}) \circ \underline{u}$. Because $\underline{u}$ is a monomorphism we conclude that $\underline{w} \circ \underline{v} = \underline{x}$. The property $\underline{r}^2 = \underline{r}$ gives $\underline{v} \circ \underline{u} \circ \underline{v} \circ \underline{u} = \underline{v} \circ \underline{u}$. Multiplying it by $\underline{w}$ from the left side and cancelling $\underline{u}$ from the right side we get $\underline{u} \circ \underline{v} = x$, which means that $\underline{u}$ is invertible from the right side, which completes the proof.

A group G is not a Hopf group if it is isomorphic to a proper quotient group. The natural homomorphism onto this quotient group followed by the isomorphism of the quotient group onto G constitutes a surjective endomorphism $\underline{u}$ which is not an automorphism. Conversely, if $\underline{u} \in \mathrm{Sur} \setminus \mathrm{Aut}$, then $G \cong G/\mathrm{Ker}\ \underline{u}$ and G is not a Hopf group. Baer's theorem [1] says that, if a relatively free group G is isomorphic to a proper quotient group, then it is also isomorphic to a proper subgroup. We can extend this theorem a little further.

2.9. Theorem. If a relatively free group G is isomorphic to a proper quotient group, then it is also isomorphic to a proper retract.

Proof. Since G is not a Hopf group, there exists a surjective endomorphism $\underline{v}$ which is not an automorphism. By 2.5, $\underline{v} \in L$ that is $\underline{u} \circ \underline{v} = \underline{x}$ for an endomorphism $\underline{u}$ such that $\underline{u} \in R \subseteq \mathrm{Mono}$. The $\underline{u}$ is not a surjection, because otherwise, by 2.6, $\underline{u}$ and hence $\underline{v}$ would be automorphisms which is not true. Now, by 2.8, $gp(\underline{u})$ is a proper retract of G freely generated by the set $\underline{u}$. Since the endomorphism $\underline{v}$ maps the base of $gp(\underline{u})$ onto the base $\underline{x}$ of G, so that $u_i\underline{v} = x_i$, we conclude that $gp(\underline{u}) \cong G$, which completes the proof.

2.10. Definition. A vector $\underline{u}$ is called free if the set of its non-unit components generates $gp(\underline{u})$ freely. By a unit in a relatively free group we mean a word which, interpreted as an element of F, is congruent to 1 modulo V.

The free vector may contain finite or infinite sets of units and non-unit components. Multiplying the free vector by a proper permutation from the left side we can get a so-called standard vector of one of the types given below:

1. $\underline{u} = (1,\dots,1,u_{k+1},u_{k+2},\dots)$, $k \geq 0$,
2. $\underline{u} = (u_1,1,u_3,1,\dots)$,
3. $\underline{u} = (u_1,\dots,u_k,1,1,1,\dots)$, $k \geq 0$,

where $u_i \neq 1$. We say that $\underline{u}$ and $\underline{v}$ are of the same type if the corresponding standard vectors are of the same form for the same k.

2.11. The set of free vectors without units represents the semigroup Mono.

Proof. Let $\underline{u}$ be a free vector without units and $\underline{v} \circ \underline{u} = \underline{w} \circ \underline{u}$, then $v_i\underline{u} = w_i\underline{u}$ and hence $v_i w_i^{-1}\underline{u} = 1$ for all i. This means that $v_i w_i^{-1}$ interpreted as element of F is congruent to 1 mod V. Thus, $\underline{v}$ and $\underline{w}$ represent the same endomorphism of G, and $\underline{u} \in \mathrm{Mono}$. Conversely, if $\underline{u} \in \mathrm{Mono}$, then, by 2.3, $\underline{u}$ is an injection and hence $\underline{u}$ is free without units. The proof is complete.

2.12. The subsets R, L, R\Aut, L\Aut, End\(R∪L), End\R, End\L form the subsemigroups in End. ([6], VI, 1.8).

2.13. For our future needs we introduce also a subsemigroup M of the vectors, where every finite subset of components is independent modulo F'. Obviously $M \in$ Mono. The natural homomorphism $\alpha: F \to F/F'$ induces an isomorphism of M onto the semigroup of monomorphisms of F/F'. We note also that $Aut \subseteq M$, because every automorphism of F induces an automorphism of F/F'.

3. Equivalence in the semigroup of endomorphisms of F

In this section we consider endomorphisms of the free group F and use the word vector instead of the word endomorphism.

3.1. D e f i n i t i o n . Vectors $\underline{u}$ and $\underline{v}$ are called equivalent if there exists an invertible vector $\underline{a}$ such that $\underline{u} = \underline{a} \circ \underline{v}$.

By 2.10, every free vector is equivalent to a standard vector.

3.2. Equivalent vectors generate the same subgroup.

P r o o f . If $\underline{u} = \underline{a} \circ \underline{v}$, then $gp(\underline{u}) = gp(\underline{a} \circ \underline{v}) \subseteq gp(\underline{v}) = gp(\underline{a}^{-1} \circ \underline{u}) \subseteq gp(\underline{u})$, which gives $gp(\underline{u}) = gp(\underline{v})$. The converse is not true, since the vectors $\underline{x}$ and $\underline{u} = (1, x_1, x_2, \dots)$ generate F, but are not equivalent.

3.3. The subsemigroups from 2.12, 2.13 are closed under equivalence.

3.4. T h e o r e m . Every vector is equivalent to a standard vector with the Nielsen-reduced set of non-unit components.

P r o o f . By [7], for every vector $\underline{u}$ with components in F there exists an invertible vector $\underline{a}$ such that the set of non-unit components in $\underline{a} \circ \underline{u}$ is Nielsen-reduced and hence $\underline{a} \circ \underline{u}$ is a free vector equivalent (by 2.10) to a standard vector, thus completing the proof.

Note that the Nielsen-Schreier subgroup theorem for F follows from 3.4 and 3.2.

3.5. Theorem. Every subgroup of F is a free group.

Proof. Let $H \subseteq F$ and $\underline{u}$ be any set of generators in H. Then, by 3.4, $\underline{u}$ is equivalent to a free vector $\underline{v}$, which generates the free subgroup $gp(\underline{v})$. By 3.2, $H = gp(\underline{u}) = gp(\underline{v})$ is free, which completes the proof.

Using Theorem 3.4 we can generalise Proposition 2.12 [5] and Theorem 3.3 [8] from F_n to F. A complicated proof of the following theorem is given in [3], Theorem 6.4.

3.6. Theorem. Let $\underline{u}$ be a homomorphism from F onto a free group H. Then F has a basis $Z = Z_1 \cup Z_2$ such that $\underline{u}$ maps $gp(Z_1)$ isomorphically onto H and maps $gp(Z_2)$ into 1.

Proof. Since the extention F of a group by the free group H splits, we can treat H as a subgroup of F and $\underline{u}$ as the endomorphism of F onto $H = gp(\underline{u})$. By 3.4, there exists an invertible vector $\underline{a}$ such that $\underline{v} = \underline{a} \circ \underline{u}$ is a standard vector. We consider the case of $\underline{v} = (v_1, 1, v_3, 1, v_5, 1 \ldots)$. For other forms of $\underline{v}$ the proof is similar. Now, by 3.2, $H = gp(\underline{u}) = gp(\underline{v})$ is freely generated by the set $\langle v_1, v_3, v_5, \ldots \rangle$. Since $\underline{a}$ corresponds to an automorphism, the set of its components $\langle a_1, a_2, a_3, \ldots \rangle =: Z$ is a free base in F such that under $\underline{u}$ its elements with the odd indices are mapped onto the free generators $v_1, v_3, v_5, \ldots$ of H and the elements with the even indices into 1. The above suggests taking $Z_1 = \langle a_1, a_3, a_5, \ldots \rangle$, $Z_2 = \langle a_2, a_4, a_6, \ldots \rangle$. Just as in the case of F_n [8] it follows that Ker $\underline{u}$ coincides with the normal closure $\overline{gp(Z_2)}$ of Z_2 in F. The proof is complete.

3.7. Two free vectors are equivalent if and only if they are of the same type and generate the same subgroup.

Proof. It follows from the previous proof that the orders of the sets Z_1 and Z_2 are in the one-to-one correspondence with the type of the standard vector $\underline{v}$ to which $\underline{u}$ is equivalent. This shows that if $\underline{u}$ and $\underline{w}$ are equivalent free vectors, then they are equivalent to the same standard vector and hence have the same type. By 3.2, they generate the same subgroup. If now $gp(\underline{u}) = gp(\underline{w})$, where $\underline{u}$ and $\underline{w}$ are standard vectors of the same type, then we shall show their equivalence. Every word u_i can be expressed through the non-unit elements of $\underline{w}$ and every w_i can be expressed through the non-unit elements of $\underline{u}$, giving $\underline{u} = \underline{a} \circ \underline{w}$ and $\underline{w} = \underline{b} \circ \underline{u}$, where we take $a_i = b_i = x_i$ if $u_i = w_i = 1$. We have now $\underline{u} = (\underline{a} \circ \underline{b}) \circ \underline{u}$, $\underline{w} = (\underline{b} \circ \underline{a}) \circ \underline{w}$. Since the non-unit elements of $\underline{u}$ and $\underline{w}$ build the sets of free generators in $gp(\underline{u})$, $gp(\underline{w})$, we conclude that $(\underline{a} \circ \underline{b})_i = x_i = (\underline{b} \circ \underline{a})_i$ if $u_i, w_i \neq 1$ and, if $u_i = w_i = 1$, then we also have $(\underline{a} \circ \underline{b})_i = a_i(\underline{b}) = x_i(\underline{b}) = b_i = x_i$. Similarly, $(\underline{b} \circ \underline{a})_i = x_i$. This shows that $\underline{a}$ is invertible and $\underline{u}$, $\underline{w}$ are equivalent, which completes the proof.

4. Relations for subsemigroups in End F

We introduce here the simplest vectors which are surjections but not the automorphisms.

(1) $\underline{p} = (1, x_1, x_2, x_3, \ldots)$,

(2) $\underline{p}^{\infty} = (x_1, 1, x_2, 1, x_3, 1, \ldots)$.

The k-th power of $\underline{p}$ contains k units $\underline{p}^k = (1, 1, \ldots, 1, x_1, x_2, x_3, \ldots)$ and $\underline{p}^0 = \underline{x}$. We shall show that every vector which is invertible from the left side is equivalent to one of $\underline{p}^k$, $0 \leq k \leq \infty$.

4.1. Every standard vector of the form (1) or (2) can be written as $\underline{u} = \underline{p}^k \circ \underline{u}'$, where $\underline{u}'$ is a free vector containing non-unit components of $\underline{u}$ in the same order. This gives immediately for the set of End_{∞} of endomorphisms with infinite images the following result.

4.2. T h e o r e m . $End_{\infty} = Sur \circ Inj$.

P r o o f . If $\underline{v} \in End_{\infty}$, then, by 3.4 and 4.1, $\underline{v} = \underline{a} \circ \underline{u} = \underline{a} \circ (\underline{p}^k \circ \underline{u}') = (\underline{a} \circ \underline{p}^k) \circ u' \in Sur \circ Inj$. Conversely, if $\underline{w} = \underline{u} \circ \underline{v}$ where $\underline{u} \in Sur = L$, $\underline{v} \in Inj = Mono$, then there exists a vector $\underline{s}$ which is a left inverse for $\underline{u}$, which gives $\underline{v} = \underline{s} \circ \underline{w}$. This implies $gp(\underline{w}) = gp(\underline{u} \circ \underline{v}) \subseteq gp(\underline{v}) = gp(\underline{s} \circ \underline{w}) \subseteq gp(\underline{w})$, that is, $gp(\underline{w}) = gp(\underline{v})$ is infinitely generated and hence $\underline{w} \in End_{\infty}$.

4.3. T h e o r e m . $L = \bigcup_k Aut \circ p^k$, $0 \leq k \leq \infty$, where the union is disjoint.

P r o o f . By 3.3, L is a disjoint union of classes of equivalent vectors. By 3.4, every class contains a standard vector with a Nielsen-reduced set of non-unit components which generate F, since by 2.6, elements of L are surjections. By ([8], p.122) the Nielsen-reduced set of generators for F consists of $x_i^{\delta_i}$, $\delta_i = \pm 1$, $i = 1,2,\ldots$. This implies that every class contains a vector $\underline{p}^k$, $0 \leq k \leq \infty$. Since, by 3.7, $\underline{p}^k$ and $\underline{p}^i$ are not equivalent for $k \neq i$, the result follows.

4.4. The set of left inverse vectors for $\underline{p}^k$ consists of the vectors of the form $\underline{s}(k) = (x_{k+1}c_1, x_{k+2}c_2, \ldots)$, where $c_i \in \overline{gp(x_1,x_2,\ldots,x_k)} = \operatorname{Ker} \underline{p}^k$ for $k < \infty$, and $\underline{s}(\infty) = (x_1c_1, x_3c_3, \ldots)$, where $c_i \in \overline{gp(x_2,x_4,\ldots)}$.

P r o o f . We consider the case $k < \infty$, then $F = N \operatorname{Ker} \underline{p}^k$, where $N = gp(x_{k+1}, x_{k+2} \ldots)$. Let now $\underline{u} \circ \underline{p}^k = \underline{x}$, then we express u_i as n_ic_i, where $n_i \in N$, $c_i \in \operatorname{Ker} \underline{p}^k$. Because $x_n\underline{p}^k = x_{n-k}$, and $u_i\underline{p}^k = n_i\underline{p}^k = \underline{x}_i$, we conclude that $n_i = x_{k+i}$ and $\underline{u} = (u_1, u_2, \ldots) = (x_{k+1}c_1, x_{k+2}c_2, \ldots)$, which completes the proof. In the case of $k = \infty$ the proof is similar.

4.5. $R = \bigcup \underline{s}(k) \circ \operatorname{Aut}$, $0 \leq k \leq \infty$.

P r o o f . If $\underline{u} \in R$, then there exists $\underline{v} \in L$ such that $\underline{u} \circ \underline{v} = \underline{x}$. By 4.3, $\underline{v} = \underline{a} \circ \underline{p}^k$ for an invertible $\underline{a}$, $0 \leq k \leq \infty$. We get then $\underline{x} = \underline{u} \circ \underline{v} = \underline{u} \circ (\underline{a} \circ \underline{p}^k) = (\underline{u} \circ \underline{a}) \circ \underline{p}^k$ and, by 4.4, $\underline{u} \circ \underline{a} = \underline{s}(k)$ which gives $\underline{u} = \underline{s}(k) \circ \underline{a}^{-1}$ completing the proof. The union above is not disjoint; we can take $\underline{s}(2) = (x_3x_1x_2, x_4, x_5, \ldots)$ and $\underline{s}'(2) = (x_3x_2x_1, x_4, x_5, \ldots)$ which are different but $\underline{s} = \underline{s}' \circ \underline{a}$, where $\underline{a}$ permutes x_1 and x_2. So, $\underline{s} \circ \underline{x}$ and $\underline{s}' \circ \underline{x}$ give two presentations of the same endomorphism.

4.6. We note here that the set of components of a $\underline{s}(k)$ is independent modulo F' and hence $R \subseteq M$.

4.7. T h e o r e m . The subgroup H of F is a retract if and only if there exists a free base $\underline{a}$ of F and a free base $\langle h_j, j \in J, |J| \leq \infty \rangle$ of H such that for the partition of the set of natural numbers $N = J \cup K$, we get $h_j = a_j c_j$, $c_j \in gp(a_k,\ k \in K)$, $j \in J$.

P r o o f . If H is finitely generated, then we can restrict ourselves to a F_n and the result follows from ([8], p. 149). If H is infinitely generated and $\underline{u}$ is any base of H, then $\underline{u}$ defines also a monomorphism of F onto H. By 2.3, H is a retract if and only if $\underline{u} \in R$ and, by 4.5, if and only if $\underline{u} = \underline{s}(k) \circ \underline{a}$, $0 \leqslant k \leqslant \infty$ which implies the result.

In the set of vectors $\underline{s}(k)$ which are left inverses for $\underline{p}^k$ we fix $\underline{s}_k = (x_{k+1}, x_{k+2}, x_{k+3}, \ldots)$ for $k < \infty$ and $\underline{s}_\infty = (x_1, x_3, x_5, \ldots)$.

4.8. We have the inclusions: (a) $\underline{v} \circ \mathrm{Mono} \subseteq \mathrm{Mono} \circ \underline{v}$, (b) $\underline{v} \circ M \subseteq M \circ \underline{v}$, (c) $\underline{v} \circ L \subseteq L \circ \underline{v}$, (d) $\underline{v} \circ R \subseteq R \circ \underline{v}$ for any vector $\underline{v} \in L$. In particular for $k > 0$:

(a') $\underline{p}^k \circ \mathrm{Mono} \subset \mathrm{Mono} \circ \underline{p}^k$, (b') $\underline{p}^k \circ M \subset M \circ \underline{p}^k$,

(c') $\underline{p}^k \circ L \subset L \circ \underline{p}^k$, (d') $\underline{p}^k \circ R \subset R \circ \underline{p}^k$.

P r o o f . We shall prove first (a')-(d'). For a given vector $\underline{u}$ we define $\underline{u}(k) = (x_1, x_2, \ldots, x_k, u_1(\underline{s}_k), u_2(\underline{s}_k), \ldots)$ for $k < \infty$ and $\underline{u}_\infty = (u_1(\underline{s}_\infty), x_2, u_2(\underline{s}_\infty), x_4, u_3(\underline{s}_\infty), x_6, \ldots)$. It is easy to see that $\underline{p}^k \circ \underline{u} = (1, \ldots, 1, u_1, u_2, \ldots) = \underline{u}(k) \circ \underline{p}^k$, for $k < \infty$ and $\underline{p}^\infty \circ \underline{u} = (u_1, 1, u_2, 1, u_3, 1, \ldots) = \underline{u}_\infty \circ \underline{p}^\infty$. Now, if $\underline{u} \in \mathrm{Mono}$, that is $\underline{u}$ is free without units, or if $\underline{u} \in M$, then obviously the same is true for $\underline{u}(k)$ and $\underline{u}_\infty$ which proves (a') and (b'). If $\underline{u} \in L$ ($\underline{u} \in R$) and $\underline{v} \circ \underline{u} = \underline{x}$ ($\underline{u} \circ \underline{v} = \underline{x}$), then the left (right) inverse for $\underline{u}(k)$ is $(x_1, \ldots, x_k, v_1(\underline{s}_k), v_2(\underline{s}_k), v_3(\underline{s}_k), \ldots)$ and the left (right) inverse for $\underline{u}_\infty$ is $(v_1(\underline{s}_\infty), x_2, v_2(\underline{s}_\infty), x_4, v_3(\underline{s}_\infty), x_6, \ldots)$. This means that $\underline{u}(k)$ and $\underline{u}_\infty$ are in L (in R) which proves (c') and (d'). The unequalities (a')-(d') are proper, because every vector $\underline{p}^k \circ \underline{u}$ from the left side has the first component equal to 1 for $k < \infty$, and the second component equal to 1 for $\underline{p}_\infty \circ \underline{u}$, while the right sides contain vectors without this property. The unequalities (a)-(d) follow from (a')-(d'), because, by 4.3, any vector $\underline{v}$ from L is equal to $\underline{a} \circ \underline{p}^k$ for an invertible a and $0 \leqslant k \leqslant \infty$.

4.9. It follows from 4.8 that ($\tilde{a}$) L∘Mono ⊆ Mono∘L, ($\tilde{b}$) L∘M ⊆ M∘L, ($\tilde{c}$) L∘R ⊆ R∘L. The unequalities ($\tilde{a}$), ($\tilde{b}$) are proper, because, by 4.2, L∘Mono = End_∞ while Mono∘L = End which is shown below and the similar situation for ($\tilde{b}$) will follow from 5.3 and 5.4.

4.10. Theorem. End = Inj∗Sur and hence End = = Mono∘Epi.

Proof. By 2.3 and 2.5, Inj = Mono, Sur = L so, since End_∞ = L∘Mono ⊆ Mono∘L, we shall only consider a vector $\underline{u}$ with a finitely generated image. Because of 3.3 and 3.4 we take $\underline{u} = (u_1, u_2, \dots, u_k, 1, 1, \dots)$ as a standard vector. One can check that $\underline{u} = (u_1(\underline{s}_\infty), \dots, u_k(\underline{s}_\infty), x_2, x_4, x_6, \dots) \circ \underline{p}^\infty$. Since $\langle u_1, u_2, \dots, u_k \rangle$ is a free set and $\underline{s}_\infty$ is a monomorphism onto $gp(x_1, x_3, x_5, \dots)$, we come to the conclusion that $\underline{u} \in \mathrm{Mono} \circ \underline{p}^\infty \subseteq \mathrm{Mono} \circ L$ completing the proof.

4.11. Corollary. The endomorphisms which have infinitely generated kernels form the semigroup $\mathrm{Mono} \circ \underline{p}^\infty$. The endomorphisms which have finitely generated kernels form the semigroup $\bigcup_k \mathrm{Mono} \circ \underline{p}^k$, $k < \infty$.

5. Endomorphisms of a free abelian group

We shall consider a free abelian group of a countably infinite rank as the quotient group F/F' with the abelian base $\underline{x}F' = (x_1F', x_2F', \dots)$. Every endomorphism of F/F' is uniquely defined by the set of images of elements from the base, so we denote the endomorphism as $\underline{u}F' = (u_1F', u_2F', \dots)$, where the vector $\underline{u}$ is not fixed. The natural homomorphism $\alpha' : F \to F/F'$ induces the homomorphism $\alpha : \mathrm{End}\, F \to \mathrm{End}\, F/F'$ which is obviously onto and maps $\underline{u}$ into $\underline{u}F'$. The kernel of α consists of all vectors $\underline{u} = (x_1c_1, x_2c_2, \dots)$, where $c_i \in F'$, that is Ker $\alpha \in$ M. We denote Ker $\alpha \cap$ Aut by I.

5.1. Ker $\alpha \cap$ R = Ker $\alpha \cap$ L = I, that is every element from Ker α which is invertible from one side must be invertible from the other.

P r o o f . By 2.6, $\mathrm{Ker}\,\alpha \cap L \subseteq M \cap L \subseteq \mathrm{Aut}$, which gives $\mathrm{Ker}\,\alpha \cap L = I$. Now, if $\underline{u} \in \mathrm{Ker}\,\alpha \cap R$, then there exists $\underline{v} \in L$ such that $\underline{u} \circ \underline{v} = \underline{x}$. This implies $\underline{v} \in \mathrm{Ker}\,\alpha \cap L = I$, and hence $\underline{u} \in I$, which was required.

5.2. The homomorphism α maps R, L and Aut onto the corresponding semigroups in End(F/F′), α maps M onto Mono(F/F′) and maps $R \setminus \mathrm{Aut}$, $L \setminus \mathrm{Aut}$ isomorphically into End(F/F′).

We shall prove now a theorem analogous to 4.2 for endomorphisms of F/F′. It is not an immediate consequence of 4.2, because the image of a monomorphism under α is not necessarily a monomorphism.

5.3. T h e o r e m . $\mathrm{End}_\infty(F/F') = \mathrm{Sur}(F/F') \circ \mathrm{Inj}(F/F') \subseteq \mathrm{Inj}(F/F') \circ \mathrm{Sur}(F/F')$ and hence, by 2.5, $\mathrm{End}_\infty(F/F') = \mathrm{Epi}(F/F') \circ \mathrm{Mono}(F/F') \subseteq \mathrm{Mono}(F/F') \circ \mathrm{Epi}(F/F')$.

P r o o f . Let $\underline{u}F'$ be an endomorphism of F/F′. To prove the theorem we find a vector $\underline{u}$ such that $\underline{u}\alpha = \underline{u}F'$ and $\underline{u} \in L \circ M \subseteq M \circ L$; then, using α we get the result because of 5.2. So, in the subgroup $gp(uF')$ we choose an abelian base $\underline{w}F' = (w_1F', w_2F', \dots)$. Since $gp(\underline{u}F') = gp(\underline{w}F')$, every u_iF' is a word in generators $\underline{w}F'$, let us say $u_iF' = s_i(w_1F', w_2F', \dots)$. We fix now any vector $\underline{w}$ such that $\underline{w}\alpha = \underline{w}F'$ and introduce the vector $\underline{u}$, where $u_i = s_i(w_1, w_2, \dots)$. Obviously then $\underline{u}\alpha = \underline{u}F'$ and $gp(\underline{u}) \subseteq gp(\underline{w})$. Because α maps the components of $\underline{w}$ onto a set of free abelian generators, we get $\underline{w} \in M$. In fact $gp(\underline{u}) = gp(\underline{w})$, because $gp(\underline{u}F') = gp(\underline{w}F')$ implies that there exist such elements $c_j \in F'$ that every w_i is a word in $u_1c_1, u_2c_2, \dots$, that is $w_i = t_i(u_1c_1, u_2c_2, \dots) \equiv t_i(u_1, u_2, \dots) \bmod F' \equiv t_i(s_1(\underline{w}), s_2(\underline{w}), \dots) \bmod F'$ and, since components of $\underline{w}$ are independent modulo F′, we conclude that $w_i = t_i(s_1(\underline{w}), s_2(\underline{w}), \dots) = t_i(u_1, u_2, \dots) \in gp(\underline{u})$. Now, by theorem 3.4, there exists an invertible vector $\underline{a}$ such that $\underline{a} \circ \underline{u}$ is a free standard vector with infinite number of non-unit components and hence, if we denote by $\underline{r}$ the vector of these non-unit components, we get, by 4.1, $\underline{a} \circ \underline{u} = \underline{p}^k \circ \underline{r}$, $0 \leq k \leq \infty$. We have then $\underline{u} = (\underline{a}^{-1} \circ \underline{p}^k) \circ \underline{r}$ and $\underline{r} = (\underline{s}_k \circ \underline{a}) \circ \underline{u}$. Since $\underline{w}$ and $\underline{r}$ are two free bases in $gp(\underline{u})$

there exists an invertible vector $\underline{b}$ such that $\underline{r} = \underline{b} \circ \underline{w}$ and then $\underline{u} = (\underline{a}^{-1} \circ \underline{p}^k) \circ \underline{r} = (\underline{a}^{-1} \circ \underline{p}^k \circ \underline{b}) \circ \underline{w} = \underline{v} \circ \underline{w} \in L \circ M$. By 4.9.($\tilde{b}$), we have $\underline{u} \in L \circ M \subseteq M \circ L$ which completes the proof.

5.4. Theorem. $End(F/F') = Inj(F/F') \circ Sur(F/F') = Mono(F/F') \circ Epi(F/F')$.

Proof. Because of the previous theorem we need to consider only an endomorphism $\underline{u}F'$ with a finitely generated image $gp(\underline{u}F')$. As in the previous proof we take an abelian base $\underline{w}F'$ in $gp(\underline{u}F')$ and fix a vector $\underline{w}$. Then we fix a vector $\underline{u}$ with components in $gp(\underline{w})$. By 3.4, $\underline{u}$ is equivalent to a standard vector $\underline{u}' = \underline{a} \circ \underline{u} = (u_1', u_2', \ldots, u_k', 1, 1, \ldots)$. Since $\underline{w}$ and $\underline{u}'$ are two free vectors generating the same finitely generated subgroup, they are of the same type and, by 3.7, are equivalent. Thus, for an invertible $\underline{a}$, $\underline{u} = \underline{a} \circ \underline{w} = \underline{a} \circ (w_1(\underline{s}_\infty), \ldots \ldots, w_k(\underline{s}_\infty), x_2, x_4, x_6, \ldots) \circ \underline{p}^\infty \in M \circ \underline{p}^\infty \subseteq M \circ L$ and using α gives the result.

5.5. Definition. ([2], 1.3). An infinite matrix over Z is called row-finite if every row contains only a finite number of non-zero components. Row-finite matrices form a semigroup. The matrix is called row-bounded if there exists an n such that all columns with indices greater than n consist of zeros.

5.6. The semigroup $End(F/F')$ is isomorphic to the semigroup of row-finite matrices over Z. The set $End_\infty(F/F')$ is in one-to-one correspondence with the set of row-bounded matrices.

Proof. Let $\underline{u}F' \in End(F/F')$, $\underline{u}F' = (u_1F', u_2F', \ldots)$, where $u_i = \prod x_j^{k_{ij}} \bmod F'$. We introduce the exponent matrix $U = (k_{ij})$ to represent $\underline{u}F'$. This correspondence defines the required isomorphism.

The reformulation of 5.3 and 5.4 gives the following theorem.

5.7. Theorem. Every row-finite matrix over Z is a product of a matrix with linearly independent rows and a left invertible matrix. Every row-finite matrix over Z which is not row-bounded is also a product of a left invertible matrix and a matrix with linearly independent rows.

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